Principal Lyapunov Exponent and Principal Floquet Bundle of Stochastic/Random Parabolic Equations

Wenxian Shen
Dept. of Mathematics and Statistics
Auburn University, USA

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Outline of Talk

- Problem – Extension of Principal Eigenvalue/Principal Eigenfunction Theory for Time Independent Parabolic Equations to Random Parabolic Equations
- Principal Eigenvalue/Principal Eigenfunction Theory for Time Independent Parabolic Equations
- Existing Extensions
- Main Results
- Idea of Proofs
- Applications – Spreading Speed in KPP Models

Problem

Problem:

Extend the *principal eigenvalue/principal eigenfunction theory* for

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} \\ \qquad\qquad +c_0(x)u, \qquad x \in D \\ u = 0, \qquad x \in \partial D \end{cases} \tag{1}$$

to

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} a_{ij}(\theta_t\omega, x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(\theta_t\omega, x)\frac{\partial u}{\partial x_i} \\ \qquad\qquad +c_0(\theta_t\omega, x)u, \qquad x \in D \\ u = 0, \qquad x \in \partial D \end{cases} \tag{2}$$

$D \subset \mathbb{R}^N$ – smooth bounded domain
$(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ – ergodic metric dynamical system

Problem

$(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ – ergodic metric dynamical system:

$(\Omega, \mathcal{F}, \mathbb{P})$ – probability space

$\theta_t : \Omega \to \Omega$

$\theta_t\omega$ – measurability in t and ω

$\theta_{t+s} = \theta_t \circ \theta_s \ \forall t, s \in \mathbb{R}$

$\mathbb{P}$ – ergodic invariant measure of θ_t

Problem

Example – nonautonomous deterministic equation

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(t,x)\frac{\partial u}{\partial x_i} \\ \qquad\qquad +c_0(t,x)u, \qquad\qquad x \in D \\ u = 0, \qquad\qquad x \in \partial D \end{cases}$$

$a_{ij}(t,x)$, $b_i(t,x)$, $c_0(t,x)$ – smooth functions

Problem

Example – nonautonomous deterministic equation

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$a_{ij}(t,x)$, $b_i(t,x)$, $c_0(t,x)$ – smooth functions

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} \tilde{a}_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \tilde{b}_i(t,x)\frac{\partial u}{\partial x_i} \\ \qquad\qquad +\tilde{c}_0(t,x)u, \qquad x \in D \\ u = 0, \qquad x \in \partial D \end{cases}$$

$\tilde{a} = (\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_0) \in \Omega$

$\Omega = \mathrm{cl}\{a(\cdot+t,\cdot) = (a_{ij}(\cdot+t,\cdot), b_i(\cdot+t,\cdot), c_0(\cdot+t,\cdot)) \mid t \in \mathbb{R}\}$

Problem

$\Omega = \mathrm{cl}\{a(\cdot + t, \cdot) = (a_{ij}(\cdot + t, \cdot), b_i(\cdot + t, \cdot), c_0(\cdot + t, \cdot)) \mid t \in \mathbb{R}\}$

$\theta_t \tilde{a}(\cdot, \cdot) = \tilde{a}(\cdot + t, \cdot)$, $\tilde{a} = (\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_0) \in \Omega$

$\mathcal{F}$ – Borel σ-algebra

$\mathbb{P}$ – ergodic invariant measure of θ_t

(it exists if Ω is compact with the open compact topology)

$(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ – ergodic metric dynamical system

Problem

$\Omega = \mathrm{cl}\{a(\cdot + t, \cdot) = (a_{ij}(\cdot + t, \cdot), b_i(\cdot + t, \cdot), c_0(\cdot + t, \cdot)) \mid t \in \mathbb{R}\}$

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(it exists if Ω is compact with the open compact topology)

$(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ – ergodic metric dynamical system

Let $A_{ij}(\theta_t \tilde{a}, x) = \tilde{a}_{ij}(t, x)$, $B_i(\theta_t \tilde{a}, x) = \tilde{b}_i(t, x)$, $C_0(\theta_t \tilde{a}, x) = \tilde{c}_0(t, x)$.

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^N A_{ij}(\theta_t \tilde{a}, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N B_i(\theta_t \tilde{a}, x) \frac{\partial u}{\partial x_i} \\ \qquad\qquad + C_0(\theta_t \tilde{a}, x) u, \qquad x \in D \\ u = 0, \qquad x \in \partial D \end{cases}$$

Problem

Example – (Stratonovich) stochastic equation

$$\begin{cases} dv = \Big(\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial v}{\partial x_i} \\ \qquad + c_0(x) v \Big) dt + v \circ dW, \qquad x \in D \\ v = 0, \qquad x \in \partial D \end{cases}$$

W – two-sided real-valued Wiener process

$\Omega = \{\omega(\cdot) \in C(\mathbb{R}, \mathbb{R}) | \omega(0) = 0\}$

$\mathcal{F}$ – Borel σ-algebra generated by the open compact topology

$\mathbb{P}$ – Wiener measure on $\mathcal{F}$

$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$

$(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\})$ – ergodic metric dynamical system

Problem

$z-$ stationary solution process of

$$dz + z = dW$$

$$z(\theta_t\omega) = -\int_{-\infty}^{0} e^s(\theta_t\omega)(s)ds = -\int_{-\infty}^{0} e^s\omega(s+t)ds + \omega(t)$$

$u = ve^{-z}$

$\Longrightarrow$

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^N a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x)\frac{\partial u}{\partial x_i} \\ \qquad +(c_0(x) + z(\theta_t\omega))u, \qquad x \in D \\ u = 0, \qquad x \in \partial D \end{cases}$$

Problem

Importance:

Basic tool for the study of nonlinear equations

Linearization of a stochastic/random parabolic equation at an invariant set (e.g. a stationary solution process, a random attractor)

$\Longrightarrow$

Stochastic/random linear parabolic equation

$\Longrightarrow$

Characterizing the rate of attractiveness of an invariant set

$\cdots\cdots$

Principal Eigenvalue/Principal Eigenfunction Theory

Equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} \\ \qquad\qquad +c_0(x)u, \qquad\qquad x \in D \\ u = 0, \qquad\qquad x \in \partial D \end{cases} \tag{1}$$

$a_{ij},\ b_i,\ c_0 : \bar{D} \to \mathbb{R}$ – smooth

Basics:

$X = L^2(D)$

$X^+ = \{u \in L^2(D) | u(x) \geq 0 \text{ a.e. } x \in D\}$

$u \geq 0$ if $u \in X^+$

$U(t)u_0 = u(t, \cdot; u_0)$ – solution of (1) with $U(0)u_0 = u_0$

$u_0 \geq 0 \longrightarrow U(t)u_0 \geq 0\ \forall t > 0$ (*positivity*)

$E \subset X$ bounded $\Longrightarrow U(t)E\ (t > 0)$ precompact (*compactness*)

Principal eigenvalue of the generator:

σ – eigenvalues of

$$\begin{cases} \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c_0(x)u = \lambda u, & x \in D \\ u = 0, & x \in \partial D \end{cases} \quad (3)$$

$\sigma = \sigma_1 \cup \sigma_2$

$\sigma_1 = \{\lambda_1\}$, $\lambda_1 \in \mathbb{R}$

$\forall \lambda \in \sigma_2$, $\mathrm{Re}\lambda < \lambda_1$

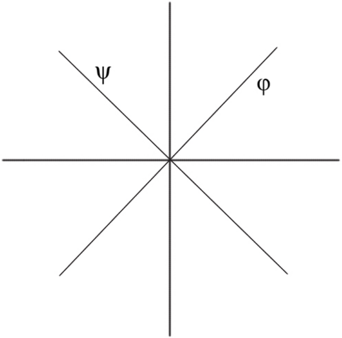
λ_1 – *principal eigenvalue* – real, simple

ϕ – eigenfunction associated to λ_1 – *principal eigenfunction*

$\phi \in X^+$

Principal Eigenvalue/Principal Eigenfunction Theory

If ψ is an eigenfunction associated to $\lambda \in \sigma_2$,
$\psi \notin X^+ \cup (-X^+)$



X

Principal Eigenvalue/Principal Eigenfunction Theory

Principal eigenvalue of the evolution operator:

$\tilde{\sigma}$ – eigenvalues of $U(1)$

$\tilde{\sigma} = \tilde{\sigma}_1 \cup \tilde{\sigma}_2$

$\tilde{\sigma}_1 = \{\mu_1\}$, $\mu_1 > 0$

$\forall \mu \in \tilde{\sigma}_2$, $|\mu| < \mu_1$

$\mu_1 = e^{\lambda_1}$– *principal eigenvalue of* $U(1)$ – real, simple

$\phi \in X^+$ – eigenfunction of $U(1)$ associated to μ_1

If ψ is an eigenfunction associated to $\mu \in \tilde{\sigma}_2$

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Principal Eigenvalue/Principal Eigenfunction Theory

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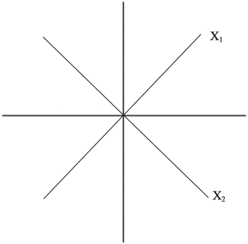
(**Krein-Rutman Theorem**)

Principal Eigenvalue/Principal Eigenfunction Theory

Exponential separation: $X = X_1 \oplus X_2$

$X_1 = \mathrm{span}\{\phi\}$, $\phi \in X^+$ – one dimensional

$X_2 \cap X^+ = \{0\}$



$U(t)X_1 = X_1 \ \forall t \in \mathbb{R}$
$U(t)X_2 \subset X_2 \ \forall t > 0$

$\exists M > 0,\ \gamma > 0$ s.t. $\forall w \in X_2,\ w \neq 0$
$\frac{\|U(t)w\|}{\|U(t)\phi\|} \leq Me^{-\gamma t}\frac{\|w\|}{\|\phi\|} \ \forall t > 0$

$\lambda_1 = \lim_{t\to\infty} \frac{\ln \|U(t)\phi\|}{t}$ – principal eigenvalue

Multiplicative ergodic theorem:

$\exists l_1 > l_2 > l_3 > \cdots$ – *Lyapunov exponents*

$\exists\ E_1, E_2, E_3, \cdots \subset X$, $\mathrm{dim} E_i < \infty$ – *Oseledets spaces or Floquet spaces*

$\exists F_1, F_2, \cdots \subset X$, $\mathrm{codim} F_i < \infty$

$X = E_1 \oplus E_2 \oplus E_3 \oplus \cdots \oplus E_i \oplus F_i$,

$U(t)E_i = E_i\ \forall t \in \mathbb{R}$

$U(t)F_i \subset F_i\ \forall t > 0$

$\lim_{t\to\pm\infty} \frac{\ln \|U(t)u_0\|}{t} = l_i$ for $u_0 \in E_i \setminus \{0\}$

$l_1 = \lambda_1$ – *principal Lyapunov exponent*

$E_1 = X_1$ (hence $\mathrm{dim} E_1 = 1$) – *principal Floquet bundle*

Existing Extensions

Equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} a_{ij}(\theta_t\omega, x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(\theta_t\omega, x)\frac{\partial u}{\partial x_i} \\ \qquad\qquad + c_0(\theta_t\omega, x)u, \qquad\qquad x \in D \\ u = 0, \qquad\qquad x \in \partial D \end{cases} \tag{2}$$

$D \subset \mathbb{R}^N$ – smooth bounded domain

$(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ – ergodic metric dynamical system

$\exists k_1(\omega), k_2(\omega) > 0$ ($k_1(\theta_t\omega), k_2(\theta_t\omega)$ continuous in t) such that

$$k_1(\omega)\sum_{i=1}^{N}\xi_i^2 \le \sum_{i,j=1}^{N} a_{ij}(\omega, x)\xi_i\xi_j \le k_2(\omega)\sum_{i=1}^{N}\xi_i^2$$

$a_{ij},\ b_i,\ c_0 : \Omega \times \bar{D} \to \mathbb{R}$ – measurable in ω and smooth in x

$a^\omega(t,x) = a_{ij}(\theta_t\omega, x),\ b^\omega(t,x) = b_i(\theta_t\omega, x),\ c_0^\omega(t,x) = c_0(\theta_t\omega, x)$ – C^1 in t

Existing Extensions

Basics:

$X = L^2(D)$

$U(t,\omega)u_0 = u(t,\cdot;u_0,\omega)$ – solution of (2) with $U(0,\omega)u_0 = u_0$

$U(t,\omega)u_0$ – continuous in $t \geq 0$, $u_0 \in X$, and measurable in $\omega \in \Omega$

$U(0,\omega) = Id$

$U(t+s,\omega) = U(t,\theta_s\omega) \circ U(s,\omega)$

$\{U(t,\omega)\}$ – random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\})$

$u_0 \geq 0 \Longrightarrow U(t,\omega)u_0 \geq 0 \ \forall t > 0,\ \omega \in \Omega$ (*positivity*)

$E \subset X$ bounded $\Longrightarrow U(t,\omega)E$ precompact $(t > 0,\ \omega \in \Omega)$ (*compactness*)

Existing Extensions

Multiplicative ergodic theorem: (Z. Lian and K. Lu)

Assume

(A1) $f_1, f_2 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $f_1(\omega) := \sup_{0 \leq s \leq 1} \ln^+ \|U(s, \omega)\|$, $f_2(\omega) := \sup_{0 \leq s \leq 1} \ln^+ \|U(1 - s, \theta_s \omega)\|$.

Existing Extensions

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Assume

(A1) $f_1, f_2 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $f_1(\omega) := \sup_{0\le s\le 1} \ln^+ \|U(s,\omega)\|$,
$f_2(\omega) := \sup_{0\le s\le 1} \ln^+ \|U(1-s, \theta_s\omega)\|$.

$\Longrightarrow$

$\exists\ \Omega_0 \subset \Omega$, $\theta_t\Omega_0 \subset \Omega_0$ $(t \in \mathbb{R})$, $\mathbb{P}(\Omega_0) = 1$ s.t. for $\omega \in \Omega_0$, one of the following cases holds

(i) $\forall u_0 \in X \setminus \{0\}$, $\lim_{t\to\infty} \frac{\ln \|U(t,\omega)u_0\|}{t} = -\infty$

Existing Extensions

(ii) $\exists\ k \geq 1$

$\exists\ \lambda_1 > \lambda_2 > \cdots > \lambda_k$ – Lyapunov exponents

$\exists E_1(\omega),\ E_2(\omega),\ \cdots,\ E_k(\omega),\ F_\infty(\omega)$ – subspaces of X,
$\mathrm{dim} E_i(\omega) < \infty$, $\mathrm{codim} F_\infty(\omega) < \infty$

$\Longrightarrow$

$X = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F_\infty(\omega)$

$U(t,\omega)E_i(\omega) = E_i(\theta_t\omega)$ for $t \in \mathbb{R}$

$U(t,\omega)F_\infty(\omega) \subset F_\infty(\omega)$ for $t > 0$

$\lim_{t\to\pm\infty} \frac{\ln \|U(t,\omega)u_0\|}{t} = \lambda_i$ for $u_0 \in E_i(\omega) \setminus \{0\}$

$\lim_{t\to\infty} \frac{\ln \|U(t,\omega)u_0\|}{t} = -\infty$ for $u_0 \in F_\infty(\omega)$

Existing Extensions

(iii) $\exists \lambda_1 > \lambda_2 > \lambda_3 > \cdots$ – Lyapunov exponents

$\exists\ E_1(\omega),\ E_2(\omega),\ \cdots$ – subspaces of X, $\mathrm{dim} E_i(\omega) < \infty$

$\exists\ F_1(\omega),\ F_2(\omega),\ \cdots$ – subspaces of X, $\mathrm{codim} F_i(\omega) < \infty$

$\Longrightarrow$

$X = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_i(\omega) \oplus F_i(\omega)$

$U(t,\omega)E_i(\omega) = E_i(\theta_t\omega)$ for $t \in \mathbb{R}$

$U(t,\omega)F_i(\omega) \subset F_i(\omega)$ for $t > 0$

$\lim_{t\to\pm\infty} \frac{\ln \|U(t,\omega)u_0\|}{t} = \lambda_i$ for $u_0 \in E_i(\omega) \setminus \{0\}$

Existing Extensions

In case of (1), $\lim_{t\to\infty} \frac{\ln \|U(t,\omega)\|}{t} = -\infty$

In case of (2) or (3),

$\lambda_1 = \lim_{t\to\infty} \frac{\ln \|U(t,\omega)\|}{t}$

–*principal Lyapunov exponent*

$E_1(\omega)$ – *principal Floquet bundle*

Existing Extensions

In case of (1), $\lim_{t\to\infty} \frac{\ln \|U(t,\omega)\|}{t} = -\infty$

In case of (2) or (3),

$\lambda_1 = \lim_{t\to\infty} \frac{\ln \|U(t,\omega)\|}{t}$

–*principal Lyapunov exponent*

$E_1(\omega)$ – *principal Floquet bundle*

Question: In the case (2) or (3),

$$\dim E_1(\omega) = ?$$

$$E_1(\omega) \cap (X^+ \setminus \{0\}) \neq \emptyset ?$$

$$F_\infty(\omega) \cap X^+ = \{0\} \text{ or } F_2(\omega) \cap X^+ = \{0\} ?$$

Existing Extensions

Exponential Separation (J. Mierczynski and W. Shen, P. Polacik, et al.)

Assume

(H1) $\exists M_0 > 0$ such that $\|a_{ij}(\omega,\cdot)\|_{L^\infty(D)}$, $\|b_i(\omega,\cdot)\|_{L^\infty(D)}$, $\|c_0(\omega,\cdot)\|_{L^\infty(D)} \le M_0$ for $\omega \in \Omega$.

Existing Extensions

Exponential Separation (J. Mierczynski and W. Shen, P. Polacik, et al.)

Assume

(H1) $\exists M_0 > 0$ such that $\|a_{ij}(\omega,\cdot)\|_{L^\infty(D)}$, $\|b_i(\omega,\cdot)\|_{L^\infty(D)}$, $\|c_0(\omega,\cdot)\|_{L^\infty(D)} \le M_0$ for $\omega \in \Omega$.

$\Longrightarrow$

$\exists\ X_1(\omega)$, $X_2(\omega)$, $\mathrm{dim} X_1(\omega) = 1$, $\mathrm{codim} X_2(\omega) = 1$

$X_1(\omega) = \mathrm{span}\{\phi(\omega)\}$, $\phi(\omega) \in X^+$

$X_2(\omega) \cap X^+ = \{0\}$

$X = X_1(\omega) \oplus X_2(\omega)$

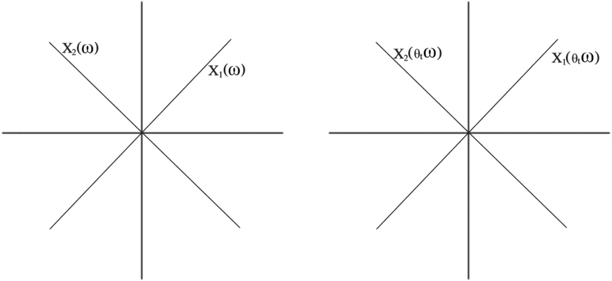
$\exists M > 0$, $\gamma > 0$

such that

Existing Extensions

$U(t,\omega)X_1(\omega) = X_1(\theta_t\omega)\ \forall t \in \mathbb{R}$

$U(t,\omega)X_2(\omega) \subset X_2(\theta_t\omega)\ \forall t > 0$



$\frac{\|U(t,\omega)w\|}{\|U(t,\omega)u\|} \le Me^{-\gamma t}\frac{\|w\|}{\|u\|} \quad \forall t > 0\ \forall u \in X_1(\omega)\setminus\{0\}$,
$w \in X_2(\omega)\setminus\{0\}$

(case (2) or (3) occurs in the MET

$E_1(\omega) = X_1(\omega)$ – one dimensional, $E_1(\omega)\cap(X^+\setminus\{0\}) \ne \emptyset$

$F_\infty(\omega)\cap X^+ = \{0\}$ or $F_2(\omega)\cap X^+ = \{0\}$)

Existing Extensions

Question:

In the case (2) or (3) of the Multiplicative Ergodic Theorem, whether there is $\phi(\omega) \in X^+$ such that

$E_1(\omega) = \mathrm{span}\{\phi(\omega)\}$

whether $F_\infty(\omega) \cap X^+ = \{0\}$ or $F_2(\omega) \cap X^+ = \{0\}$

Main Results

Consider

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} a_{ij}(\theta_t\omega, x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(\theta_t\omega, x)\frac{\partial u}{\partial x_i} \\ \qquad\qquad +c_0(\theta_t\omega, x)u, \qquad x \in D \\ u = 0, \qquad x \in \partial D \end{cases} \tag{2}$$

$\exists k_1(\omega), k_2(\omega) > 0$ ($k_1(\theta_t\omega), k_2(\theta_t\omega)$ continuous in t) such that

$$k_1(\omega)\sum_{i=1}^{N} \xi_i^2 \le \sum_{i,j=1}^{N} a_{ij}(\omega, x)\xi_i\xi_j \le k_2(\omega)\sum_{i=1}^{N} \xi_i^2$$

$a_{ij},\ b_i,\ c_0 : \Omega \times \bar{D} \to \mathbb{R}$ – measurable in ω and smooth in x

$a^\omega(t,x) = a_{ij}(\theta_t\omega, x),\ b^\omega(t,x) = b_i(\theta_t\omega, x),\ c_0^\omega(t,x) = c_0(\theta_t\omega, x)$
– C^1 in t

Main Results

Let $X = L^2(D)$

$U(t, \omega)u_0$ – solution of (2) with $U(0, \omega)u_0 = u_0$

$u_0 \geq 0 \Longrightarrow U(t, \omega)u_0 \geq 0$ for $t > 0$

Main Results

Let $X = L^2(D)$

$U(t,\omega)u_0$ – solution of (2) with $U(0,\omega)u_0 = u_0$

$u_0 \geq 0 \Longrightarrow U(t,\omega)u_0 \geq 0$ for $t > 0$

Let $\mathbf{e} \in X$, $\|\mathbf{e}\| = 1$, be a positive principal eigenfunction of

$$\begin{cases} \Delta u = \lambda u, & x \in D \\ u = 0, & x \in \partial D \end{cases}$$

$\mathbf{e}(x) > 0$ for $x \in D$

$\frac{\partial \mathbf{e}}{\partial n}(x) < 0$ for $x \in \partial D$

Main Results

Assume

(A1) $f_1, f_2 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $f_1(\omega) := \sup_{0\le t\le 1} \ln^+ \|U(s,\omega)\|$,
$f_2(\omega) := \sup_{0\le s\le 1} \ln^+ \|U(1-s, \theta_s\omega)\|$.

(A2) $\lim_{t\to\infty} \frac{\ln \|U(t,\omega)\|}{t} > -\infty$ for a.e. ω ($\Longrightarrow$ case (2) or (3) in the MET occurs)

(A3) $\forall \omega \in \Omega$, $\exists \kappa(\omega) > 1$ such that $\forall u_0 \in X^+ \setminus \{0\}$, $\exists \alpha(u_0, \omega) > 0$,

$$\alpha(u_0,\omega)\mathbf{e} \le U(1,\omega)u_0 \le \kappa(\omega)\alpha(u_0,\omega)\mathbf{e}$$

Main Results

Remark on (A3):

Hilbert projective metric:

$u, v \in X^+$, $u \sim v$ if $\exists \alpha, \beta > 0$ s.t. $\alpha v \le u \le \beta v$

$m(u/v) = \sup\{\alpha | \alpha v \le u\}$

$M(u/v) = \inf\{\beta | u \le \beta v\}$

$d(u, v) = \ln \frac{m(u/v)}{M(u/v)}$ $(u \ne tv)$– Hilbert projective metric

Main Results

Remark on (A3):

Hilbert projective metric:

$u, v \in X^+$, $u \sim v$ if $\exists \alpha, \beta > 0$ s.t. $\alpha v \leq u \leq \beta v$

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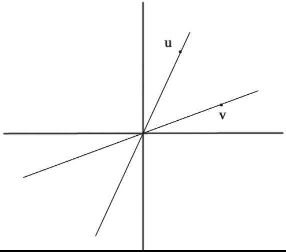
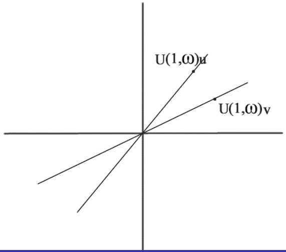
$d(u, v) = \ln \frac{m(u/v)}{M(u/v)}$ $(u \neq tv)$– Hilbert projective metric

(A3) $\Longrightarrow$

$$d(U(1,\omega)u, U(1,\omega)v) \leq 2 \ln \kappa(\omega) < \infty \quad \forall u, v \in X^+ \backslash \{0\} \quad (u \neq tv)$$





Main Results

Theorem 1. (J. Mierczynski and W. Shen)
Assume (A1)-(A3). There are measurable $\phi : \Omega \to X^+ \setminus \{0\}$ and $\phi^* : \Omega \to X^+ \setminus \{0\}$ such that

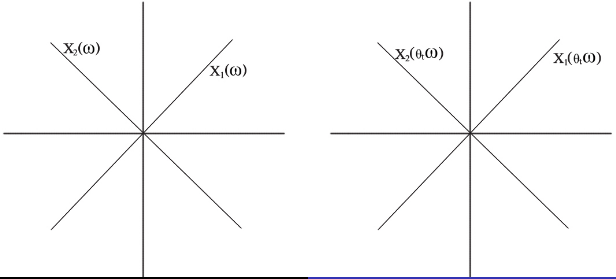
$X_1(\omega) := \mathrm{span}\{\phi(\omega)\}$

$X_2(\omega) := \{u \in L^2(D) | \langle u, \phi^*(\omega)\rangle = 0\}$ satisfy

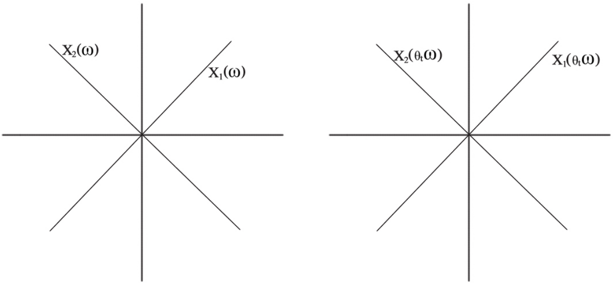
$X = X_1(\omega) \oplus X_2(\omega)$

$U(t,\omega)X_1(\omega) = X_1(\theta_t\omega)$ for $t > 0$

$U(t,\omega)X_2(\omega) \subset X_2(\theta_t\omega)$ for $t > 0$



Main Results



Moreover,

$\exists \gamma > 0$ and a tempered random variable $M(\omega) > 0$ such that for any $u_0 \in X_2(\omega)$

$$\frac{\|U(t,\omega)u_0\|}{\|U(t,\omega)\phi(\omega)\|} \leq M(\omega)e^{-\gamma t}\frac{\|u_0\|}{\|\phi(\omega)\|} \;\; \forall t > 0$$

$$(\Longrightarrow E_1(\omega) = X_1(\omega),\ F_\infty(\omega) \cap X^+ = \{0\} \text{ or } F_2(\omega) \cap X^+ = \{0\})$$

Main Results

Corollary 2. (J. Mierczynski and W. Shen)

Under the assumptions of Theorem 1,
for a.e. $\omega \in \Omega$, any $u_0 \in X^+ \setminus \{0\}$,

$\lim_{t\to\infty} \frac{\ln \|U(t,\omega)u_0\|}{t} = \lambda_1$ – principal Lyapunov exponent

Main Results

Theorem 3. (J. Mierczynski and W. Shen)

Assume (A1)-(A3).

Assume $a_{ij}(\omega, x) = a_{ij}(x)$, $b_i(\omega, x) = b_i(x)$.

Let $\lambda_1 = \lim_{t\to\infty} \frac{\|U(t,\omega)\|}{t}$.

Let $\hat{\lambda}_1$ be the principal eigenvalue of

$$\begin{cases} \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + \hat{c}_0(x)u = \lambda u, & x \in D \\ u = 0, & x \in \partial D \end{cases}$$

$\hat{c}_0(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t c_0(\theta_s\omega, x)ds$.

$\Longrightarrow$

$\lambda_1 \geq \hat{\lambda}_1$

with equality iff $c_0(\theta_t\omega, x) = c_{01}(\theta_t\omega) + c_{02}(x)$ for a.e. $\omega \in \Omega$ and any $t \in \mathbb{R}$, $x \in D$

Main Results

Remarks

(1) Theorems 1 and 3 also hold for the Neumann or Robin or periodic boundary conditions.

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Krein-Rutman Theorem. Assume that X is an ordered Banach space with a positive cone X^+ and $A : X \to X$ is a positive compact linear operator (i.e. $Au \geq 0$ if $u \geq 0$ and AE is precompact for any bounded set $E \subset X$). If X^+ has nonempty interior X^{++} and A is strongly positive ($Au \in X^{++}$ if $u \in A^+ \setminus \{0\}$), then the spectral radius $r(A)$ is a simple eigenvalue of A with a positive eigenfunction and for any other eigenvalue λ of A, $|\lambda| < r(A)$.

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(3) The arguments in proving Theorem 1 can be used to extend the Krein-Rutman Theorem to general ranodm dynamical systems

Idea of Proofs

Idea of proof of Theorem 1.

• Multiplicative Ergodic Theorem $\Longrightarrow$ for a.e. $\omega \in \Omega$

$\exists X_1(\omega)$, $X_2(\omega)$ with $\mathrm{dim} X_1(\omega) < \infty$, $\mathrm{codim} X_2(\omega) < \infty$ s.t.

$X = X_1(\omega) \oplus X_2(\omega)$

$U(t,\omega)X_1(\omega) = X_1(\theta_t\omega)$ for $t > 0$

$U(t,\omega)X_2(\omega) \subset X_2(\theta_t\omega)$ for $t > 0$

$\lim_{t\to\pm\infty} \frac{\ln \|U(t,\omega)u_0\|}{t} = \lambda_1$ $\forall u_0 \in X_1(\omega) \setminus \{0\}$

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(*it suffices to prove that* $\exists \phi(\omega)$, $\phi^*(\omega) \in X^+$ *s.t.*

$X_1(\omega) = \mathrm{span}\{\phi(\omega)\}$

$X_2(\omega) = \{u \in L^2(D) | \langle u, \phi^*(\omega)\rangle = 0\}$)

Idea of Proofs

- Consider the adjoin problem of (2)

$$\begin{cases} -\frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} a_{ji}(\theta_t\omega, x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^*(\theta_t\omega, x)\frac{\partial u}{\partial x_i} \\ \qquad\qquad +c_0^*(\theta_t\omega, x)u, \qquad x \in D \\ u = 0, \qquad x \in \partial D \end{cases} \tag{2*}$$

where $b_i^*(\theta_t\omega, x) = \sum_{j=1}^{N} \frac{\partial a_{ij}}{\partial x_j}(\theta_t\omega, x) - b_i(\theta_t\omega, x)$

$c_0^*(\theta_t\omega, x) = c_0(\theta_t\omega, x) - \sum_{j=1}^{N} \frac{\partial b_j(\theta_t\omega, x)}{\partial x_j}$

$U^*(t,\omega)u_0$ $(t < 0)$ – solution of $(2)^*$ with $U^*(0,\omega)u_0 = u_0 \in X$

$u_0 \geq 0 \implies U^*(t,\omega)u_0 \geq 0$ for $t < 0$

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where $b_i^*(\theta_t\omega, x) = \sum_{j=1}^{N} \frac{\partial a_{ij}}{\partial x_j}(\theta_t\omega, x) - b_i(\theta_t\omega, x)$

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$(U(t,\omega))^* = U^*(-t, \theta_t\omega)$ for $t > 0$

$\langle U(t,\omega)u_0, v_0\rangle = \langle u_0, U^*(-t, \theta_t\omega)v_0\rangle$

Idea of Proofs

Multiplicative Ergodic Theorem $\Longrightarrow$ for a.e. $\omega \in \Omega$

$\exists X_1^*(\omega)$, $X_2^*(\omega)$ with $\mathrm{dim} X_1^*(\omega) < \infty$, $\mathrm{codim} X_2^*(\omega) < \infty$ s.t.

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$U^*(t,\omega)X_2^*(\omega) \subset X_2^*(\theta_t\omega)$ for $t < 0$

$\lim_{t\to\pm\infty} \frac{\ln \|U^*(-t,\omega)u_0\|}{t} = \lambda_1 \ \forall u_0 \in X_1^*(\omega) \setminus \{0\}$

$\lim_{t\to-\infty} \frac{\ln \|U^*(-t,\omega)|_{X_2^*(\omega)}\|}{t} < \lambda_1$

Idea of Proofs

- Prove $X_1(\omega) \cap (X^+ \setminus \{0\}) \neq \emptyset$ for a.e. $\omega \in \Omega$

(prove $X_1^*(\omega) \cap (X^+ \setminus \{0\}) \neq \emptyset$)

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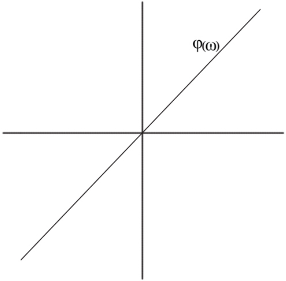
(prove $X_1^*(\omega) \cap (X^+ \setminus \{0\}) \neq \emptyset$)

Take $\phi(\omega) \in X_1(\omega) \cap (X^+ \setminus \{0\})$



(take $\phi^*(\omega) \in X_1^*(\omega) \cap (X^+ \setminus \{0\})$)
(*it suffices to prove* $X_1(\omega) = \mathrm{span}\{\phi(\omega)\}$

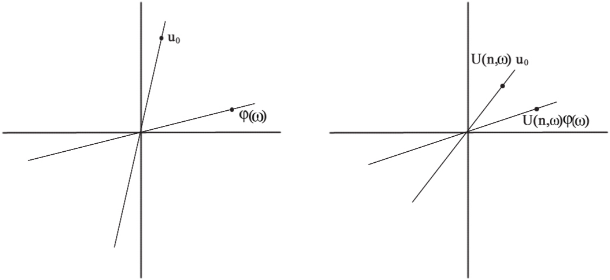
$X_2(\omega) = \{u \in L^2(D) | \langle u, \phi^*(\omega)\rangle = 0\}$)

Idea of Proofs

• **Important Proposition.** For a.e. $\omega \in \Omega$, $\forall u_0 \in X_1(\omega)$, $\exists M(u_0, \omega)$ s.t.

$$\frac{U(n,\omega)u_0}{\|U(n,\omega)\phi(\omega)\|} - M(u_0,\omega)\frac{U(n,\omega)\phi(\omega)}{\|U(n,\omega)\phi(\omega)\|} \to 0$$

exponentially as $n \to \infty$



Idea of Proofs

- Prove that if $u_0 \in X_1(\omega)$, $\langle u_0, \phi^*(\omega) \rangle = 0$, then $M(u_0, \omega) = 0$

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- Prove that if $u_0 \in X_1(\omega)$, $\langle u_0, \phi^*(\omega)\rangle = 0$, then $M(u_0,\omega) = 0$

Let $\phi_n^*(\omega) = \frac{U^*(n,\omega)\phi^*(\omega)}{\|U^*(n,\omega)\phi^*(\omega)\|}$

$\langle u_0, \phi^*(\omega)\rangle = 0 \Longrightarrow \langle U(n,\omega)u_0, \phi_n^*(\omega)\rangle = 0$

By the proposition,

$M(u_0,\omega)\frac{\langle U(n,\omega)\phi(\omega),\phi_n^*(\omega)\rangle}{\|U(n,\omega)\phi(\omega)\|} \to 0$ exponentially.

But

$\frac{\langle U(n,\omega)\phi(\omega),\phi_n^*(\omega)\rangle}{\|U(n,\omega)\phi(\omega)\|} = \langle \phi(\omega), \phi^*(\omega)\rangle \frac{1}{\|U(n,\omega)\phi(\omega)\|} \cdot \frac{1}{\|U^*(n,\omega)\phi^*(\omega)\|}$

$\lim_{n\to\infty} \frac{\ln \|U(n,\omega)\phi(\omega)\|}{n} = -\lim_{n\to\infty} \frac{\ln \|U^*(n,\omega)\phi^*(\omega)\|}{n} = \lambda_1$

$\Longrightarrow$

$M(u_0,\omega) = 0$

Idea of Proofs

- Prove $\dim X_1(\omega) = 1$.

Idea of Proofs

- Prove $\mathrm{dim} X_1(\omega) = 1$.

Assume $\mathrm{dim} X_1(\omega) > 1$.

$\exists u_0 \in X_1(\omega)$ s.t. $\langle u_0, \phi^*(\omega)\rangle = 0$

$\Longrightarrow M(u_0, \omega) = 0$

$\Longrightarrow \frac{U(n,\omega)u_0}{\|U(n,\omega)\phi(\omega)\|} \to 0$ exponentially

Hence $\limsup_{n\to\infty} \frac{1}{n} \ln \frac{\|U(n,\omega)u_0\|}{\|U(n,\omega)\phi(\omega)\|} < 0$ and

$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \ln \|U(n,\omega)u_0\| < \lim_{n\to\infty} \frac{1}{n} \ln \|U(n,\omega)\phi(\omega)\| = \lambda_1$

A contradiction!

$\Longrightarrow$

$\mathrm{dim} X_1(\omega) = 1$ and $X_1(\omega) = \mathrm{span}\{\phi(\omega)\}$.

Idea of Proofs

• Prove $X_2(\omega) \cap X^+ = \{0\}$

Assume that $X_2(\omega) \cap X^+ \neq \{0\}$ for a.e. $\omega \in \Omega$

Take a random variable $u_0 : \Omega \to X_2(\omega) \cap (X^+ \setminus \{0\})$

Then $\langle u_0(\omega), \phi^*(\omega) \rangle \neq 0$

$\Longrightarrow$

$\lambda_1 > \lim_{t\to\infty} \frac{\ln \|U(t,\omega)u_0(\omega)\|}{t} = \lim_{t\to\infty} \frac{\ln \|U(t,\omega)\phi(\omega)\|}{t} = \lambda_1$

A contradiction!

Applications – Spreading Speed in KPP Models

KPP (Kolmogorov, Petrowsky, Piscunov) or Fisher equation in random environment:

$$u_t = u_{xx} + m(\theta_t\omega, x)u(1-u), \quad x \in \mathbb{R} \tag{4}$$

population model for spatial spread of an allele in a migrating diploid with two type alleles

$u(t,x)$ – fraction of one of the two alleles at (t,x)

$m(\theta_t\omega, x)$, $m(\omega, x+p) = m(\omega, x)$ – fitness coefficients

$\omega \in \Omega$, $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\})$ – ergodic metric dynamical system

$u^- = 0$ – unstable (w.r.t. spatially periodic perturbation)

$u^+ = 1$ – stable (w.r.t. spatially periodic perturbation)

Applications – Spreading Speed in KPP Models

Characterization of the stability of $u^{\pm}$:

λ_- – principal Lyapunov exponent of

$$\begin{cases} u_t = u_{xx} + m(\theta_t\omega, x)u, & x \in \mathbb{R} \\ u(t, x+p) = u(t,x), & x \in \mathbb{R} \end{cases}$$

λ_+ – principal Lyapunov of

$$\begin{cases} u_t = u_{xx} - m(\theta_t\omega, x)u, & x \in \mathbb{R} \\ u(t, x+p) = u(t,x), & x \in \mathbb{R} \end{cases}$$

$\lambda_- > 0 \Longrightarrow u = u^-$ – unstable

$\lambda_+ < 0 \Longrightarrow u = u^+$ – stable

Spreading phenomena:

$u_0 \in C_{\rm unif}(\mathbb{R}) = \{u \in C(\mathbb{R}) | u$ is bounded and uniformly continuous on $\mathbb{R}\}$

$u(t, \cdot; u_0, \omega)$ – solution of (4) with $u(0, \cdot; u_0, \omega) = u_0$

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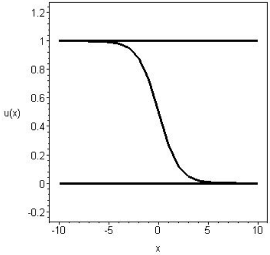
$X_0 = \{u \in C_{\rm unif}(\mathbb{R}) | u(x) \geq 0, \quad \liminf_{x \to -\infty} u(x) > 0, \; u(x) = 0$ for $x \gg 1\}$

If $u_0 \in X_0$,



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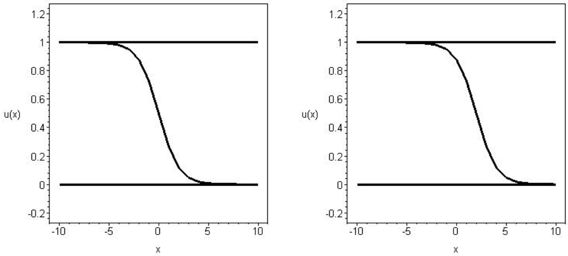

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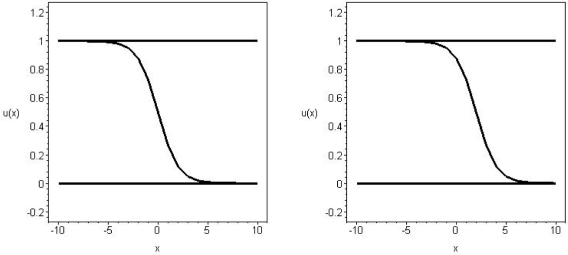


Question: How fast the solutions spread?

Applications – Spreading Speed in KPP Models

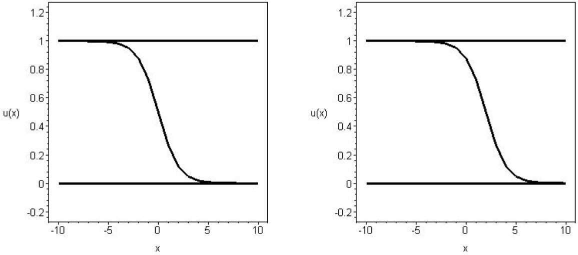
Theorem (H. Weinberger)

In the periodic case, i.e., $m(\theta_t\omega, x) = m(t, x)$ is periodic in t,

$\exists c^*$ s.t. for any $u_0 \in X_0$

$\lim_{x \le c' t, t \to \infty} [u(t, x; u_0, \omega) - u^+] = 0 \ \forall c' < c^*$

$\lim_{x \ge c'' t, t \to \infty} u(t, x; u_0, \omega) = 0 \ \forall c'' > c^*$



c^* – spreading speed

Applications – Spreading Speed in KPP Models

If further $m(t,x) \geq 0$, then

$$c^* = \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}$$

$\lambda(\mu)$ – principal eigenvalue of

$$\begin{cases} u_t = u_{xx} - 2\mu u_x + [m(t,x) + \mu^2]u \\ u(t,x) = u(t,x+p) \end{cases}$$

Theorem 4. (W. Shen)

Assume $m(t, x) \geq 0$.

$c^* \geq \hat{c}^*$

$\hat{c}^*$ is the spreading speed of

$$u_x = u_{xx} + \hat{m}(x)u(1-u)$$

$\hat{m}(x) = \frac{1}{T}\int_0^T m(t, x)dt.$

Applications – Spreading Speed in KPP Models

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Spatial-temporal variation speeds up the spatial spread!

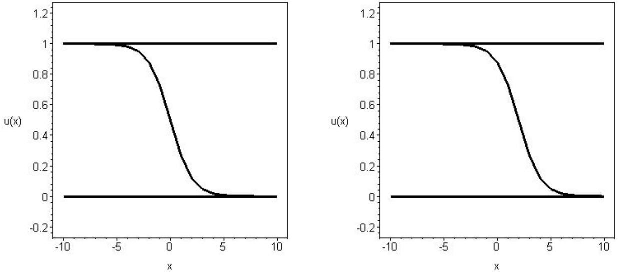
(*Both the principal eigenvalue and the positivity of a principal eigenfunction are needed in the proof*)

Spreading speed interval: (W. Shen) (J. Huang and W. Shen in the deterministic case)

$X_0 = \{u \in C_{\rm unif}(\mathbb{R}) | u(x) \geq 0, \quad \liminf_{x \to -\infty} u(x) > 0, \; u(x) = 0$ for $x \gg 1\}$

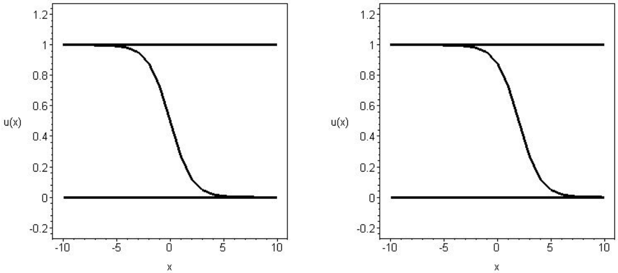
$C^*_{\rm inf} = \{c | \lim_{x \leq ct, t \to \infty} [u(t, x; u_0, \omega) - u^+] = 0, \quad \forall u_0 \in X_0, \quad a.e. \omega \in \Omega\}$

$c^*_{\rm inf} = \sup\{c \in C^*_{\rm inf}\}$



Applications – Spreading Speed in KPP Models

$$C_{\sup}^* = \{c | \lim_{x \ge ct, t \to \infty} u(t,x;u_0,\omega) = 0, \forall u_0 \in X_0, a.e. \omega \in \Omega\}$$



$$c_{\sup}^* = \inf\{c \in C_{\sup}^*\}$$

$[c_{\inf}^*, c_{\sup}^*]$ – *spreading speed interval* in the positive direction

Applications – Spreading Speed in KPP Models

Remark. In the periodic case, $c^*_{\inf} = c^*_{\sup} = c^*$

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Theorem 5 (W. Shen)

If $m(\omega, x) \geq 0$, then $c^*_{\sup} \leq \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}$

where $\lambda(\mu)$ is the principal Lyapunov exponent of

$$\begin{cases} u_t = u_{xx} - 2\mu u_x + (m(\theta_t\omega, x) + \mu^2)u \\ u(t, x+p) = u(t, x) \end{cases}$$

Remark. In the periodic case, $c^*_{\inf} = c^*_{\sup} = c^*$

Theorem 5 (W. Shen)

If $m(\omega, x) \geq 0$, then $c^*_{\sup} \leq \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}$

where $\lambda(\mu)$ is the principal Lyapunov exponent of

$$\begin{cases} u_t = u_{xx} - 2\mu u_x + (m(\theta_t \omega, x) + \mu^2)u \\ u(t, x+p) = u(t, x) \end{cases}$$

Open problem: $c^*_{\inf} \geq \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}$?

If yes, $c^*_{\inf} = c^*_{\sup} = \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}$