Identification of Second-Order Stochastic Dynamical Systems

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Based on joint work with Wei Liu

First Order vs Second Order in Time

First-order SODE: OU process $dX + aX(t)dt = dW(t)$
Three options: $a > 0$, $a = 0$, $a < 0$.

First-order SPDE: $du = au_{xx}dt + dW(t,x)$
One option: $a > 0$ (Infinite-dimensional stable OU process.)

Second-order SODE: $\ddot{X}(t) + a\dot{X}(t) + bX(t) = \dot{W}(t)$
How many options?

Second-order SPDE: $u_{tt} = \mathcal{A}u + \mathcal{B}u_t + \dot{W}(t,x)$.
How many options? (Certainly more than one...)

Basic SPDE Model

Stochastic wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial t} + \dot{W}(t,x),\ 0 < t < T,\ 0 < x < \pi;$$

zero initial and boundary conditions.

What is what:

- $u = u(t,x)$ — displacement of a string
- $a > 0$ — propagation speed
- $b \in \mathbb{R}$ — damping/amplification coefficient:

$$\frac{d}{dt}\int_0^\pi \big(u_t^2(t,x) + a^2 u_x^2(t,x)\big)dx = 2b\int_0^\pi u_t^2(t,x)dx$$

(amplification is $b > 0$; damping is $b < 0$).

- $\dot{W}(t,x)$ is space-time white noise.

$u_{tt} = a^2 u_{xx} + b u_t + \dot{W}(t,x)$

Motivation:

- Guitar in the sand storm: Walsh (1984)
- Interest rate models: Santa-Clara and Sornette (2001)

Parameter Estimation:

- Huebner, Khasminskii, and Rozovskii (1992) — Heat equation
- Huebner and Rozovskii (1995) — Beyond the heat equation.

Our Objectives:

- Wave equation
  (1) Existence and uniqueness of solution.
  (2) Estimating a^2 and b.
- Beyond the wave equation.

The equation: Definition of solution

$u_{tt} = a^2 u_{xx} + b u_t + \dot{W}(t,x),\ 0 < t < T,\ 0 < x < \pi.$

Space-time white noise: $\dot{W}(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k \geq 1} \sin(kx) \dot{w}_k(t)$

$W_f(t) = \sum_{k \geq 1} f_k w_k(t),\ f \in L_2((0,\pi)).$

Sobolev spaces H^γ: $\|f\|_\gamma^2 = \sum_{k \geq 1} k^{2\gamma} f_k^2;$

Solution of the equation:

$u \in L_2\big(\Omega \times (0,T) \times (0,\pi)\big), \quad v \in L_2\big(\Omega; L_2((0,T); H^{-1})\big),$

$$(u(t,\cdot), f) = \int_0^t (v(t,\cdot), f)(s) ds,$$

$$(v(t,\cdot), f) = \int_0^t \big(a^2 (u(t,\cdot), f'') - 2b(v(t,\cdot), f)\big) ds + W_f(t)$$

The equation: Existence of solution

$u_{tt} = a^2 u_{xx} + b u_t + \dot{W}(t,x),\ 0 < t < T,\ 0 < x < \pi.$

Fundamental solution: $\varphi_k''(t) - b\varphi_k'(t) + k^2 a^2 \varphi_k(t) = 0,$

$\varphi_k(0) = 0,\ \varphi_k'(0) = 1.$

Fourier coefficients:

$$u_k(t) = \int_0^t \varphi_k(t-s)\,dw_k(s), \quad v_k(t) = \int_0^t \varphi_k'(t-s)\,dw_k(s).$$

Theorem.

$$u(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} u_k(t)\ \sin(kx), \quad v(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} v_k(t)\ \sin(kx),$$

$$u \in L_2\big(\Omega; L_2((0,T); H^\gamma)\big);\ \ v \in L_2\big(\Omega; L_2((0,T); H^{\gamma-1})\big),\ \gamma < 1/2.$$

Note: If $4a^2k^2 > b^2$ and $\ell_k = \sqrt{4a^2k^2 - b^2}$, then

$$\varphi_k(t) = \frac{2}{\ell_k} \exp(bt/2)\ \sin(\ell_k t/2).$$

$u_{tt} = \theta_1 u_{xx} + \theta_2 u_t + \dot{W}(t,x),\ 0 < t < T,\ 0 < x < \pi,\ \theta_1 > 0,\ \theta_2 \in \mathbb{R}.$

Observations: $\big(u_k(t),\ u_k'(t)\big),\ 0 < t < T,\ k = 1, \ldots, N.$

Notation: $v_k(t) = u_k'(t)$.

To get an MLE:

$dv_k(t) = \left(-k^2\theta_1 \int_0^t v_k(s)ds + \theta_2 v_k(t)\right) dt + dw_k(t).$

The likelihood ratio:

$$\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_{1,\ldots,N}) = \exp\left(\sum_{k=1}^{N}\int_0^T \big(-\theta_1 k^2 u_k(t) + \theta_2 v_k(t)\big) dv_k(t) - \frac{1}{2}\sum_{k=1}^{N}\int_0^T \big(-\theta_1 k^2 u_k(t) + \theta_2 v_k(t)\big)^2 dt\right);$$

Explicit formulas

$$\widehat{\theta}_{1,N} = \frac{B_{1,N}J_{2,N} + B_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}, \quad \widehat{\theta}_{2,N} = \frac{B_{1,N}J_{12,N} + B_{2,N}J_{1,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}.$$

where

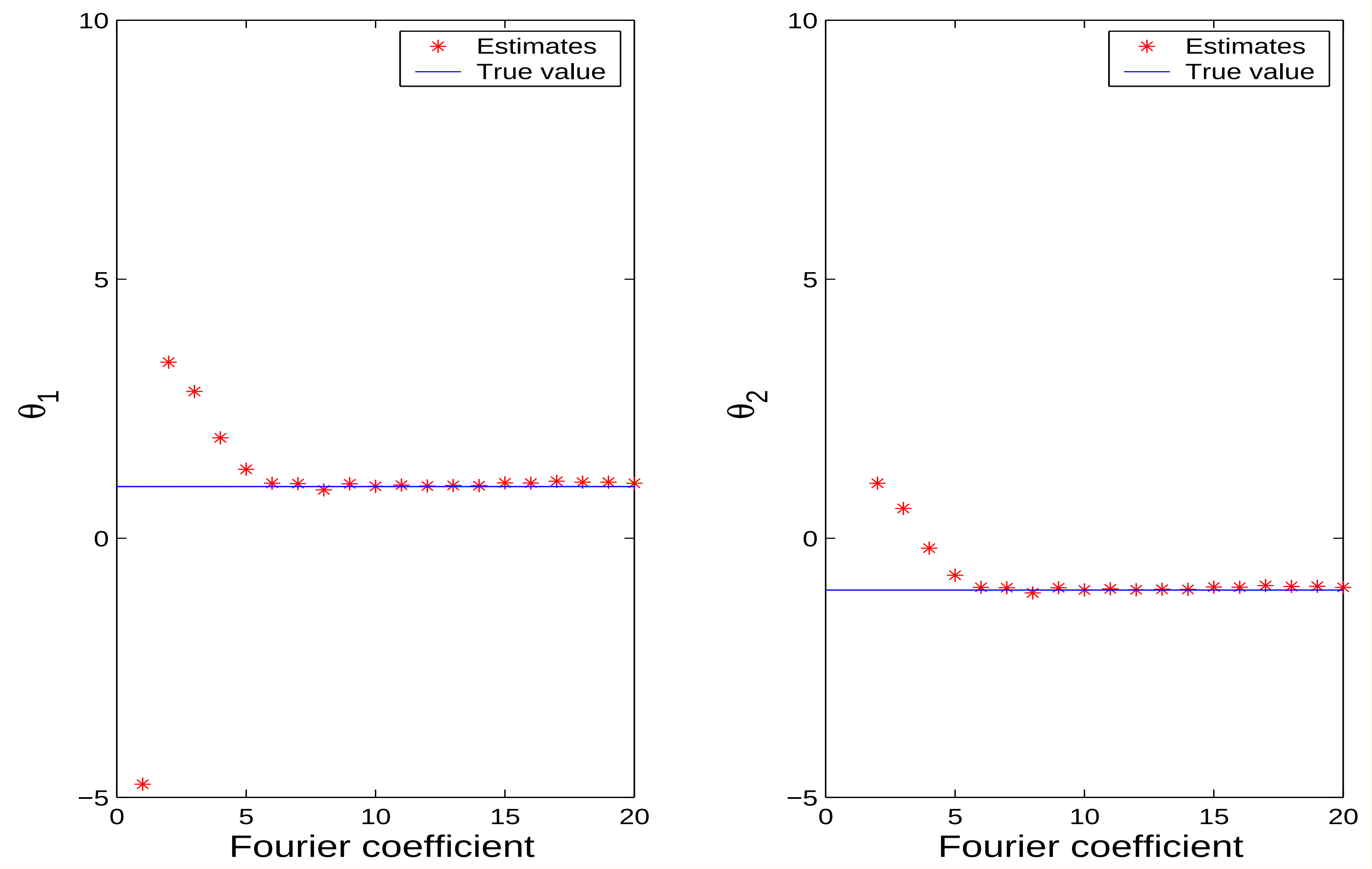
$$J_{1,N} = \sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)dt, \quad J_{2,N} = \sum_{k=1}^{N} \int_0^T v_k^2(t)dt,$$

$$J_{12,N} = \sum_{k=1}^{N} k^2 \int_0^T u_k(t)v_k(t)dt;$$

$$B_{1,N} = -\sum_{k=1}^{N} k^2 \int_0^T u_k(t)dv_k(t), \; B_{2,N} = \sum_{k=1}^{N} \int_0^T v_k(t)dv_k(t).$$

Some pictures

$$u_{tt} = \mathbf{1} \cdot u_{xx} \,\mathbf{-1} \cdot u_t + \dot{W}$$



Theorem. We have

$$\lim_{N\to\infty} \widehat{\theta}_{1,N} = \theta_1, \quad \lim_{N\to\infty} \widehat{\theta}_{2,N} = \theta_2$$

with probability one and

$$\lim_{N\to\infty} N^{3/2}(\widehat{\theta}_{1,N} - \theta_1) = \mathfrak{N}\left(0, \frac{3\theta_1}{T^2 M(\theta_2 T)}\right),$$

$$\lim_{N\to\infty} N^{1/2}(\widehat{\theta}_{2,N} - \theta_2) = \mathfrak{N}\left(0, \frac{1}{T^2 M(\theta_2 T)}\right)$$

in distribution, where

$$M(x) = \begin{cases} \dfrac{e^x - x - 1}{2x^2}, & \text{if } x \neq 0; \\ \dfrac{1}{4}, & \text{if } x = 0. \end{cases}$$

About the proof

$$D_N = \frac{J_{12,N}^2}{J_{1,N} J_{2,N}}, \ \xi_{1,N} = \sum_{k=1}^{N} k^2 \int_0^T u_k(t)dw_k(t),$$

$$\xi_{2,N} = \sum_{k=1}^{N} \int_0^T v_k(t)dw_k(t)$$

$$\widehat{\theta}_{1,N} = \theta_1 + \frac{1}{1-D_N}\left(\frac{\xi_{1,N}}{J_{1,N}} + \xi_{2,N}\frac{J_{12,N}}{J_{1,N}J_{2,N}}\right),$$

$$\widehat{\theta}_{2,N} = \theta_2 + \frac{1}{1-D_N}\left(\frac{\xi_{2,N}}{J_{2,N}} + \xi_{1,N}\frac{J_{12,N}}{J_{1,N}J_{2,N}}\right).$$

Strong Law of Large Numbers:

$$D_N \to 0, \ J_{1,N} \asymp N^3, \ J_{2,N} \asymp N.$$

CLT: $\frac{\xi_{i,N}}{\sqrt{J_{1,N}}} \sim \mathfrak{N}(0,1), \ i = 1, 2.$

A generalization

$$\ddot{u} + (\mathcal{A}_0 + \theta_1 \mathcal{A}_1)u = (\mathcal{B}_0 + \theta_2 \mathcal{B}_1)\dot{u} + \dot{W}$$

Evolution operator: $\mathcal{A} = \mathcal{A}_0 + \theta_1 \mathcal{A}_1$;
Dissipation operator: $\mathcal{B} = \mathcal{B}_0 + \theta_2 \mathcal{B}_1$; "$\mathcal{B} > 0$" is amplification.
Diagonalizable: common system of eigenfunctions, ensures

$$\ddot{u}_k = (\varrho_k + \theta_1 \tau_k)u_k + (\rho_k + \theta_2 \nu_k)\dot{u}_k + \dot{w}_k.$$

Hyperbolic: "$\mathcal{A} > 0$", "not too much amplification"
Examples

$$u_{tt} = \theta_1 \mathbf{\Delta} u + \theta_2 u_t + \dot{W} \text{ (just considered.)}$$

$$u_{tt} = \theta_1 \mathbf{\Delta} u + \theta_2 \mathbf{\Delta} u_t + \dot{W} \text{ (Why not if } \theta_2 > 0?)$$

$$u_{tt} = \theta_1 \mathbf{\Delta} u - \theta_2 \mathbf{\Delta}^2 u_t + \dot{W} \text{ (Even more so if } \theta_2 > 0.)$$

Second-Order ODE

Fundamental solution:

$\ddot{y}(t) - 2b\dot{y}(t) + a^2 y(t) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1.$

$$y(t) = \begin{cases} \dfrac{\sin(\ell t)}{\ell}\, e^{bt}, & a^2 > b^2; \\ te^{bt}, & a^2 = b^2; \\ \dfrac{\sinh(\ell t)}{\ell}\, e^{bt}, & a^2 < b^2; \end{cases} \quad \ell = \sqrt{|a^2 - b^2|}$$

SODE: $\ddot{X}(t) - 2b\dot{X} + a^2 X(t) = \dot{w}(t)$, $X(0) = \dot{X}(0) = 0.$

$X(t) = \int_0^t y(t-s)dw(s)$, $\mathbb{E}|X(t)|^2 = \int_0^t |y(s)|^2 ds,$

$$\mathbb{E}|\dot{X}(t)|^2 = \int_0^t |\dot{y}(s)|^2 ds.$$

What is (Stochastic) Hyperbolic?

$$\ddot{u}_k + (\varrho_k + \theta_1 \tau_k) u_k = (\rho_k + \theta_2 \nu_k)\dot{u}_k + \dot{w}_k,\ 0 < t < T.$$

$W(t) = \sum_{k \geq 1} w_k(t) h_k$, h_k CONS in H

$W(t) \in X$ if $H \subset X$; Hilbert-Schmidt embedding: $\sum_k \|h_k\|_X^2 < \infty$.

Hyperbolic Diagonalizable SPDE:

- $\varrho_k + \theta_1 \tau_k \nearrow +\infty$ ("$\mathcal{A} > 0$")
- $u(t) \in X$, $t \in [0, T]$ ("well-posed")

Theorem (a) If $T(\rho_k + \theta_2 \nu_k) \leq \ln(\varrho_k + \theta_1 \tau_k) + C$ then $\sup_{k,T} \mathbb{E}|u_k(t)|^2 < \infty$, and so $u(t) \in L_2(\Omega; X)$ for all $t \in [0, T]$.

(b) If $(\rho_k + \theta_2 \nu_k) \leq C$ (*bounded amplification*), then $u(t), \dot{u}(t) \in L_2(\Omega; X)$ for all $t \in [0, T]$.

$\ddot{u}_k + (\varrho_k + \theta_1 \tau_k) u_k = (\rho_k + \theta_2 \nu_k)\dot{u}_k + \dot{w}_k, \ 0 < t < T.$

Expressions for $\widehat{\theta}_{1,N}, \ \widehat{\theta}_{2,N}$: *really* do not want to see them...

Important terms:

$$\Psi_{1,N} = \sum_{k=1}^{N} \mathbb{E} \int_0^T \tau_k^2 u_k^2(t)dt, \ \Psi_{2,N} = \sum_{k=1}^{N} \mathbb{E} \int_0^T \nu_k^2 v_k^2(t)dt.$$

The reason: $\widehat{\theta}_{1,N} - \theta_1 \approx \dfrac{\sum_{k=1}^N \zeta_k}{\Psi_{1,N}}$, ζ_k are independent, zero-mean,

$\mathbb{E}\zeta_k^2 = \mathbb{E} \int_0^T \tau_k^2 u_k^2(t)dt$. Same for $\widehat{\theta}_{2,N} - \theta_2$.

Goals

- **Consistency**: $\lim_{N\to\infty} \widehat{\theta}_{i,N} = \theta_i, \ i = 1, 2.$
- **Asymptotic normality**: $\lim_{N\to\infty} \sqrt{\Psi_{i,N}}\big(\widehat{\theta}_{i,N} - \theta_i\big) = \mathfrak{N}(0, 1).$

A reasonable guess: Need $\lim_{N\to\infty} \Psi_{i,N} = +\infty$.

Sometimes, that is all we need, sometimes not...

How to study the MLE?

The key relations: $\lambda_k(\theta) = \varrho_k + \theta\tau_k,\ \mu_k(\theta) = \rho_k + \theta\nu_k,$

$$\mathbb{E}\int_0^T u_k^2(t)dt \sim \frac{T^2 M\big(T\mu_k(\theta_2)\big)}{\lambda_k(\theta_1)},\ \mathrm{Var}\int_0^T u_k^2(t)dt \sim \frac{T^4 V\big(T\mu_k(\theta_2)\big)}{\lambda_k^2(\theta_1)},$$

$$\mathbb{E}\int_0^T v_k^2(t)dt \sim T^2 M\big(T\mu_k(\theta_2)\big),\ \mathrm{Var}\int_0^T v_k^2(t)dt \sim T^4 V\big(T\mu_k(\theta_2)\big),$$

where

$$M(x) = \begin{cases} \dfrac{e^x - x - 1}{2x^2}, & \text{if } x \neq 0, \\ \dfrac{1}{4}, & \text{if } x = 0; \end{cases}$$

$$V(x) = \begin{cases} \dfrac{e^{2x} + 4e^x - 4xe^x - 2x - 5}{4x^4}, & \text{if } x \neq 0, \\ \dfrac{1}{24}, & \text{if } x = 0. \end{cases}$$

Algebraic case

$\ddot{u} + (\mathcal{A}_0 + \theta_1 \mathcal{A}_1)u + (\mathcal{B}_0 + \theta_2 \mathcal{B}_1)\dot{u} + \dot{W}$

For "real-life" (positive-definite elliptic self-adjoint) operators

$$(\text{eigenvalue})_k \asymp k^{\left(\frac{\text{order of the operator}}{\text{dimension of the space}}\right)}$$ (order can be fractional)

Example. If $\boldsymbol{\Delta}$ is the Laplace operator in a smooth bounded domain $G \in \mathbb{R}^d$ with zero boundary conditions, and $\gamma \in \mathbb{R}$, then

$$k\text{-th eigenvalue of } (1 - \boldsymbol{\Delta})^{\gamma/2} \asymp k^{\gamma/d}.$$

Theorem. • $\lim_{N\to\infty} \widehat{\theta}_{1,N} = \theta_1 \ (\mathbb{P} - a.s) \ \Leftrightarrow$

$\text{order}(\mathcal{A}_1) \geq \frac{1}{2}\Big(\text{order}(\mathcal{A}_0 + \theta_1 \mathcal{A}_1) + \text{order}(\mathcal{B}_0 + \theta_2 \mathcal{B}_1) - d\Big);$

- $\lim_{N\to\infty} \widehat{\theta}_{2,N} = \theta_2 \ (\mathbb{P} - a.s) \ \Leftrightarrow \text{order}(\mathcal{B})_1 \geq \frac{1}{2}\Big(\text{order}(\mathcal{B}_0 + \theta_2 \mathcal{B}_1) - d\Big).$
- Consistency $\Longrightarrow$ Asymptotic normality

Many parameters

The results extend to a more general estimation problem $\ddot{u} + \sum_{i=0}^{n} \theta_{1i} \mathcal{A}_i u = \sum_{j=0}^{m} \theta_{2j} \mathcal{B}_j u_t + \dot{W}$, as long as all the operators $\mathcal{A}_i, \mathcal{B}_j$ have a common system of eigenfunctions:

- the coefficient θ_{1p} can be consistently estimated if and only if

$$\operatorname{order}(\mathcal{A}_p) \geq \frac{1}{2}\left(\operatorname{order}\left(\sum_{i=0}^{n} \theta_{1i}\mathcal{A}_i\right) + \operatorname{order}\left(\sum_{j=0}^{m} \theta_{2j}\mathcal{B}_j\right) - d\right).$$

- the coefficient θ_{2q} can be consistently estimated if and only if

$$\operatorname{order}(\mathcal{B}_q) \geq \frac{1}{2}\left(\operatorname{order}\left(\sum_{j=0}^{m} \theta_{2j}\mathcal{B}_j\right) - d\right).$$

Parabolic case: Huebner (1997).

Examples

1. $u_{tt} = \theta_1 \mathbf{\Delta} u + \theta_2 \mathbf{\Delta} u_t + \dot{W}$ in $G \subset \mathbb{R}^2$
$\lambda_k = \theta_1 \tau_k \asymp k,\ \mu_k = \theta_2 \nu_k \asymp k;$

$$N^{1/2}(\widehat{\theta}_{1,N} - \theta_1) \sim \mathfrak{N}(0, \sigma_1^2),\ N(\widehat{\theta}_{2,N} - \theta_2) \sim \mathfrak{N}(0, \sigma_2^2).$$

Note: $\mathbf{\Delta} u$ is more regular that $\mathbf{\Delta} u_t$.

2. $u_{tt} + \mathbf{\Delta}^2 u = \theta_1 \mathbf{\Delta} u + \mathbf{\Delta} u_t + \theta_2 u_t + \dot{W}$ in $G \subset \mathbb{R}^2$
$\lambda_k \asymp k^2,\ \tau_k \asymp k,\ \mu_k \asymp k,\ \nu_k = 1;$

$$(\ln N)^{1/2}(\widehat{\theta}_{1,N} - \theta_1) \sim \mathfrak{N}(0, \sigma_1^2),\ (\ln N)^{1/2}(\widehat{\theta}_{2,N} - \theta_2) \sim \mathfrak{N}(0, \sigma_2^2).$$

Note 1: $\mathbf{\Delta} u$ is as regular as u_t.
Note 2: In $\mathbb{R}^1$, no consistency for either $\widehat{\theta}_{1,N}$ or $\widehat{\theta}_{2,N}$.

Non-algebraic case

Example. $\tau_k = e^k$, $\nu_k = \ln\ln(k+3)$.

Why?

- Why not?
- Multi-channel observations (Korostelev and Yin (2006))

What do we gain? Generality

What do we lose? Strong consistency, simplicity.

Algebraic vs General

$$b_N \nearrow +\infty \Longrightarrow \lim_{N\to\infty} \frac{1}{b_N}\sum_{k=1}^{N}(\xi_k - \mathbb{E}\xi_k) = 0 \text{ (?)}$$

Weak Law of Large Numbers: Need

$$\lim_{N\to\infty} \frac{1}{b_N^2}\sum_{k=1}^{N} \operatorname{Var}\xi_k = 0 \text{ (to apply Chebyshev's Inequality)}$$

Strong Law of Large Numbers: Need

$$\sum_{k\geq 1} \frac{\operatorname{Var}\xi_k}{b_k^2} < \infty \text{ (to apply Kolmogorov's SLLN)}$$

Slowly increasing sequence: $a_k > 0$, $\sum_{k\geq 1} a_k = +\infty$ AND

Either $\dfrac{\sum_{k=1}^{N} a_k^2}{\left(\sum_{k=1}^{N} a_k\right)^2} \to 0$ or $\displaystyle\sum_{n\geq 1} \frac{a_n^2}{\left(\sum_{k=1}^{n} a_k\right)^2} < \infty$.

- Not an issue in algebraic case
- Could be worse (Adler, Rosansky (1991))

$\ddot{u}_k + (\varrho_k + \theta_1 \tau_k) u_k = (\rho_k + \theta_2 \nu_k)\dot{u}_k + \dot{w}_k,\ 0 < t < T.$

Theorem

- $\left\{ \dfrac{\tau_k^2 M\big(\mu_k(\theta_2)\big)}{\lambda_k(\theta_1)},\ k \geq 1 \right\}$ is slowly increasing $\Longrightarrow$ consistency (in probability) and asymptotic normality of $\widehat{\theta}_{1,N}$;
- $\left\{ \nu_k^2 M\big(\mu_k(\theta_2)\big),\ k \geq 1 \right\}$ is slowly increasing $\Longrightarrow$ consistency (in probability) and asymptotic normality of $\widehat{\theta}_{2,N}$.

A useful relation: $M(x) \sim \begin{cases} (2|x|)^{-1}, & x \to -\infty \\ (2x)^{-2}\, e^x, & x \to +\infty. \end{cases}$

An example: $\tau_k = e^k,\ \lambda_k = \theta_1 e^k + e^{2k}$,
$\nu_k = \ln \ln (k+3),\ \mu_k = \theta_2 \ln \ln (k+3).$

Equations that are second-order time are *more than twice* the fun of the first-order equations!

(And even the first-order fun is not over yet...)