A Study of Stochastic Differential Equations

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Joint work with M. Scheutzow

Stochastic Differential Equation(SDE)

X^i, C^1 vector fields on a state space $\mathbf{R}^n$ (or on a manifold).

$$dx_t = \sum_{i=1}^{m} X_i(x_t)dB_t^i + X_0(x_t)dt. \tag{1}$$

- If there is a global solution $F_t(x,\omega)$, SDE is complete, also called non-explosive, conservative.
- If $(t,x) \in [0,\infty) \times \mathbf{R}^n \mapsto F_t(x,\omega)$ is continuous a.s., SDE is strongly complete, also called strict conservative=smooth flow exists.
- Flow of diffeomorphism: if in addition to continuous flow, $x \mapsto F_t(x,\omega)$ is a diffeomorphism a.s. Flow of diffeomorphism exists if the SDE and the adjoint SDE are strongly complete.

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Why strong completeness?

1. In numerical computation, the initial point is often only an approximate value.
2. It is required when an SDE is studied as a dynamical system.
3. If strong complete, we have a homotopy of maps. The solution flow could be used to feel the shape of the underlying space.
   e.g. If the solution sends a C^1 curve to a C^1 curve and that it shrinks the curve, Moment stable, ?technicalities, we have $\pi_1(M) = \{0\}$.
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Completenss = Strong Completeness

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + X_0(x_t)dt$$

1. ODE.
2. 1-dimensional SDE.
3. 1 vector field X, If $\phi(t,x) = X(\phi_t(x))$,
   $\phi(B_t(\omega), x)$ is solution to SDE.
4. Commuting vector field: $[X_i, X_j] = 0$, $\dot{\phi}^j(t,x) = X_j(\phi^j(t,x))$

   $$\tilde{\phi}_1(t,x) = \phi^1(B_t^1, x), \qquad \tilde{\phi}_{j+1}(t,x) = \phi(B_t^j, \tilde{\phi}_j(t,x)),$$

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Standard Results:

- Directional Linear growth at $\infty \Longrightarrow$ complete.
- Lipschitz continuous (Bounded derivatives), $\Longrightarrow$ strongly complete. Compute $\mathbf{E}|F_t(x) - F_t(y)|^p$ and use Kolmogrov's continuity theorem (a trick working only in linear spaces).
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A Basic tool : Partial Flow

$\exists$ a partial smooth flow $F_t(x,\omega), t < \zeta(x,\omega)$ which is a maximal solution such that

- $M_t(\omega) = \{x : \zeta(x,\omega) > t\}$ is an open set.
- $(s,x) \mapsto F_s(x,\omega)$ is continuous on $M_t(\omega)$.
- For any compact set K, set

$$\zeta^K = \inf_{x \in K} \zeta(x,\omega).$$

On $\{\zeta^K < \infty\}$

$$\lim_{t \to \zeta^K} \sup_{x \in K} d(F_t(x,\omega), O) = \infty.$$

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This works for C^1 vector fields for Itô SDE.

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The Characteristics of the two concepts

- Conservation of an SDE says that the measures μ_x on the path space are probability measures. It is a property of the infinitesimal generator $\mathcal{A}$ of the S.D.E., c.f. Varadhan-Stroock's martingale formulation.
- Strong completeness is related to the Hörmander form decomposition of the second order differential operator $\mathcal{A}$:

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An example

A complete, but not strongly complete SDE.

- on $\mathbf{R}^n - \{0\}$ (Elworthy 78), $dx_t = dB_t$.
- The equation $dx_t = dB_t$ on $\mathbf{R}^n - \{0\}$ is conservative.
- $B_t(\omega) + x$ is a maximal solution. $P(x + B_t(\omega) = 0 \text{ for some } x\ ) = 1$.
- Planer Brownian motion has capacity 0 on co-dimension 2 sets, hence the SDE is strongly $n-2$ complete. [Li95].

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Two SDEs with the same generator, Carverhill

Generator $\mathcal{A} = r^4\Delta$, Lyapunov function $\log r$. Hence conservative.

- Strongly complete.

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dx_t &= \frac{x_t}{r_t} dB_t^1 - \frac{y_t}{r_t} dB_t^2 \\
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  It is $dz_t = \frac{z_t}{|z_t|} dB_t$ on C.

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The derivative flow

- Consider the S.D.E. in Stratnovitch form and its linearization:

$$dv_t = \sum_{i=1}^{m} DX^i(x_t)(v_t) \circ dB_t + DX_0(x_t)(v_t)dt.$$

  The solution from v_0 is $v_t = T_{x_0}F_t(v_0)$.
  Moments: $\mathbf{E}|TF_t|^p \chi_{t<\tau}$. See [Li95-2] for estimates.

- Assume that $F_t(x_0)$ does not explode, then the SDE is strongly complete, [Li92, 95], if

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A criterion for SDEs on $\mathbf{R}^n$

$$dx_t = \sum_{i=1}^{m} X_i(x_t)dB_t^i + X_0(x_t)dt.$$

We apply the theorem to the above SDE in $\mathbf{R}^n$. The following is a reformulation of that given in [Li92], [Li95].

Let g be a Lyapunov function. Then the SDE is strongly complete if it is complete from one point, if the following holds,

$$2\frac{\langle DX_0(x)(v), v\rangle}{|v|^2} + \sum_{i=1}^{m} \frac{|DX_i(x)(v)|^2}{|v|^2} + (n-2)\sum_{i=1}^{m} \frac{\langle DX_i(x)(v), v\rangle^2}{|v|^4}$$

$$\leq \ln\left(\frac{1}{6n^2} g(x)\right)$$

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A little geometry

For elliptic SDE: suppose that $m = n$ and there is no drift, and that the linear connection associated to the SDE is the Levi-Civita connection then the SDE is strongly complete if it is complete. The linear connection is as defined in [Elworthy-LeJan-Li 97, 99.

Objectives: Li+ Scheutzow

- To gain some understanding why solutions of SDE lose regularity. Does it depend on the smoothness of the vector fields or on the growth of vector fields at infinity?
- Construct a 2-dimensional SDE with coefficients smooth, bounded, locally Lipschitz continuous for which strong completeness does not hold.

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Example[Li+ Scheutzow]

- An elliptic SDE run by 1 Brownian motion:

$$\begin{aligned} dX(t) &= \sigma(X(t), Y(t))dW(t) \\ dY(t) &= 0 \end{aligned}$$

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Exit Times and Completeness

- Let $U_1 \subset U_2 \subset U_3 \subset$ be a sequence of relatively compact open sets. Let τ_n be the first exit time of the solution, starting from a point in U_{n-1}, from U_n. If there exists $\sum \delta_n = \infty$,

$$P(\tau_n \le t) \le ct^2, \qquad t \le \delta_n$$

$\Longrightarrow$ SDE is complete. [Li89], [Li94]

- Sharp.
- E.g. A Brownian motion on a Riemannian manifold with $K(s) = -\inf_{B_s} Ricci(x) \wedge 0$, $\int_1^\infty \frac{1}{\sqrt{K(s)}} ds = \infty$ has the above property and so does SDE on $\mathbf{R}^n$ with linear growth.

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- E.g. A Brownian motion on a Riemannian manifold with $K(s) = -\inf_{B_s} Ricci(x) \wedge 0$, $\int_1^\infty \frac{1}{\sqrt{K(s)}} ds = \infty$ has the above property and so does SDE on $\mathbf{R}^n$ with linear growth.

Exit Times and Strong Completeness

We work on the smooth partial flow $F_t(x,\omega), t < \zeta(x)$, ζ^K.

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Basic Observations

$P(\tau_n \leq t) \leq ct^2$ implies completeness.

- If $\zeta \leq \sum_{n=1}^{\infty} T_n$, T_n finite stopping times, and $\{a_n\}$, $\{b_n\}$ be two summable sequences such that

$$P(T_n > a_n) < b_n$$

  then $\zeta < \infty$ almost surely. This can be used to show explosion or the failing of strong completeness.
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Intuitive Idea

The following illustrates why strong completenss could fail while completeness holds.

- $(B_t^1, B_t^2, \dots)$, Independent Brownian motions on $\mathbf{R}^n$.

$$P(\sup_{s\le a}|B_s^i|\ge 1)\to Ca^2, \qquad \text{for } a \text{ small }.$$

  $\to$ BM is complete.

- $$P(\sup_{1\le i\le n}\sup_{s\le a}|B_s^i|\le 1)\to 0, \qquad \text{as } n\to\infty.$$

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A Key Lemma

A SDE on $\mathbf{R}^1$

$$dX^\varepsilon(t) = H(\frac{1}{\varepsilon}X^\varepsilon(t))dW(t)$$
$$X^\varepsilon(0) = 0.$$

- $H : \mathbf{R} \to (0, \infty)$, continuous with period 1.
- $\varepsilon_{n+1}/\varepsilon_n \to 0$ as $n \to \infty$
- $(X^{\varepsilon_n}, X^{\varepsilon_{n+1}}, ...)$ converges weakly to $(\alpha B_0 + \beta B_1, \alpha B_0 + \beta B_2, ...)$ as $n \to \infty$.
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Example

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\begin{aligned}
dX(t) &= \sigma(X(t), Y(t))dW(t), \qquad x(0) \text{ fixed} \\
dY(t) &= 0, \qquad y(0) \in [0,1]
\end{aligned}
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- Define $\sigma(x,y) = H(x)$ for $x, y \in [0,1]$, H smooth period 1, bounded away from 0.
- Define $\sigma(x,y) = H(\frac{1}{\epsilon(x,y)}x)$.
- $$R = (-\infty, 0) \cup [0,1] \cup \cdots \cup [n, n+1] \cup \ldots.$$
- At level n, partition $[0,1]$ into M_n equal pieces:
$$[0,1] = \Delta_1^n \cup \cdots \cup \Delta_{M_n}^n$$
- If $(x,y) \in [n, n+1] \times \Delta_i^n$, define
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Key Point

Let $K_{n,i} = [n, n+1] \times \Delta_{n,i}$. $T_{n,i}$ =first time that $\{\phi(x, y), (x, y) \in K_{n,i}\}$ is greater than $n + 1$. We wish to prove something like following

- Lemma: $\sum_n t_n < \infty$, $\sum_n \beta_n < \infty$, Then $\exists M_n$, a_n s.t.

$$P\left(\inf_{1\leq i\leq M_n} T_{n.i} \leq t_n)\right) \leq \beta_n.$$

- That is

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Lemma

For two convergent series $\sum \beta_n$ *and* $\sum t_n$, $\exists N_n$ *s.t. if* $n > N_n$,

$$P\left(\sup_{1\le j\le N_n} \sup_{0\le s\le t_n} (B_s^j + W_s) \ge 1, \inf_{1\le j\le N_n} \inf_{0\le s\le t_n} (B_s^j + W_s) \ge -\delta_n \right) \ge 1-\beta_n.$$

This can be show by large deviation result for a Brownian path deviate from a Cameron-Martin path and an approximation of continuous path from a Cameron-Martin path. The distribution of our diffusion converge to that of the independent Brownian motions by an earlier lemma. The only problem is that if the diffusion crosses backward the speed changes and we no longer have control over the convergence speed.

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A control Strip

- Choose $\delta_n > 0$ where we smooth up $\epsilon(x, y)$.
- Each diffusion at level n has a positive probability p_0 of travelling δ_n distance to the right.
- The average time each martingale spent on the right half of the strip is positive. By continuity there is a time slot where it spend time there.
- Show that at least N_n diffusions will be on the right half of the strip at a given random time.

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An unanswered Question

How does adding a drift affect strong completeness? In particular,

- What about adding an arbitrary drift to the previous example? (Stratonovitch correction term gives a strongly complete SDE.)
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Trivial path space transformations preserve strong completeness: orthogonal transform and translation by a Cameron-Martin vector.

- If $dx_t = X(x_t) \circ dB_t$ is strongly complete,
  so is $dx_t = X(x_t) \circ dB_t + X(x_t)\dot{h}_t dt$.
  h=adapted Cameron-Martin on Wiener space $C_0(\mathbf{R}^m)$. This is not sufficient to transform a drift term. Girsanov does not do the trick.