PRE-LIE ALGEBRAS AND THE ROOTED TREES OPERAD

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Abstract. A pre-Lie algebra is a vector space L endowed with a bilinear product $\cdot : L \times L \to L$ satisfying the relation $(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x \cdot z) \cdot y - x \cdot (z \cdot y)$, $\forall x, y, z \in L$. We give an explicit combinatorial description in terms of rooted trees of the operad associated to this type of algebras and prove that it is a Koszul operad.

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Introduction

We study here a type of algebra which deserves more attention than it has been given. People have been using these algebras, under various names, for a long time. They appeared under the name of left-symmetric algebras in the work of Vinberg on convex homogeneous cones [V], and so were dubbed Vinberg algebras in some papers. They also appeared in the study of affine manifolds, under the name of right-symmetric algebras [Mat]. We propose to adopt the name of pre-Lie algebras, which has been used by Gerstenhaber [G]: the Lie bracket involved in the Gerstenhaber structure on the Hochschild cohomology comes from a pre-Lie algebra structure on the cochains. Besides, rooted trees have shown their interest in the study of vector fields, numerical analysis (see e.g. the paper of C. Brouder [B] and the references therein) and more recently in quantum field theory [Connes, Kreimer]. We define in this paper the underlying operad of pre-Lie algebras in terms of rooted trees which should shed light on the relationships between these different topics. The first author is indebted to M. Kontsevich for a talk about Hochschild complex and Deligne's conjecture which inspired the link between rooted trees and pre-Lie algebras.

The description of the operad defining pre-Lie algebras in terms of rooted trees is the subject of the first section. The operad arising here should not be confused with the structure on rooted trees appearing in [B-V]. The second section is devoted to the definition of the operadic homology of pre-Lie algebras. Finally, we prove in the third section that the operad associated to pre-Lie algebras is a Koszul operad. To that end, we prove in fact that a free pre-Lie algebra L is a free module over the enveloping algebra of the Lie algebra underlying L. Combined with the first section, it gives a new interpretation of the Hopf algebra appearing in the works of A. Connes and D. Kreimer [C-K].

1. A description of the operad defining pre-Lie algebras

This section is devoted to the description of the operad defining pre-Lie algebras. We prove in the theorem 1.9 that this operad is the operad of rooted trees.

We recall briefly some facts about operads (see [Gi-K],[Ge-J]). An *operad* $\mathcal{P}$ is a sequence of vector spaces $\mathcal{P}(n)$, for $n \geq 1$, such that $\mathcal{P}(n)$ is a module over the symmetric group S_n, together with composition maps $\gamma : \mathcal{P}(n) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_n) \to \mathcal{P}(i_1 + \cdots + i_n)$ satisfying some relations of associativity, unitarity and equivariance with respect to the symmetric group, called May axioms [May]. Note that giving a S_n-module for all n is equivalent to giving a $\mathbb{S}$-*module*, i.e. a functor from the category of (finite sets, bijections) to the category of vector spaces. Hence an operad can be defined by a $\mathbb{S}$-module $\mathcal{P}$ together with composition maps $\gamma : \mathcal{P}(I) \otimes \mathcal{P}(J_1) \otimes \cdots \otimes \mathcal{P}(J_n) \to \mathcal{P}(J_1 \sqcup \cdots \sqcup J_n)$ where n is the cardinal of I. An *algebra* over an operad $\mathcal{P}$ is a vector space A together with maps $\mathcal{P}(n) \otimes A^{\otimes n} \to A$ satisfying some relations of associativity, unitarity and equivariance.

1.1. PRE-LIE ALGEBRAS. A *pre-Lie algebra* is a vector space L together with a bilinear map $\cdot : L \times L \to L$ satisfying the relation
$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x \cdot z) \cdot y - x \cdot (z \cdot y), \ \forall x, y, z \in L.$$
When the vector space L is graded we define a *graded pre-Lie algebra* by the relation
$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (-1)^{|y||z|}((x \cdot z) \cdot y - x \cdot (z \cdot y)), \ \forall x, y, z \in L.$$

1.2. PROPOSITION. *Let $(L, \cdot)$ be a pre-Lie algebra. The bracket defined by*
$$[a, b] = a \cdot b - b \cdot a, \ \forall a, b \in L$$
endows L with a structure of Lie algebra. In the sequel $L_{\mathcal{L}\mathrm{ie}}$ will denote the Lie algebra $(L, [-, -])$.

1.3. EXAMPLES. M. Gerstenhaber [G] introduced a structure of pre-Lie algebra on the Hochschild complex of an associative algebra A as follows: denote by $C^m(A, A)$ the space $\mathrm{Hom}(A^{\otimes m}, A)$ in degree $m - 1$; let $f \in C^m$ and $g \in C^n$, then the product
$$(f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1}) =$$
$$\sum_{i=1}^{m} (-1)^{(n-1)(i-1)} f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1})$$
satisfies the graded relation defining pre-Lie algebras.

The structure of pre-Lie algebra also appears in the study of affine structures on manifolds [Mat]. An affine structure on a n-manifold is an atlas whose coordinate changes are in the group of affine motions of $\mathbb{R}^n$. It can also be given by a linear connection ∇ whose torsion and curvature vanish. The product $X \circ Y = -\nabla_Y X$ then defines a structure of pre-Lie algebra on the set of vector fields, such that the associated Lie bracket is the usual bracket of vector fields.

1.4. THE OPERAD $\mathcal{PL}$. From the definition 1.1, it is clear that a pre-Lie algebra is an algebra over a binary quadratic operad, denoted by $\mathcal{PL}$. We recall briefly how to construct $\mathcal{PL}$ [Gi-K]. Let $\mathcal{F}$ be the free operad generated by the regular representation of S_2. A basis of $\mathcal{F}(n)$, as a vector space, is given by "parenthesized products" on n variables indexed by $\{1, \cdots, n\}$. For instance, a basis of $\mathcal{F}(2)$ is given by $(x_1 x_2)$ and $(x_2 x_1)$, and a basis of $\mathcal{F}(3)$ is given by $((x_1 x_2) x_3)$, $(x_1(x_2 x_3))$ and all their permutations. Let R be the S_3-sub-module of $\mathcal{F}(3)$ generated by the relation $r = ((x_1 x_2) x_3) - (x_1(x_2 x_3)) - ((x_1 x_3) x_2) + (x_1(x_3 x_2))$. Then $\mathcal{PL} = \mathcal{F}/(R)$, where (R) denotes the ideal of $\mathcal{F}$ generated by R. The operadic composition on $\mathcal{PL}$ is induced by the one on $\mathcal{F}$, given by

$$\gamma : \mathcal{F}(n) \otimes \mathcal{F}(i_1) \otimes \cdots \otimes \mathcal{F}(i_n) \to \mathcal{F}(i_1 + \cdots + i_n)$$

which assigns to $(\mu, \nu_1, \cdots \nu_n)$ the word obtained by substituting ν_i for $x_i \in \mu$. Notice that the concatenation $(\rho \rho')$ is the particular case of the composition $\gamma((x_1 x_2), \rho, \rho')$.

1.5. THE OPERAD OF ROOTED TREES $\mathcal{RT}$. Let $n > 0$. A *rooted tree* of degree n, or n-rooted tree, is a non-empty connected graph without loops whose vertices are labelled by the set $[n] = \{1, \cdots, n\}$, together with a distinguished element in this set called the root. Edges of this graph are oriented towards the root. We denote by $\mathcal{RT}(n)$ the free $\mathbb{Z}$-module generated by n-rooted trees. We can endow $\mathcal{RT} = (\mathcal{RT}(n))_{n \geq 1}$ with an operad structure, as explained below.

The action of the symmetric group is the natural one, by permutation of indices. Let T be a n-rooted tree; denote by $\mathrm{In}(T, i)$ the set of incoming edges at the vertex i of T. Let S be a m-rooted tree. In order to define the operadic composition, describing the compositions $\circ_i : \mathcal{RT}(n) \times \mathcal{RT}(m) \to \mathcal{RT}(n + m - 1)$, for $1 \leq i \leq n$ (see e.g. [Lo]) is enough. We define the composition of T and S along the vertex i of T by

$$T \circ_i S = \sum_{f : \mathrm{In}(T,i) \to [m]} T \circ_i^f S,$$

where $T \circ_i^f S$ is the rooted tree obtained by substituting the tree S for the vertex i in T : the outgoing edge of i, if exists, becomes the outgoing edge of the root of S; incoming edges of i are grafted on the vertices of S following the map f. Then, it is easy to check that these compositions endow $\mathcal{RT}$ with a structure of operad. Let us give an example. A rooted tree is drawn with its root at the bottom. Let
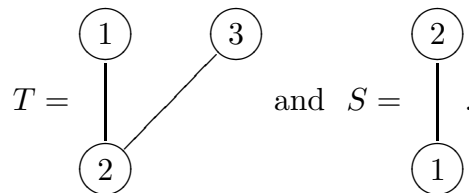
By describing maps from $\{1, 3\}$ to $\{1, 2\}$ and by reindexing the vertices of S and T, one

gets

$$T \circ_2 S = \quad + \quad + \quad + \quad .$$

$$f: \begin{cases} 1 \to 2 \\ 3 \to 2 \end{cases} \qquad f: \begin{cases} 1 \to 2 \\ 3 \to 1 \end{cases} \qquad f: \begin{cases} 1 \to 1 \\ 3 \to 2 \end{cases} \qquad f: \begin{cases} 1 \to 1 \\ 3 \to 1 \end{cases}$$

1.6. THE POINCARE SERIES ASSOCIATED TO $\mathcal{RT}$. The Poincare series is defined by

$$g_{\mathcal{RT}}(x) = \sum_{n \geq 1} \dim(\mathcal{RT}(n))(-x)^n/n!$$

Using classical results in combinatorics [W], one obtains that $\dim(\mathcal{RT}(n)) = n^{n-1}$ and that $g_{\mathcal{RT}}$ is the inverse map of $x \mapsto -xe^{-x}$.

1.7. A PRODUCT IN THE OPERAD $\mathcal{RT}$. Recall that the operadic composition for $\mathcal{RT}$, $\gamma : \mathcal{RT}(n) \otimes \mathcal{RT}(i_1) \otimes \cdots \otimes \mathcal{RT}(i_n) \to \mathcal{RT}(i_1 + \cdots + i_n)$ is given in terms of the $\circ_i$ compositions by $\gamma(\mu, \nu_1, \cdots, \nu_n) = (\cdots (\mu \circ_n \nu_n) \circ_{n-1} \nu_{n-1}) \cdots \circ_1 \nu_1)$.

We may then define a particular composition, which corresponds to the concatenation in $\mathcal{PL}$-case (see 1.4): for any rooted trees T_1 and T_2, let

$$T_1 \star T_2 = \gamma(\,\raisebox{-0.5ex}{\overset{\textstyle 2}{\underset{\textstyle 1}{\vert}}}\,, T_1, T_2) = \sum_{s \in I_1} \underset{T_1}{\overset{T_2}{s\!\downarrow}} \quad .$$

More explicitly, the operation $T_1 \star T_2$ consists of grafting the root of T_2 on every vertex of T_1.

1.8. LEMMA. *The following relation holds for any rooted trees T_j, $1 \leq j \leq 3$:*

$$(T_1 \star T_2) \star T_3 - T_1 \star (T_2 \star T_3) = (T_1 \star T_3) \star T_2 - T_1 \star (T_3 \star T_2).$$

Proof. By computing

$$(T_1 \star T_2) \star T_3 - T_1 \star (T_2 \star T_3) = \sum_{s \in I_1} \sum_{t \in I_2} \underset{T_1}{\overset{T_3}{\underset{T_2}{t\!\downarrow}}_{s\!\downarrow}} + \sum_{s \in I_1} \sum_{s \in I_2} \overset{T_3 \quad T_2}{\underset{T_1 \quad s}{t\!\diagdown}} - \sum_{s \in I_1} \sum_{t \in I_2} \underset{T_1}{\overset{T_3}{\underset{T_2}{t\!\downarrow}}_{s\!\downarrow}}$$

and inverting the roles of T_2 and T_3, the required equality is obtained. $\square$

1.9. THEOREM. *The operad $\mathcal{PL}$ defining pre-Lie algebras is isomorphic to the operad of rooted trees $\mathcal{RT}$.*

Proof. Firstly, we define an operadic morphism $\Phi : \mathcal{PL} \to \mathcal{RT}$. Since $\mathcal{PL} = \mathcal{F}/(R)$ (1.4) it is sufficient to define Φ on $\mathcal{PL}(2) = \mathcal{F}(2)$, then to extend it on $\mathcal{F}$ by the universal property of the free operad and to check that $\Phi(r) = 0$. Set

$$\Phi((x_1 x_2)) = \begin{matrix} \boxed{2} \\ | \\ \boxed{1} \end{matrix} \quad \text{and} \quad \Phi((x_2 x_1)) = \begin{matrix} \boxed{1} \\ | \\ \boxed{2} \end{matrix}.$$

Hence

$$\Phi(r) = 0.$$

Remark that, since Φ is an operadic morphism, it sends the concatenation product (see 1.4) to the product $\star$ defined in 1.7.

The proof relies on the existence of an inverse Ψ of Φ. As we explained in the introduction of the section, it is more convenient for the proof to deal with $\mathbb{S}$-modules. For a finite set I, a *I-rooted tree* is a rooted tree whose vertices are labelled by I; its degree is the cardinal of I. We denote by $\Phi_I : \mathcal{PL}(I) \to \mathcal{RT}(I)$ the natural extension of Φ.

Claim. For any finite set I, there exists a map $\Psi_I : \mathcal{RT}(I) \to \mathcal{PL}(I)$ such that $\Psi_I \Phi_I = \mathrm{Id}$ and $\Phi_I \Psi_I = \mathrm{Id}$.

We prove the claim by induction on $\#I$. If $I = \{i\}$, then it is trivial. Assume that the claim is true for any I such that $\#I \leq n$ and let I be a finite set of cardinal $n+1$. Let T be a I-rooted tree and i be its root. Up to a permutation, we can write uniquely

$$T = B(i, T_1, \cdots, T_p) = \begin{matrix} \boxed{T_1} \quad \boxed{T_2} \quad \cdots\cdots \quad \boxed{T_p} \\ \diagdown \quad | \quad \diagup \\ \boxed{i} \end{matrix},$$

where T_i, for $1 \leq i \leq p$, is a rooted tree of degree strictly less than $n+1$. Let us define the map Ψ_I by induction on p.

If $p = 1$, then $T = B(i, T_1) = \boxed{i} \star T_1$ and $\Psi_I(T) = (i \Psi_I(T_1))$ is well defined. Moreover $\Phi_I \Psi_I(T) = T$, because Φ_I sends concatenation to the product $\star$ and because of the induction hypothesis. For $p \geq 2$, one has

$$T = B(i, T_2, \cdots T_p) \star T_1 - \sum_{j=2}^{p} B(i, T_2, \cdots, T_j \star T_1, \cdots T_p).$$

Consequently, we may define by induction:

$$\Psi_I(T) = (\Psi_I(B(i,T_2,\cdots T_p))\Psi_I(T_1)) - \sum_{j=2}^{p} \Psi_I(B(i,T_2,\cdots,T_j \star T_1,\cdots T_p)).$$

Moreover, as in the case $p=1$, $\Phi_I \Psi_I(T) = T$.

A priori, since T is uniquely determined only up to a permutation, this definition depends on the choice of the edge we ungraft in the tree T. Let us prove by induction on p that it is not the case. For $p = 0,1$ there is no choice; for $p > 1$, we prove that ungrafting the edge where T_1 lies, then ungrafting edges where T_2 lies in the trees involved in the sum, gives the same definition of $\Psi_I(T)$ as doing it inverting T_1 and T_2. By ungrafting T_2 in the previous relation, we get
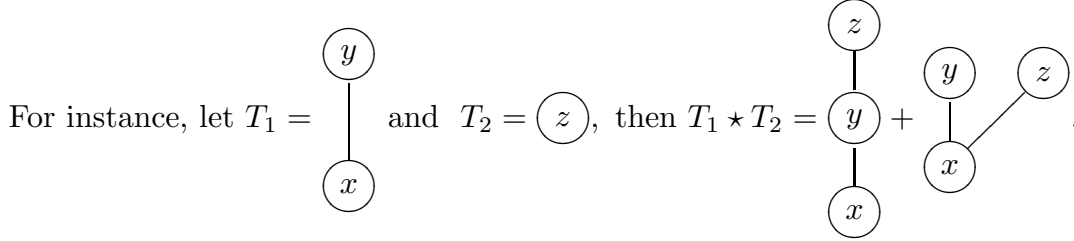
$$T = (B(i,T_3,\cdots T_p) \star T_2) \star T_1 - \sum_{k=3}^{p} B(i,T_3,\cdots,T_k \star T_2,\cdots,T_p) \star T_1$$

$$-B(i,T_3,\cdots T_p) \star (T_2 \star T_1) + \sum_{j=3}^{p} B(i,T_3,\cdots,T_j \star (T_2 \star T_1),\cdots T_p)$$

$$-\sum_{j=3}^{p} B(i,T_3,\cdots,T_j \star T_1,\cdots T_p) \star T_2 + \sum_{j=3}^{p} B(i,T_3,\cdots,(T_j \star T_1) \star T_2,\cdots T_p)$$

$$+\sum_{j=3}^{p}\sum_{\substack{k=3\\k\neq j}}^{p} B(i,T_3,\cdots,T_k \star T_2,\cdots,T_j \star T_1,\cdots,T_p).$$

Let $A_{12} = ((\Psi_I B(i,T_3,\cdots T_p)\Psi_I(T_2))\Psi_I(T_1)) - (\Psi_I B(i,T_3,\cdots T_p)(\Psi_I(T_2)\Psi_I(T_1)))$ and let A_{21} be the same term with T_1 and T_2 inverted; since $A_{21} - A_{12} \in (R)$, these terms coincide in $\mathcal{PL}$. It is clear that the terms $B_{12}^k = (\Psi_I B(i,T_3,\cdots,T_k \star T_2,\cdots,T_p)\Psi_I(T_1)) + (\Psi_I B(i,T_3,\cdots,T_k \star T_1,\cdots,T_p)\Psi_I(T_2))$ and B_{21}^k coincide, as well as the terms $C_{12}^{jk} = \Psi_I(B(i,T_3,\cdots,T_k \star T_2,\cdots,T_j \star T_1,\cdots,T_p))$ and C_{21}^{kj}; finally, the terms $D_{12}^j = \Psi_I(B(i,T_3,\cdots,(T_j \star T_1) \star T_2,\cdots T_p)) + \Psi_I(B(i,T_3,\cdots,T_j \star (T_2 \star T_1),\cdots T_p))$ and D_{21}^j coincide thanks to lemma 1.8.

Furthermore $\Psi\Phi = \mathrm{Id}$. In fact, one has $\Psi(T \star T') = (\Psi(T)\Psi(T'))$. Indeed when $T = \textcircled{i}$ the result comes from the case $p=1$; if $T = B(i,T_1,\cdots,T_p)$, then $T \star T' = B(i,T',T_1,\cdots,T_p) + \sum_{k=1}^{p} B(i,T_1,\cdots,T_k \star T',\cdots,T_p)$, and by choosing to ungraft T' in order to define Ψ, we get the result. As a consequence, let μ be a word in $\mathcal{PL}$, hence μ can be uniquely decomposed in a concatenation $\mu = (\rho\rho')$; since $\Phi(\mu) = \Phi(\rho) \star \Phi(\rho')$, then $\Psi\Phi(\mu) = (\Psi\Phi(\rho)\Psi\Phi(\rho'))$ and we can conclude by induction on the degree of μ. This ends the proof of the theorem. $\square$

The description of free pre-Lie algebras is a direct consequence of the previous results.

1.10. COROLLARY. *Let V be a vector space. The free pre-Lie algebra generated by V is the vector space generated by the rooted trees labelled by a basis of V, with the product $\star$ defined in 1.7 : let T_1 and T_2 be two trees labelled by a basis of V then $T_1 \star T_2$ is the sum over the vertices v of T_1 of trees obtained by linking with an edge the root of T_2 to the vertex v of T_1.*

For instance, let $T_1 = \begin{smallmatrix} y \\ | \\ x \end{smallmatrix}$ and $T_2 = z$, then $T_1 \star T_2 = \begin{smallmatrix} z \\ | \\ y \\ | \\ x \end{smallmatrix} + \begin{smallmatrix} y\ \ z \\ \diagdown \diagup \\ x \end{smallmatrix}$.

2. Homology of pre-Lie algebras

Since the homology of pre-Lie algebras has been already defined by A. Nijenhuis [N], and extended by A. Dzhumadil'daev [D], the aim of this section is to understand how the operad theory can lead naturally to the definition of the operadic homology of a type of algebra, which coincides hopefully with some definitions given earlier.

Following Ginzburg and Kapranov [Gi-K], in order to define the operadic homology of a pre-Lie algebra, it is necessary to introduce the complex built on the free coalgebra on the dual operad of $\mathcal{PL}$ whose differential is the only coderivation induced by the pre-Lie product.

Notation. In the sequel, the ground field K will be of characteristic zero.

A $(k_1, \cdots, k_p)$-*shuffle* is a permutation $\sigma \in S_{k_1 + \cdots + k_p}$ such that $\sigma(1) < \cdots < \sigma(k_1)$, $\sigma(k_1 + 1) < \cdots < \sigma(k_1 + k_2)$, etc... We denote by $\mathrm{Sh}_{k_1, \cdots, k_p}$ the set of all $(k_1, \cdots, k_p)$-shuffles. A permutation $\sigma \in S_n$ acts on $V^{\otimes n}$ by $\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, and in case V is a graded vector space we denote by $\epsilon(\sigma, \bar{v})$ the sign appearing in this action. For instance $\epsilon((12), \bar{v}) = (-1)^{|v_1||v_2|}$.

Let V be a graded vector space; we denote by $S(V)$ the *graded symmetric algebra* generated by V, that means $S(V)$ is the quotient of the free associative algebra $T(V)$ generated by V by the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$, $\forall x, y \in V$. If V is concentrated in degree 1, it becomes the exterior algebra on V and it is usually denoted by $\Lambda(V)$. The *suspension* of V, denoted by sV is defined by $(sV)_n = V_{n-1}$.

2.1. PROPOSITION. *The dual operad of the pre-Lie operad is the operad* Perm *defined in [Ch]: a* Perm-*algebra is a vector space A together with a bilinear product $\cdot : A \times A \to A$ satisfying the relations*

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$
$$a \cdot b \cdot c = a \cdot c \cdot b, \ \ \forall a, b, c \in A.$$

Proof. Recall the definition of the quadratic dual operad $\mathcal{P}^!$ of an operad $\mathcal{P}$ in our framework, with the notation of 1.4. If $\mathcal{P} = \mathcal{F}/(R)$, where $\mathcal{F}$ is the free operad generated by the regular representation of S_2, then there is a scalar product on $\mathcal{F}(3)$ defined by

$$< i(jk), i(jk) >= \mathrm{sgn} \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

$$< (ij)k, (ij)k >= -\mathrm{sgn} \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

and $\mathcal{P}^! = \mathcal{F}/(R^\perp)$ where $R^\perp$ is the annihilator of R with respect to this scalar product [Gi-K].

Let R be the S_3-sub-module of $\mathcal{F}(3)$ defined in 1.4 and R' be the S_3-sub-module of $\mathcal{F}(3)$ generated by the relations $s = ((x_1 x_2)x_3) - (x_1(x_2 x_3))$ and $t = ((x_1 x_2)x_3) - ((x_1 x_3)x_2)$, since $< R, R' >= 0$, $\dim(R) = 3$, and $\dim(R') = 9$, we can conclude that $R' = R^\perp$. $\qquad\square$

2.2. DEFINITION. A *graded* Perm-*coalgebra* C is a positively graded vector space equipped with a comultiplication $\Delta : C \rightarrow C \otimes C$ of degree 0, satisfying the following identities

$$(\mathrm{Id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{Id})\Delta,$$
$$(\mathrm{Id} \otimes \Delta)\Delta = (\mathrm{Id} \otimes T)(\mathrm{Id} \otimes \Delta)\Delta$$

where $T(a \otimes b) = (-1)^{|a||b|}b \otimes a$.

A *coderivation* of C is a linear map $d : C \rightarrow C$ satisfying $\Delta d = (\mathrm{Id} \otimes d)\Delta + (d \otimes \mathrm{Id})\Delta$. The space of all coderivations of C is denoted by $\mathrm{Coder}(C)$.

The proof of the following lemma is left to the reader.

2.3. LEMMA. *Let V be a reduced graded vector space, i.e. $V_0 = 0$, then the free* Perm-*coalgebra generated by V, denoted by $\mathrm{Perm}^c(V)$, is the vector space $V \otimes S(V)$ equipped with the following comultiplication :*

$$\Delta(v \otimes 1) = 0, \forall v \in V$$

$$\Delta(v_0 \otimes v_1 \cdots v_n) = \sum_{\substack{0 \leq k \leq n-1 \\ \sigma \in \mathrm{Sh}_{k,1,n-1-k}}} \epsilon(\sigma, \bar{v}) \, v_0 \otimes v_{\sigma(1)} \cdots v_{\sigma(k)} \otimes v_{\sigma(k+1)} \otimes v_{\sigma(k+2)} \cdots v_{\sigma(n)},$$

$$\forall v_0 \in V, \ v_1 \cdots v_n \in S^n(V).$$

The space $\mathrm{Perm}^c(V)$ comes with a natural projection onto V, denoted by π.

2.4. PROPOSITION. *The following isomorphism of vector spaces holds:*

$$\phi : \quad \mathrm{Coder}(\mathrm{Perm}^c(V)) \quad \rightarrow \quad \mathrm{Hom}(\mathrm{Perm}^c(V), V)$$

$$d \qquad\qquad \mapsto \qquad\qquad \pi \circ d.$$

Moreover, there exists a graded pre-Lie algebra structure on $\mathrm{Hom}(\mathrm{Perm}^c(V), V)$ *such that* ϕ *is an isomorphism of graded Lie algebras; the Lie bracket on* $\mathrm{Coder}(\mathrm{Perm}^c(V))$ *is defined by* $[d^1, d^2] = d^1 d^2 - (-1)^{|d^1||d^2|} d^2 d^1$. *The pre-Lie product* $\circ$ *is the following one: let* $l, m \in \mathrm{Hom}(\mathrm{Perm}^c(V), V)$; *we set* $l = \sum_i l_i$ *with* $l_i : V \otimes S^i(V) \to V$, *then*

$$(m \circ l)_n(v_0 \otimes v_1 \cdots v_n) = \sum_{\substack{0 \le i \le n \\ \sigma \in \mathrm{Sh}_{i,n-i}}} \epsilon(\sigma, \bar{v}) m_{n-i}(l_i(v_0 \otimes v_{\sigma(1)} \cdots v_{\sigma(i)}) \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)})$$

$$+ (-1)^{|v_0||l|} \sum_{\substack{0 \le i \le n-1 \\ \sigma \in \mathrm{Sh}_{1,i,n-i-1}}} \epsilon(\sigma, \bar{v}) m_{n-i}(v_0 \otimes l_i(v_{\sigma(1)} \otimes v_{\sigma(2)} \cdots v_{\sigma(i+1)}) v_{\sigma(i+2)} \cdots v_{\sigma(n)}),$$

Proof. The proof consists essentially of computation. The first part of the proposition relies on the fact that the map $\psi : \mathrm{Hom}(\mathrm{Perm}^c(V), V) \to \mathrm{Coder}(\mathrm{Perm}^c(V))$ defined by

$$\psi(l)(v_0 \otimes v_1 \cdots v_n) = \sum_{i=0}^{n} \sum_{\sigma \in \mathrm{Sh}_{i,n-i}} \epsilon(\sigma, \bar{v}) \, l_i(v_0 \otimes v_{\sigma(1)} \cdots v_{\sigma(i)}) \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)}$$

$$+ (-1)^{|v_0||l|} \sum_{i=0}^{n-1} \sum_{\sigma \in \mathrm{Sh}_{1,i,n-i-1}} \epsilon(\sigma, \bar{v}) \, v_0 \otimes l_i(v_{\sigma(1)} \otimes v_{\sigma(2)} \cdots v_{\sigma(i+1)}) v_{\sigma(i+2)} \cdots v_{\sigma(n)},$$

is the inverse map of ϕ; this definition implies also that ϕ is a Lie algebra morphism. The fact that the product $\circ$ is a pre-Lie algebra morphism is a straightforward calculation. □

This proposition yields naturally the definition of a pre-Lie algebra up to homotopy.

2.5. DEFINITION. A graded vector space V is a *pre-Lie algebra up to homotopy* or a $\mathcal{PL}_\infty$-*algebra* if it is equipped with a map $l \in \mathrm{Hom}(\mathrm{Perm}^c(sV), sV)$ of degree -1 such that $l \circ l = 0$, or equivalently $[l, l] = 0$.

Furthermore, this proposition leads to the definition of the homology of a pre-Lie algebra. Indeed, let $(L, \cdot)$ be a pre-Lie algebra and let $\mu : (sL \otimes sL) \to (sL)$ be the map of degree -1 defined by $\mu(sx \otimes sy) = (-1)^{|sx|} s(x \cdot y)$. Hence, by the definition 1.1 of a pre-Lie algebra one gets $\mu \circ \mu = 0$, then $[\mu, \mu] = 0$; thus the coderivation d, induced by the isomorphism, satisfies $d^2 = 0$. The complex so obtained is the one defining the operadic homology of a pre-Lie algebra, in the sense of [Gi-K]. Using Koszul sign rules and the isomorphism $sL \simeq e \otimes L$, with e a formal element of degree 1, one gets the definition of the homology of a pre-Lie algebra.

2.6. HOMOLOGY OF PRE-LIE ALGEBRAS. Let L be a pre-Lie algebra. The *pre-Lie homology* of L, denoted by $\mathrm{HPL}(L)$ is the homology of the complex $(\mathrm{CPL}_n(L), d)$, where $\mathrm{CPL}_n(L) = L \otimes \Lambda^{n-1}(L)$ and

$$d(v_0 \otimes v_1 \wedge \cdots \wedge v_n) = \sum_{1 \le j \le n} (-1)^j v_0 \cdot v_j \otimes v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n$$

$$+ \sum_{1 \le i < j \le n} (-1)^{i+j-1} v_0 \otimes [v_i, v_j] \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n,$$

where the bracket is the Lie bracket in L (see 1.2).

This complex coincides with the one defined by A. Nijenhuis [N], and for a complete definition of the cohomology of a pre-Lie algebra with coefficients in a representation we will refer to [D].

2.7. ON THE LINK BETWEEN A PRE-LIE ALGEBRA AND ITS INDUCED LIE ALGEBRA.

Let $(L, \cdot)$ be a pre-Lie algebra and denote by $(L_{\mathcal{L}ie}, [-,-])$ its induced Lie algebra (see 1.2). Then the relation defining the pre-Lie algebra structure implies that L is a right module over $L_{\mathcal{L}ie}$ via the action

$$
\begin{aligned}
L \times L_{\mathcal{L}ie} &\rightarrow L \\
(v, g) &\mapsto v \cdot g.
\end{aligned}
$$

Hence L is a right module over the enveloping algebra $\mathcal{U}(L_{\mathcal{L}ie})$ of $L_{\mathcal{L}ie}$, with the usual definition $l \cdot (a_1 \otimes \cdots \otimes a_n) = (\cdots (l \cdot a_1) \cdot a_2) \cdots \cdot a_n)$.

As a consequence, there is a nice interpretation of the pre-Lie homology of L in terms of the Chevalley-Eilenberg homology of $L_{\mathcal{L}ie}$ with coefficients in L; one has the isomorphisms:

$$
\mathrm{HPL}_{n+1}(L) \simeq H_n^{CE}(L_{\mathcal{L}ie}, L) \simeq \mathrm{Tor}_n^{\mathcal{U}(L_{\mathcal{L}ie})}(L, K).
$$

3. Koszulness of the operad defining pre-Lie algebras

The aim of this section is to prove that the operad $\mathcal{PL}$ is a Koszul operad. As explained in [Gi-K], it is enough to prove that for any free pre-Lie algebra L, its homology $\mathrm{HPL}(L)$ is concentrated in degree 1. In fact, the main point is that a free pre-Lie algebra L is a free right $\mathcal{U}(L_{\mathcal{L}ie})$-module (theorem 3.3). Then the Koszulness of the operad follows with the help of remark 2.7.

Before proving theorem 3.3, we would like to point out some interesting remarks on free pre-Lie algebras.

3.1. A LINK WITH THE CONNES-KREIMER HOPF ALGEBRA. As a Lie algebra, the free pre-Lie algebra on a single generator (see 1.10) has already appeared in the work of Alain Connes and Dirk Kreimer on the combinatorics of renormalization. They consider a commutative Hopf algebra of polynomials in rooted trees. By the Milnor-Moore theorem, the dual Hopf algebra is the universal enveloping algebra of some Lie algebra, which they calculated in [C-K]. This Lie algebra has a basis indexed by rooted trees, and one can check that the bracket is the same as the one induced by the pre-Lie structure of the free pre-Lie algebra.

3.2. LEMMA. *Let L be a pre-Lie algebra, V a vector space and $\sigma : V \to L$ a morphism.*
The right $\mathcal{U}(L_{\mathcal{L}ie})$-module $V \otimes \mathcal{U}(L_{\mathcal{L}ie})$ can be equipped with a structure of pre-Lie alge-
bra such that the map $\tilde{\sigma} : V \otimes \mathcal{U}(L_{\mathcal{L}ie}) \to L$ defined by $\tilde{\sigma}(v \otimes u) = \sigma(v) \star u$, where the
action of $\mathcal{U}(L_{\mathcal{L}ie})$ on L is denoted by $\star$, becomes a morphism of pre-Lie algebras.

Proof. An element in $V \otimes U(L_{\mathcal{L}ie})$ will be denoted by (v, u). The product of pre-Lie
algebra on $V \otimes \mathcal{U}(L_{\mathcal{L}ie})$ is defined as follows

$$(v, u) * (v', u') = (v, u \otimes (\sigma(v') \star u')), \ \forall v, v' \in V, u, u' \in \mathcal{U}(L_{\mathcal{L}ie}).$$

Let us check the relation $R = (A * B) * C - A * (B * C) - (A * C) * B + A * (C * B) = 0$
in $V \otimes \mathcal{U}(L_{\mathcal{L}ie})$. Let $A = (v, u)$, $B = (v', u')$, $C = (v'', u'')$. Then

$$(A*B)*C - A*(B*C) = (v, u \otimes \sigma(v') \star u' \otimes \sigma(v'') \star u'') - (v, u \otimes \sigma(v') \star (u' \otimes \sigma(v'') \star u'')).$$

But, since L is a right $\mathcal{U}(L_{\mathcal{L}ie})$-module, $\sigma(v') \star (u' \otimes \sigma(v'') \star u'') = (\sigma(v') \star u') \star (\sigma(v'') \star u'')$.
Let $\alpha = \sigma(v') \star u'$ and $\beta = \sigma(v'') \star u''$. Since α and β lie in L the action $\alpha \star \beta$ coincides
with the pre-Lie product in L, thus $\alpha \star \beta - \beta \star \alpha = [\alpha, \beta] \in L_{\mathcal{L}ie}$. As a consequence

$$R = (v, u \otimes (\alpha \otimes \beta - \beta \otimes \alpha - [\alpha, \beta])) = 0.$$

Then it is clear that $\tilde{\sigma}$ is a morphism of pre-Lie algebras. $\qquad\square$

3.3. THEOREM. *Let L be a free pre-Lie algebra generated by a vector space V. Then*
there is an isomorphism of right $\mathcal{U}(L_{\mathcal{L}ie})$-module

$$L \simeq V \otimes \mathcal{U}(L_{\mathcal{L}ie}).$$

Proof. Let σ be the canonical morphism from V to L, let $U = \mathcal{U}(L_{\mathcal{L}ie})$ and let τ be the
morphism from V to $V \otimes U$ such that $\tau(v) = v \otimes 1$. The product in L is denoted by $\star$
as well as the action of U on L.

By the universal property of the free right U-module $V \otimes U$, there is a unique right
U-module morphism $\psi : V \otimes U \to L$ such that $\psi \tau = \sigma$. Indeed ψ is the morphism

$$
\begin{array}{rccc}
\psi : & V \otimes U & \to & L \\
& v \otimes u & \mapsto & \sigma(v) \star u
\end{array}
$$

and by virtue of lemma 3.2, it is a pre-Lie algebras morphism for the product on $V \otimes U$
given by $(v, u) * (v', u') = (v, u \otimes (\sigma(v') \star u'))$. Furthermore, the universal property
of the free pre-Lie algebra L implies that there is a unique pre-Lie algebras morphism
$\phi : L \to V \otimes U$ such that $\phi \sigma = \tau$. As a consequence, the fact that $\psi \phi \sigma = \sigma$ and that
$\psi \phi : L \to L$ is a pre-Lie algebra morphism, implies $\psi \phi = \mathrm{Id}$.

In order to conclude it is sufficient to prove that ϕ is a right U-module morphism, because it implies that $\phi\psi = \mathrm{Id}$. The action of U on $V \otimes U$ is the concatenation, denoted by $(v \otimes u) \cdot (a_1 \otimes \cdots \otimes a_n) = (v, u \otimes a_1 \otimes \cdots \otimes a_n)$. Since U is generated by L, it is sufficient to prove that $\phi(x \star y) = \phi(x) \cdot y$, $\forall x, y \in L$. Now ϕ is a pre-Lie algebras morphism, then $\phi(x \star y) = \phi(x) * \phi(y)$. We set $\phi(x) = \sum_i (v_i, u_i) \in V \otimes U$ and $\phi(y) = \sum_j (w_j, r_j) \in V \otimes U$. Therefore

$$\phi(x) * \phi(y) = \sum_{i,j} (v_i, u_i \otimes \sigma(w_j) \star r_j)$$
$$= \sum_{i,j} (v_i, u_i \otimes \psi(w_j \otimes r_j))$$
$$= \sum_i (v_i, u_i \otimes \psi\phi(y)).$$

But $\psi\phi = \mathrm{Id}$, hence $\phi(x) * \phi(y) = \phi(x) \cdot y$. □

3.4. THEOREM. *The operad defining pre-Lie algebras is a Koszul operad.*

Proof. Let L be the free pre-Lie algebra generated by the vector space V. By virtue of theorem 3.3, L is the free right $U(L_{\mathcal{L}ie})$-module on V, hence by virtue of remark 2.7

$$\mathrm{HPL}_n(L) = \mathrm{Tor}_{n-1}^{U(L_{\mathcal{L}ie})}(V \otimes U(L_{\mathcal{L}ie}), K) = \begin{cases} V & \text{if } n = 1, \\ 0 & \text{if not.} \end{cases} \qquad \square$$

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