Sub-Geometries of Lie Sphere Differential Geometry

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In Memory of Professor S. S. Chern

Traditional Geometries

Line-Geometries:

(i) Euclidean Geometry (Chern)

(ii) Affine Geometry (Chern)

(iii) Projective Geometry (Chern)

Sphere-Geometries:

(i) Hyperbolic Geometry (Chern)

(ii) Möbius Geometry

(iii) Laguerre Geometry

(iv) Lie Sphere Geometry (Chern)

Chern and Lie sphere geometry

Chern made important contributions in many traditional geometries, not only the line-geometries and sphere-geometries mentioned above, but also geometries like Integral geometry, web geometry, etc.

Surface theory of Lie sphere geometry and its sub-geometries were well-studied by the geometry group of W. Blaschke, see the book [1].

Around 1985 Chern focused his interesting in Lie sphere differential geometry of Dupin hypersurfaces and published two important papers in this field together with T. Cecil. The works of U. Pinkall ([4]), Chern and Cecil ([2],[3]) aroused a series of papers on the study of Dupin hypersurfaces under the Lie sphere group.

Lie sphere group

By stereographic projection we have $\mathbb{S}^m = \mathbb{R}^m \cup \{\infty\}$.

A oriented sphere (plane) in $U\mathbb{S}^m$ is the (m-1)-submanifold

$$S_r(p) = \{(x, n) \in U\mathbb{R}^m \mid x - p = -rn\};$$

$$P_\lambda(n) = \{(x, n) \in U\mathbb{R}^m \mid x \cdot n = \lambda\};$$

(i) $r > 0, S_r(p)$ is inward oriented;

(ii) $r < 0, S_r(p)$ is outward oriented;

(iii) $r = 0$, $S_r(p) = U_p\mathbb{R}^m$, called point sphere at p.

Definition: A diffeomorphism $\phi : U\mathbb{S}^m \to U\mathbb{S}^m$ preserving spheres is called a Lie sphere transformation. All Lie sphere transformations on $U\mathbb{S}^m$ form the Lie sphere group LSG.

Geometrical interpretation of LSG

Geometrically, LSG is generated by

(i) Möbius transformations on $\mathbb{S}^m$;

(ii) Elliptic flow of spheres on $\mathbb{S}^m$;

(iii) Hyperbolic flow of spheres on $\mathbb{H}^m$;

(iv) Parabolic flow of spheres on $\mathbb{R}^m$;

see: Pinkall [4], Chern and Cecil [2].

Lie sphere group was discovered by Sophus Lie when he studied the standard contact structure on $U\mathbb{S}^m$.

The total space of oriented spheres

Let $\mathbb{R}_2^{m+3}$ be the Lorentzian space with inner product

$$< X, Y >= X_1 Y_1 + \cdots + X_{m+1} Y_{m+1} - X_{m+2} Y_{m+2} - X_{m+3} Y_{m+3}.$$

$$\mathbb{Q}^{m+1} = \{[X] \in \mathbb{R}P^{m+2} \mid < X, X >= 0\}.$$

We define

$$S_r(p) \leftrightarrow [(p, (1 - |p|^2 + r^2)/2, (1 + |p|^2 - r^2)/2, r)] \in \mathbb{Q}^{m+1};$$

$$P_\lambda(n) \leftrightarrow [(n, -\lambda, \lambda, 1)] \in \mathbb{Q}^{m+1}.$$

The total space of oriented spheres in $U\mathbb{S}^m$ is exactly $\mathbb{Q}^{m+1}$.

Linearization of the LSG

A point $(x, n) \in U\mathbb{S}^m$ determines uniquely a pencil of oriented contacted spheres, which is a projective line on

$$\mathbb{Q}^{m+1} = \{[X] \in \mathbb{R}P^{m+2} \mid < X, X >= 0\}.$$

Thus

$$U\mathbb{S}^m = \{\text{total space of projective lines on } \mathbb{Q}^{m+1}\} := \Lambda^{2m-1}.$$

We note that the Lorentzian group $O(m+1, 2)$ in $\mathbb{R}_2^{m+3}$ acts on $\mathbb{Q}^{m+1}$ and Λ^{2m-1}.

Classical Theorem: The LSG on $U\mathbb{S}^m$ is isomorphic to the group $O(m+1, 2)/\{\pm 1\}$ acting on Λ^{2m-1}.

Invariants for hypersurfaces under LSG

Let $x : M^{m-1} \longrightarrow \mathbb{R}^m$ be an oriented hypersurface with the unit normal $n : M^{m-1} \longrightarrow \mathbb{S}^m$. We identify the hypersurafce x with $(x, n) : M \to U\mathbb{R}^m$.

Two hypersurfaces x and $\tilde{x}$ are Lie equivalent if and only if $(\tilde{x}, \tilde{n}) = \phi \circ (x, n)$ for some $\phi \in LSG$.

Basic invariants for hypersurfaces under LSG:

(i) Curvature spheres of hypersurface in $\mathbb{R}^m$;

(ii) Principal tangent vectors of hypersurface;

(iii) Lie curvature $L = \frac{(k_i - k_j)(k_s - k_t)}{(k_i - k_s)(k_j - k_t)}$; (R. Miyaoka 1985)

Dupin hypersurfaces

Definition: If any curvature sphere of a hypersurface is constant along its leaf of principal vectors, we call it Dupin hypersurface.

There were many results concerning Dupin hyapersurfaces in 1985-1990.

Isoparameteric hypersurfaces are special Dupin hypersurfaces.

Theorem (G. Thorbergsson): The number of distinct principal curvatures for any embedded compact Dupin hypersurface can only take the values $\{1, 2, 3, 4, 6\}$.

Question: What is the complete invariant system of hypersurfaces under LSG?

Up to now there are only partial answers!

Möbius group and Laguerre group

Let $O(m+1,2)$ be the Lorentzian group in $\mathbb{R}_2^{m+3}$. It acts on $U\mathbb{S}^m = \Lambda^{2m-1}$ as the Lie sphere group.

There are two important subgroups of $O(m+1,2)$ defined by

$$MG = \{T \in O(m+1,2) \mid Te = e\}, \;\; < e, e >= -1;$$

$$LG == \{T \in O(m+1,2) \mid Te = e\}, \;\; < e, e >= 0.$$

The subgroup MG acts on $U\mathbb{S}^m = \Lambda^{2m-1}$ as Möbius group.

The subgroup LG acts on $U\mathbb{S}^m = \Lambda^{2m-1}$ as Laguerre group.

Möbius differential geometry

Geometrically, Möbius group is the

(i) the transformation group on $\mathbb{S}^m$ preserving (m-1)-spheres;

(ii) the conformal group of the standard metric on $\mathbb{S}^m$;

(iii) the fractional linear group of $\tilde{x} = \frac{xA+a}{xb+c}$, where

$$\begin{pmatrix} A & b \\ a & c \end{pmatrix} \in O^+(m+1,1) \subset O(m+1,2);$$

(iv) the group $O^+(m+1,1)$ acting on the line-space of light-cone in $\mathbb{R}^{m+2}_1$;

(v) the subgroup of LSG preserving point spheres.

The purpose of Möbius differential geometry is to study properties of submanifolds in $\mathbb{S}^m$ which are invariant under the Möbius group.

Laguerre differential geometry

Geometrically, Laguerre group in $U\mathbb{R}^m$ is the

(i) the subgroup of LSG preserving (m-1)-planes in $\mathbb{R}^m$;

(ii) the group generated by the isometries in $\mathbb{R}^m$, parabolic flow and hyperbolic flow in $U\mathbb{R}^m$; (Li-Wang 2006)

(iii) the group of Lorentzian motions in $\mathbb{R}^{m+1}_1$.

Remark to (iii): An oriented sphere $S_r(p)$ in $\mathbb{R}^m$ corresponds to a point $(p,r) \in \mathbb{R}^{m+1}_1$. Laguerre group acting on $\{S_r(p)\}$ is exactly the group of Lorentzian motions in $\mathbb{R}^{m+1}_1$.

The purpose of Laguerre differential geometry is to study properties of hypersurafces in $\mathbb{R}^m$ which are invariant under the Laguerre group.

Möbius invariants of submanifolds in $\mathbb{R}^n$

Let $x : M^m \longrightarrow \mathbb{R}^n$ be a submanifold of co-dimension k. Then we have the following Möbius invariants for x:

(i) the mean curvature sphere $\xi : M^m \longrightarrow G_m^+(\mathbb{R}_1^{n+2})$ (called conformal Gauss map);

(ii) the Möbius metric $g = \rho^2 I$, $\rho^2 = \frac{m}{m-1}\|II - \vec{H}I\|^2$;

(iii) the Möbius second fundamental form $B = \rho^{-1}(II - \vec{H}I)$;

(iv) the normal connection $\nabla^\perp$ on the normal bundle of x.

Theorem (Wang 1997): $\{g, B, \nabla^\perp\}$ is a complete Möbius invariant system for submanifolds M^m ($m \geq 3$) in $\mathbb{R}^n$.

Willmore Conjecture

For surface $x : M^2 \to \mathbb{R}^3$ we have

$$W(x) = Vol_g(x) = \int_M (H^2 - K)\, dM.$$

It is called Willmore functional for surfaces.

Willmore Conjecture: For any torus T^2 in $\mathbb{R}^3$ we have

$$W(T^2) = \int_{T^2} (H^2 - K) dT^2 \geq 2\pi^2;$$

and the equality holds if and only if T^2 is Möbius equivalent to the Clifford torus.

The Willmore conjecture is still open (since 1965).

Willmore surfaces in $\mathbb{R}^3$

Definition: Critical surfaces to the Willmore functional is called Willmore surfaces (or Möbius minimal surfaces).

Theorem (Thomsen 1923, Bryant 1982): Any Willmore surface $x : M \longrightarrow \mathbb{R}^3$ has a dual Willmore surface $\hat{x} : M \longrightarrow \mathbb{R}^3$, they share the same mean curvature sphere ξ as a space-like minimal surface $\xi : M \longrightarrow \mathbb{S}^4_1$.

Theorem(Burstall,Ferus, Leschke, Pedit, Pinkall 2001):A Willmore surface $x : M \longrightarrow \mathbb{R}^4$ induces a left dual Willmore surface $L(x)$ ($L(L(x)) = x$) and a right dual Willmore surface $R(x)$ ($R(R(x)) = x$).

Willmore surfaces of constant curvature

According to Calabi, Bryant,Kenmotsu if the curvature K of a minimal surfaces in $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ is constant, then K can only take a discrete values.

Theorem(Ma, Wang 2006): Let M be a Willmore surface in $\mathbb{R}^3$ with constant Möbius curvature K, then $K = 1$ or $K = 0$ and M is Möbius equivalent to either a minimal surface in $\mathbb{R}^3$ or a clifford torus in $\mathbb{S}^3$.

Theorem(Ma, Wang 2006): Let M be a isotropic Willmore surface in $\mathbb{S}^4$ with constant Möbius curvature K, then $K = 2$ or $K = 1/2$ and M is Möbius equivalent to either a complex curve in $\mathbb{R}^4 = \mathbb{C}^2$ or the Veronese surface in $\mathbb{S}^4$.

Möbius isoparametric hypersurfaces

Definition: A hypersurface $x : M^{m-1} \to \mathbb{S}^m$ is called Möbius isoparametric, if all its Möbius principal curvatures are constant and Möbius form vanishes.

Theorem (Li-Liu-Wang-Zhao 2002): Any Möbius isoparametric hypersurface is Dupin. Thus the number γ of distinct principal curvatures of compact Möbius isoparametric hypersurface can only take the value in $\{2, 3, 4, 6\}$.

Conjecture (Wang): Any compact Moebius isoparameteric hypersurface in $\mathbb{S}^m$ is Moebius equivalent to an (Euclidean) isoparameteric hypersurface in $\mathbb{S}^m$.

Many results are obtained by Z. Hu, H. Li, X. Li, etc.

Theorem (Hu-Zhai 2010): The conjecture is true if $\gamma = 3$.

Hypersurfaces with constant curvatutre

Theorem (Guo-Li-Lin-Ma-Wang 2011): Let $x : M^{m-1} \longrightarrow \mathbb{S}^m$ $(m \geq 5)$ be a hypersurface with constant Möbius curvature K. Then x is Möbius equivalent to either (i) $\Gamma \times \mathbb{R}^{m-2} \subset \mathbb{R}^m$; or (ii) $C(\Gamma) \times \mathbb{R}^{m-3} \subset \mathbb{R}^m$; or (iii) $\Gamma \times \mathbb{S}^{m-2} \subset \mathbb{R}^m$; where Γ is the curve in $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$, whose curvature $k(s)$ satisfies

$$\left(\frac{d}{ds}\left(\frac{1}{k}\right)\right)^2 + \varepsilon\left(\frac{1}{k}\right)^2 = -K, \ \ \varepsilon = 0, \pm 1.$$

Theorem (Li-Ma-Wang 2011): Let $x : M^3 \longrightarrow \mathbb{S}^4$ be a hypersurface with constant Möbius curvature. Then x is either Möbius equivalent to one of the hypersurfaces given above, or to the cone over a standard flat torus in $\mathbb{S}^3$.

Möbius Deformable Hypersurfaces

Definition: A hypersurface $x : M^{m-1} \longrightarrow \mathbb{S}^m$ is called Möbius deformable, if there exists hypersurface $\tilde{x} : M^{m-1} \longrightarrow \mathbb{S}^m$ such that their Möbius metric $g = \tilde{g}$, but x and $\tilde{x}$ are not Möbius equivalent.

Theorem (Li-Ma-Wang 2010): Let $x : M^{m-1} \longrightarrow \mathbb{S}^m$ $(m \geq 5)$ be a Möbius deformable hypersurface. Then x is Möbius equivalent to

(i) a cylinder in $\mathbb{R}^m$ over a Bonnet surface F in $\mathbb{R}^3$;

(ii) a cone-cylinder $C(F) \times \mathbb{R}^{m-4}$ in $\mathbb{R}^m$ for a Bonnet surface in $\mathbb{S}^3$;

(iii) Rotation hypersurface in $\mathbb{R}^m$ generated by a Bonnet surface in $\mathbb{H}^3$.

Laguerre invariants for hypersurfaces in $\mathbb{R}^m$

Let $x : M^{m-1} \to \mathbb{R}^m$ be a hypersurface with non-degenerate third fundamental form. Then we have the following Laguerre invariants:

(i) The center sphere of x;

(ii) $L_{i,jk} = \frac{r_i - r_j}{r_i - r_k}$, where $r_i = 1/k_i$ is the principal radius.

(iii) $g =< dY, dY >= (\sum_i (r_i - r)^2) III = \rho^2 III$, where $r = \frac{1}{m-1} \sum_i r_i$.

Theorem (Li-Wang 2006): Two umbilical free oriented hypersurfaces in ^{n} $(n > 3)$ with non-degenerate third fundamental form are Laguerre equivalent if and only if they have the same Laguerre metric g and Laguerre second fundamental form $B = \rho^{-1}(S^{-1} - r\, id)$.

Laguerre minimal surfaces in $\mathbb{R}^3$

Laguerre minimal surfaces are critical surfaces to

$$L_g(x) = \int_M \frac{H^2 - K}{K} dM.$$

The Euler-Lagrange equation for $L(x)$ **(Weingarten 1888)**:

$$\Delta_{III}(r_1 + r_2) = 0.$$

Special examples: minimal surfaces in a 3-hyperplane in $\mathbb{R}^4_1$.

Classification theorem (Wang 2008): Given two meomrophic functions ϕ, ψ and a holomorphic 1-form $f(z)dz$ on a Riemann surface M we can define a space-like minimal surface

$$\xi = Re \int f(z)(\phi + \psi, -i(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi)dz$$

in $\mathbb{R}^4_1$, and ξ determines two dual Laguerre minimal surfaces $x, \hat{x} : M \to \mathbb{R}^3$.

Lie Sphere Differential Geometry

Let $x : M^{m-1} \longrightarrow \mathbb{R}^m$ be a hypersurface. Then $W_{i,jl} = \frac{k_i - k_j}{k_i - k_l}$ are Moebius invariants, and $L_{i,jl} = \frac{r_i - r_j}{r_i - r_l}$ are Laguerre invariants.

Thus the Lie invariants (Lie curvatures) defined by R. Miyaoka

$$M_{ij,ln} = \frac{k_i - k_l}{k_i - k_n} / \frac{k_j - k_l}{k_j - k_n} = \frac{r_i - r_l}{r_i - r_n} / \frac{r_j - r_l}{r_j - r_n}$$

are both Möbius invariants and Laguerre invariants.

Again the Question: What is a complete invariant system for hypersurfaces in $\mathbb{R}^m$ under the Lie sphere group?

It is a natural question following the works of Pinkall, Cecil and Chern.

References

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