ANTIPODAL ASSOCIATION SCHEMES AND PBIB DESIGNS

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• Introduction

Partially balance incomplete block (PBIB) design is based on association scheme. There are many kinds of association schemes and the structure of association scheme itself is an important object in combinatorics as well. Antipodal distance-regular graph leads a kind of association schemes, which is equivalent to a kind of association schemes, we still call it antipodal association scheme. It has some algebraic properties, the paper will discuss them. Based on the scheme, a lot of PBIB designs with some special properties can be constructed.

- **Association Scheme**

Let X be a finite set. An association scheme with m classes is a pair (X, R) such that
(i) $R = R_0, R_1, \cdots, R_m$ is a partition of $X \times X$;
(ii) $R = \{(x, x) | x \in X\}$;
(iii) $R_i = R_i^T (i.e., (x, y) \in R_i \Rightarrow (y, x) \in R_i)$ for all $i \in \{0, 1, \cdots, m\}$;
(iv) there are numbers p_{jk}^i (the intersection numbers of the scheme) such that for any pair $(x, y) \in R_i$ the number of $z \in X$ with $(z, x) \in R_j$ and $(z, y) \in R_k$ is a constant and it equals p_{jk}^i.

- **Antipodal Graph**

antipodal graph Γ: Γ_d is an equivalence relation. where d is the diameter of the graph.

• **Antipodal Association Schemes** We begin with C_{2m}, a circle with $2m$ vertices on it, the simplest antipodal association scheme. The parameters of the association scheme are:

$$n_1 = n_2 = \cdots = n_{m-1} = 2, n_m = 1.$$

the intersection matrices are defined by:

$$p^i_{jk} = \begin{cases} 1 \quad if \begin{cases} i = j + k \\ i = |j - k| \\ m = i + j + k \end{cases} \\ 2 \quad j + k = i = m \\ 0 \quad otherwise \end{cases}$$

$*$ Example Let $v = 10, (m = 5)$, we have a 5-associate-class association scheme with parameters:

$$v = 10, n_1 = n_2 = n_3 = n_4 = 2, n_5 = 1, \tag{1}$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, P_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, P_5 = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The association scheme table :

	1st asso.	2nd asso.	3rd asso.	4th asso.	5th asso.
1	2 10	3 9	4 8	5 7	6
2	1 3	4 10	5 9	6 8	7
3	2 4	1 5	6 10	7 9	8
4	3 5	2 6	1 7	8 10	9
5	4 6	3 7	2 8	1 9	10
6	5 7	4 8	3 9	2 10	1
7	6 8	5 9	4 10	1 3	2
8	7 9	6 10	1 5	2 4	3
9	8 10	1 7	2 6	3 5	4
10	1 9	2 8	3 7	4 6	5

$*$ Example (Cont.) The association scheme table of C_{10} can be cut by half:

$\longrightarrow$	1st asso.	2nd asso.	3rd asso.	4th asso.	5th asso.
1	2 10	3 9	4 8	5 7	6
2	1 3	4 10	5 9	6 8	7
3	2 4	1 5	6 10	7 9	8
4	3 5	2 6	1 7	8 10	9
5	4 6	3 7	2 8	1 9	10
5th asso.	4th asso.	3rd asso.	2nd asso.	1st asso.	$\longleftarrow$

Theorem Let there be an m-class antipodal association scheme, if the symbols x and y are mth associates, then for each i ($i = 1, 2, \cdots, m$),

$$p^{m}_{i(m-i)} = n_i = n_{m-i}$$

Algebraic Properties of Antipodal Association Schemes

- About the intersection matrices

Theorem Let there be an m-class antipodal association scheme, $P_1, P_2, \cdots, P_m$ are intersection matrices, then

$$p^i_{jk} = p^{m-i}_{(m-j)k},\ p^i_{jk} = p^{m-i}_{j(m-k)}.$$

$(j = 1, 2, \cdots, m-1)$.

Corollary Let there be an m-class antipodal association scheme with intersection matrices $P_1, P_2, \cdots, P_m$, then for $i = 1, 2, \cdots m-1$,

(i) If $m = 2t+1$,

$$P_{m-i} = E_{1,(m-1)}E_{2,(m-2)}\cdots E_{t,t+1}P_iF'_{1,(m-1)}F'_{2,(m-2)}\cdots F'_{t,t+1},$$

(ii) if $m = 2t$,

$$P_{m-i} = E_{1,(m-1)}E_{2,(m-2)}\cdots E_{t,t}P_iF'_{1,(m-1)}F'_{2,(m-2)}\cdots F'_{t,t},$$

P_t is unchanged.

• Adjacency Matrices of Association Schemes

$*$ For an m-class association scheme on v vertices, the relations R_i are described by $v \times v$ $(0,1)$-adjacency matrices $A_i (i = 1, 2, \cdots, m)$.

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

$*$ The properties of A_i:

$$(i) \sum_{i=0}^{m} A_i = J,\ (ii) A_0 = I,\ (iii) A_i = A_i^T,\ (iv) A_i A_j = \sum_{k=0}^{m} p_{ij}^k A_k.$$

$*$ A_i's are linearly independent and they form a base of a vector space. (Bailey, 2004; Godsil, 1993; Bannai and Ito, 1984)

• For antipodal association scheme, A_i have more properties:

$$(A_{m-i})_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_{m-i}, \\ 0, & \text{otherwise.} \end{cases}$$

$*$ $(x,y) \in R_{m-i}$ is equivalent to

$$(x, \sigma(y)) \in R_i,$$

so the rows in A_{m-i} are transformed from the rows in A_i.

$*$ The map σ: $x \mapsto m$-th asso. of x

$*$ The mapping defines a permutation:

$$\sigma = (x_1, \sigma(x_1))(x_2, \sigma(x_2)) \cdots (x_{v/2}, \sigma(x_{v/2}))$$

Theorem Let there be an m-class antipodal association scheme on a finite set $\{1, 2, \cdots, v\}$, $A_1, A_2, \cdots, A_m$ are m $v \times v$ adjacency matrices, then

$$A_{m-i} = PA_i, \ (i = 1, 2, \cdots, m),$$

where P is a primary transformation matrix formed by the permutation:

$$(x_1, \sigma(x_1))(x_2, \sigma(x_2)) \cdots (x_{v/2}, \sigma(x_{v/2})).$$

In which $\sigma(x_j)$ is just the mth associate of $x_j, (j = 1, 2, \cdots, v)$.

$*$ The primary transformation matrix P is defined by σ, it is just A_m, so:

Theorem Let there be an m-class antipodal association scheme, the adjacency matrices are $A_1, A_2, \cdots, A_m$, then for each i, $0 \leq i \leq m$,

$$A_{m-i} = A_m A_i$$

Corollary For an m-class antipodal association scheme with adjacency matrices $A_1, A_2, \cdots, A_m$,
(i) if $m = 2t + 1$, $J = (I + A_m)(A_0 + A_1 + \cdots + A_t)$.
(ii) if $m = 2t$, $J = (I + A_m)(A_0 + A_1 + \cdots + A_{t-1}) + A_t$.

Theorem Let there be an m-class association scheme with adjacency matrices $A_0, A_1, A_2, \cdots, A_m$, the association scheme is antipodal if and only if $A_m A_i = A_{m-i}$.

Corollary If an m-class association scheme is antipodal with adjacency matrices $A_1, A_2, \cdots, A_m$, then A_i and A_{m-i} have the same eigenvalues.

This theorem generalized the method to check whether an association scheme derived is antipodal by its adjacency matrices even though the graph is not embeddable onto sphere.

Example: The Association Scheme Derived from Torus with 16 points. the parameters of the association scheme.

$$v = 16, n_1 = 4, n_2 = 6, n_3 = 4, n_4 = 1. \tag{2}$$

$$P_1 = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, P_4 = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 6 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The adjacency matrices are too large, we ignore them, but they satisfy:

$$A_3 = A_4A_1,$$
$$A_2 = A_4A_2.$$

By the theorem we have , we can see that this association scheme is antipodal.

By the way, it can be shown,

$$J = A_0 + A_1 + A_2 + A_3 + A_4 = (I + A_4)(I + A_1) + A_2.$$

The eigenvalues of A_4 are:

$$(-1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1)',$$

the eigenvalues of A_1 are:

$$(-4, -2, -2, -2, -2, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 4)',$$

by the Corollary **??** the eigenvalues of A_3 are also

$$(-4, -2, -2, -2, -2, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 4)'.$$

The eigenvalues of A_2 are:

$$(-2, -2, -2, -2, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0, 6, 6)'.$$

Because it is antipodal, we need only to give half of the association scheme table, it is:

The association scheme derived from torus

$\rightarrow$	1st asso.	2nd asso.	3rd asso.	4th asso.
1	2 4 5 9	3 6 8 10 12 13	7 11 14 16	15
2	1 3 6 10	4 5 7 9 11 14	8 12 13 15	16
3	2 4 7 11	1 6 8 10 12 15	5 9 14 16	13
4	1 3 8 12	2 5 7 9 11 16	6 10 13 15	14
5	1 6 8 13	2 4 7 9 14 16	3 10 12 15	11
6	2 5 7 14	1 3 8 10 13 15	4 9 11 16	12
7	3 6 8 15	2 4 5 11 14 16	1 10 12 13	9
8	4 5 7 16	1 3 6 12 13 15	2 9 11 14	10
4th asso.	3rd asso.	2nd asso.	1st asso.	$\leftarrow$

Besides the property of P-polynomial, the antipodal association schemes have the property that among $A_0, \cdots, A_m$, the last matrix A_m is a permutation matrix such that half of A_i's can be expressed by the other half under the given permutation A_m.

By the special properties of antipodal association schemes, we can construct many PBIB designs.

For the scheme from torus, we have some PBIB designs:

$$v = b = 16,\ r = k = 5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 0,\ \lambda_4 = 0.$$

The plan is:

1 2 4 5 9	5 1 6 8 13	9 1 10 12 13	15 7 11 14 16
2 1 3 6 10	6 2 5 7 14	10 2 9 11 14	16 8 12 13 15
3 2 4 7 11	7 3 6 8 15	11 3 10 12 15	13 5 9 14 16
4 1 3 8 12	8 4 5 7 16	12 4 9 11 16	14 6 10 13 15

Another: $v = 16,\ b = 8,\ r = 4,\ k = 8,\ \lambda_1 = 0,\ \lambda_2 = 4,\ \lambda_3 = 0,\ \lambda_4 = 4.$

The plan is:

1 3 6 8 10 12 13 15	5 2 4 7 9 14 16 11
2 4 5 7 9 11 14 16	6 1 3 8 10 13 15 12
3 1 6 8 10 12 13 15	7 2 4 5 11 14 16 9
4 2 5 7 9 11 14 16	8 1 3 6 12 13 15 10

and so on.

THANK YOU