Statistical Inference for the Difference Between the Best Treatment Mean and a Control Mean

Jianan Peng
Department of Mathematics and Statistics
Acadia University
Wolfville, NS
Canada B4P 2R6
Email: jianan.peng@acadiau.ca

Charles Chu-In Lee
Memorial University of Newfoundland

Lin Liu
University of California at San Diego

OUTLINE

1. INTRODUCTION
2. TREATMENTS AT LEAST AS GOOD AS A CONTROL
3. NO PRIOR KNOWLEDGE OF TREATMENT MEANS
4. SUBSET OF THE BEST TREATMENT
5. A NUMERICAL EXAMPLE
6. MEASURES OF PERFORMANCE
7. DISCUSSION

1. INTRODUCTION

The experiment considered is a one-way analysis with k treatments and a control.

$$Y_{ij} = \mu_i + \epsilon_{ij}, i = 0, 1, \ldots, k, j = 1, \ldots, n_i$$

where $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$,

- μ_0: the control mean

- $\mu_1, \ldots, \mu_k$: the treatment means.

- $\mu_{best} = \max_{1 \le i \le k} \mu_i$: the best treatment mean.

- $S^2 = \sum_{i=0}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)^2/\nu$ to estimate σ^2, $\nu S^2/\sigma^2 \sim \chi^2_\nu$ and $\nu = \sum_{i=0}^{k} n_i - k - 1 > 0$.

How to construct lower confidence bounds for $\mu_{best} - \mu_0$?

- with prior knowledge that treatments are at least as good as the control. $\Omega = \{\boldsymbol{\mu} \in R^{k+1} : \mu_0 \leq \mu_i,\ i = 1, \ldots, k\}$, which is called simple tree ordering. $H_0 : \mu_0 = \mu_1 = \cdots = \mu_k$ vs $H_1 = \Omega - H_0$.

- without any prior knowledge of the effectiveness of treatments. $H_0' : \mu_i - \mu_0 \leq 0$ $(i = 1, \ldots, k)$ vs H_1' : at least one $\mu_i - \mu_0 > 0$.

Interval estimation provides a visual perspective unmatched by a point estimate or a test statistic.

Literature Survey

- Testing H_0 vs H_1
  - Bartholomew (1959) Biometrika
  - Mukerjee, Robertson & Wright (1987) JASA
  - Tang and Lin (1997) JASA

- Testing H_0 vs H_1 with Interval Estimation

  Marcus (1978) developed one-sided simultaneous lower confidence bounds (SLCB) for $\sum_{i=0}^{k} n_i c_i \mu_i$, where $\sum_{i=0}^{k} n_i c_i = 0$ and $c_0 \leq c_i$ $(i = 1, \ldots, k)$ in the case of known variance.

  - Dunnett (1955) JASA

  - Marcus (1978), Communication Stats (A)

  - Marcus and Talpetz (1992) Communication Stats (A)

  - Berk & Marcus (1996) JASA

- Testing H_0' vs H_1'

  - Dunnett (1955) JASA

For treatments versus control problems, the size of the lower bound of $\mu_{best} - \mu_0$ may be of importance.

The Idea

- When the null hypothesis is rejected in favour of the alternative hypothesis, there exists at least one treatment better than the control. Since $\mu_{best} - \mu_0$ is the largest difference between any treatment mean and the control mean, the lower confidence bound for $\mu_{best} - \mu_0$ is bounded below by those for $\mu_i - \mu_0$ $(i = 1, \ldots, k)$ or their non-negative linear combinations.

- If this optimized lower confidence bound for $\mu_{best} - \mu_0$ is positive, then μ_{best} is significantly larger than μ_0.

- Suitable test statistics are chosen so that the rejection of the null hypothesis is equivalent to the positiveness of the lower confidence bound for $\mu_{best} - \mu_0$.

2. TREATMENTS AT LEAST AS GOOD AS A CONTROL

2.1 The Likelihood Ratio Test

H_0: $\mu_0 = \mu_1 = \cdots = \mu_k$ vs $H_1 = \Omega - H_0$.

where $\Omega = \{\boldsymbol{\mu} \in R^{k+1} : \mu_0 \leq \mu_i, i = 1, \ldots, k\}$

- MLE under H_0: $\hat{\mu} = \bar{\bar{Y}} = \sum_{i=0}^{k} n_i \bar{Y}_i / \sum_{i=0}^{k} n_i$.

- MLE under Ω: $\boldsymbol{\mu}^* = (\mu_0^*, \mu_1^*, \ldots, \mu_k^*)$.

  Relabel the treatments so that $\bar{Y}_1 \leq \bar{Y}_2 \leq \ldots \leq \bar{Y}_k$. Let l be the smallest non-negative integer such that $A_l = \sum_{i=0}^{l} n_i \bar{Y}_i / \sum_{i=0}^{l} n_i < \bar{Y}_{l+1}$, then $\mu_0^* = A_l$, and $\mu_i^* = \max(A_l, \bar{Y}_i), i = 1, \ldots, k$. If $\bar{Y}_0 \geq A_{k-1} \geq \bar{Y}_k$, then $\mu_i^* = \hat{\mu}$ for $i = 0, 1, \ldots, k$.

- The LRT:

$$S_{01} = \sum_{i=0}^{k} n_i (\mu_i^* - \hat{\mu})^2 / (\sum_{i=0}^{k} n_i (\bar{Y}_i - \mu_i^*)^2 / \nu + S^2).$$

2.2 The Multiple Contrast Test Statistic T

- $$T = \max_{\mathbf{c}\in\mathbf{C}} \sum_{i=0}^{k} n_i c_i \bar{Y}_i / S(\sum_{i=0}^{k} n_i c_i^2)^{1/2}, \qquad (1)$$

$$\mathbf{C} = \left\{\mathbf{c} = (c_0, c_1, \ldots, c_k) : \sum_{i=0}^{k} n_i c_i = 0,\ c_0 \le c_i,\ i = 1, \ldots, k\right\}$$

- $$T^2 = \sum_{i=0}^{k} n_i(\mu_i^* - \widehat{\mu})^2 / S^2. \qquad (2)$$

LRT S_{01} and The Multiple Contrast Statistic T

- The LRT S_{01} can't be used to construct CI but statistic T can.

$$P_{\boldsymbol{\mu}}\Big\{\sum_{i=0}^{k} n_i c_i \mu_i \geq \sum_{i=0}^{k} n_i c_i \bar{Y}_i - t_{k,\nu,\alpha} S (\sum_{i=0}^{k} n_i c_i^2)^{1/2}, \forall \mathbf{c} \in \mathbf{C}\Big\} = 1 - \alpha. \quad (3)$$

- The statistic T^2 is asymptotically equivalent to S_{01}.

- The null distribution of T under H_0 is given by

$$P[T \geq t] = \sum_{j=2}^{k+1} P(j, k+1; \mathbf{w}) P[F_{j-1,\nu} \geq \frac{t^2}{j-1}] \quad (4)$$

2.3 The Lower Confidence Bound for $\mu_{best} - \mu_0$

- $\mathcal{K} = \{\mathbf{c} : \mathbf{c} \in \mathbf{C},\ \Sigma_{i=0}^{k} n_i c_i \mu_i \leq \mu_{best} - \mu_0, \forall\ \boldsymbol{\mu} \in \Omega\}$.

- The lower confidence bound for $\mu_{best} - \mu_0$ is given by

$$L(\mu_{best} - \mu_0) = \max_{\mathbf{c} \in \mathcal{K}} \sum_{i=0}^{k} n_i c_i \bar{Y}_i - t_{k,\nu,\alpha} S (\sum_{i=0}^{k} n_i c_i^2)^{1/2}. \tag{5}$$

Optimal Solution

- **Theorem 2.1** When $\boldsymbol{\mu} \in \Omega$, we have that $T > t_{k,\nu,\alpha}$ if and only if $L(\mu_{best} - \mu_0) > 0$.

- **Theorem 2.2** Suppose that $T > t_{k,\nu,\alpha}$ and $\bar{Y}_1 \leq \bar{Y}_2 \leq \ldots \leq \bar{Y}_k$. The vector $\tilde{\mathbf{c}} \in \mathcal{K}$ is an optimal solution to (5) if and only if there exist non-negative integers p and q, $l \leq p < q \leq k$, such that $\bar{Y}_p < \widehat{\mu} < \bar{Y}_q$,

$$\begin{aligned} \tilde{c}_i &= -N_{0p}^{-1} + b^{-1}(\mu_i^* - \bar{Y}_{0p}),\ i = 0, \ldots, p, \\ \tilde{c}_i &= 0,\ p+1 \leq i \leq q-1, \\ \tilde{c}_i &= N_{qk}^{-1} + b^{-1}(\bar{Y}_i - \bar{Y}_{qk}),\ i = q, \ldots, k, \end{aligned} \qquad (6)$$

and

$$\max\left\{N_{0p}(\mu_p^* - \bar{Y}_{0p}), N_{qk}(\bar{Y}_{qk} - \bar{Y}_q)\right\} < b \leq \min\left\{N_{0(p+1)}(\mu_{p+1}^* - \bar{Y}_{0(p+1)}), N_{(q-1)k}(\bar{Y}_{(q-1)k} - \bar{Y}_{q-1})\right\}, \qquad (7)$$

where

$$b^2 = (t_{k,\nu,\alpha}^2 S^2 - S_{0p}^2 - S_{qk}^2)/(N_{0p}^{-1} + N_{qk}^{-1}), \qquad (8)$$

and $N_{ab} = \sum_{i=a}^{b} n_i$, $\bar{Y}_{ab} = \sum_{i=a}^{b} n_i \bar{Y}_i / N_{ab}$, $S_{ab}^2 = \sum_{i=a}^{b} n_i(\mu_i^* - \bar{Y}_{ab})^2$. When $q = p+1$, the upper bound for b in (7) is replaced by $(\bar{Y}_{qk} - \bar{Y}_{0p})/(N_{0p}^{-1} + N_{qk}^{-1})$.

2.4 Iterative Algorithm I

(0) Set $i = 0$, $p_0 = \max\{0 \le j < k : \mu_j^* < \widehat{\mu}\}$ and $q_0 = \min\{1 \le j \le k : \mu_j^* > \widehat{\mu}\}$.

(i) Let $\beta_{i+1} = \max\{N_{0p_i}(\mu_{p_i}^* - \bar{Y}_{0p_i}), N_{q_ik}(\bar{Y}_{q_ik} - \mu_{q_i}^*)\}, t_{k,\nu,\alpha_{i+1}} = [S_{0p_i}^2 + S_{q_ik}^2 + (N_{0p_i}^{-1} + N_{q_ik}^{-1})\beta_{i+1}^2]^{1/2}/S$. If $t_{k,\nu,\alpha_{i+1}} < t_{k,\nu,\alpha}$, the optimal solution is $\tilde{\mathbf{c}}$ with $p = p_i$ and $q = q_i$. Otherwise, go to (ii).

(ii) If $N_{0p_i}(\mu_{p_i}^* - \bar{Y}_{0p_i}) > N_{q_ik}(\bar{Y}_{q_ik} - \mu_{q_i}^*)$, then set $p_{i+1} = \max\{j : 0 \le j < p_i, \mu_j^* < \mu_{p_i}^*\}$ and $q_{i+1} = q_i$. If $N_{q_ik}(\bar{Y}_{q_ik} - \mu_{q_i}^*) > N_{0p_i}(\mu_{p_i}^* - \bar{Y}_{0p_i})$, then set $p_{i+1} = p_i$ and $q_{i+1} = \min\{j : q_i < j \le k, \mu_j^* > \mu_{q_i}^*\}$. Otherwise, set $p_{i+1} = \max\{j : 0 \le j < p_i, \mu_j^* < \mu_{p_i}^*\}$ and $q_{i+1} = \min\{j : q_i < j \le k, \mu_j^* > \mu_{q_i}^*\}$. Set $i = i + 1$, go to step (i).

3. NO PRIOR KNOWLEDGE OF TREATMENTS

- No prior knowledge of the effectiveness of treatments. Some treatment means may be larger than the control mean while other treatment means may be smaller than the control mean.

- $H_0' : \mu_i - \mu_0 \leq 0 \ (i = 1, \ldots, k)$. vs H_1' : at least one $\mu_i - \mu_0 > 0$

3.1 The Union-Intersection Method

- $H_0' : \mu_i - \mu_0 \leq 0 \ (i = 1, \ldots, k). H_0' = \cap_{\mathbf{c} \in \mathbf{C}^o} H_{0\mathbf{c}}'$, where $H_{0\mathbf{c}}' : \sum_{i=1}^k n_i c_i(\mu_i - \mu_0) \leq 0$ with $\mathbf{c} \in \mathbf{C}^o$ and

  $\mathbf{C}^o = \{\mathbf{c} = (c_0, c_1, \ldots, c_k) : \sum_{i=0}^k n_i c_i = 0, \quad c_i \geq 0, i = 1, \ldots, k\}$.

- $H_1' = \cup^o_{\mathbf{c} \in \mathbf{C}} H_{1\mathbf{c}}'$, where $H_{1\mathbf{c}}' : \sum_{i=1}^k n_i c_i(\mu_i - \mu_0) > 0$.

- The union-intersection method: if any one of $H_{0\mathbf{c}}'$ is rejected, then H_0', which is true only if $H_{0\mathbf{c}}'$ is true for every $\mathbf{c} \in \mathbf{C}^o$, must also be rejected. Only if each of the hypotheses $H_{0\mathbf{c}}'$ is accepted as true will the intersection of H_0' be accepted as true.

The Multiple Contrast Test Statistic T^o

- $$T^o = \max_{\mathbf{c} \in \mathbf{C^o}} \sum_{i=0}^{k} n_i c_i \bar{Y}_i / S(\sum_{i=0}^{k} n_i c_i^2)^{1/2}. \quad (9)$$

- $T^{o2} = \sum_{i=0}^{k} n_i(\mu_i^o - \hat{\mu})^2 / S^2$.

- How to compute μ_i^o

  Let r be the largest positive integer such that $B_r = (n_0\bar{Y}_0 + \sum_{j=r}^{k} n_j\bar{Y}_j)/(n_0 + N_{rk}) > \bar{Y}_{r-1}$, then $\mu_i^o = \hat{\mu} = \bar{\bar{Y}}$ for $i = 1, \ldots, r-1$, $\mu_i^o = \bar{Y}_i - B_r + \hat{\mu}$ for $i = 0, r, r+1, \ldots, k$. If $\bar{Y}_0 \geq \bar{Y}_k$, then $\mu_i^o = \hat{\mu}$ for $i = 0, 1, \ldots, k$ and $T^o = 0$.

The null hypothesis distribution of T^o under the least favorable configuration $\mu_0 = \mu_1 = \ldots = \mu_k$ is given by

$$\sup_{\boldsymbol{\mu} \in H_0'} P[T^o \geq t] = \sum_{j=1}^{k} P(j, k+1; \mathbf{w}) P[F_{k+1-j,\nu} \geq \frac{t^2}{k+1-j}]$$

for any $t > 0$. The statistic T^{o2} has the same distribution as statistic S_{12} in Robertson et al. (1988).

3.3 The Lower Confidence Bound for $\mu_{best} - \mu_0$

- $\mathcal{K}^o = \{\mathbf{c} : \mathbf{c} \in \mathbf{C}^o, \ \sum_{i=0}^{k} n_i c_i \mu_i \leq \mu_{best} - \mu_0\}$.

- $$L^o(\mu_{best} - \mu_0) = \max_{\mathbf{c} \in \mathcal{K}^o} \sum_{i=0}^{k} n_i c_i \bar{Y}_i - t^o_{k,\nu,\alpha} S (\sum_{i=0}^{k} n_i c_i^2)^{1/2}. \qquad (10)$$

Optimal Solution

- **Theorem 3.1**

  $T^o > t^o_{k,\nu,\alpha}$ if and only if $L^o(\mu_{best} - \mu_0) > 0$.

- **Theorem 3.2**

  Suppose $T^o > t^o_{k,\nu,\alpha}$ and $\bar{Y}_1 \leq \bar{Y}_2 \leq \ldots \leq \bar{Y}_k$. Then $\tilde{\mathbf{c}}^o \in \mathcal{K}^o$ is an optimal solution to (10) if and only if there exists a positive integer q, $r \leq q \leq k$ such that

$$\begin{aligned} \tilde{c}^o_0 &= -\frac{1}{n_0}, \\ \tilde{c}^o_i &= 0, 1 \leq i \leq q-1 \\ \tilde{c}^o_i &= N^{-1}_{qk} + b^{o-1}(\bar{Y}_i - \bar{Y}_{qk}),\ i = q, \ldots, k, \end{aligned} \tag{11}$$

  with

$$N_{qk}(\bar{Y}_{qk} - \bar{Y}_q) < b^o \leq N_{(q-1)k}(\bar{Y}_{(q-1)k} - \bar{Y}_{q-1}), \tag{12}$$

  where $b^{o2} = (t^{o2}_{k,\nu,\alpha}S^2 - S^2_{qk})/({n_0}^{-1} + N^{-1}_{qk})$. When $q = r$, the upper bound for b^o in (12) is replaced by $(\bar{Y}_{qk} - \bar{Y}_0)/(n^{-1}_0 + N^{-1}_{qk})$.

3.4 Iterative Algorithm II

(0) Set $i = 0$ and $q_0 = r$.

(i) Let $\beta_{i+1} = N_{q_i k}(\bar{Y}_{q_i k} - \bar{Y}_{q_i})$, $t^o_{k,\nu,\alpha_{i+1}} = [S^2_{q_i k} + \beta^2_{i+1}(n_0^{-1} + N^{-1}_{q_i k})]^{1/2}/S$. If $t^o_{k,\nu,\alpha_{i+1}} \leq t^o_{k,\nu,\alpha}$, the optimal solution is $\tilde{\mathbf{c}}^o$ with $q = q_i$. Otherwise, go to (ii).

(ii) Set $q_{i+1} = \min\{j : q_i < j \leq k, \bar{Y}_j > \bar{Y}_{q_i}\}$ and set $i = i + 1$. Go to (i).

4. SUBSET OF THE BEST TREATMENT

Let $(1), (2), \ldots, (k)$ denote the (unknown) indices for which $\mu_{(1)} \leq \mu_{(2)} \leq \cdots \leq \mu_{(k)} = \mu_{best}$. Then treatment $\pi_{(k)}$ is the best treatment associated with the largest mean $\mu_{(k)} = \mu_{best}$.

Gupta and Huang (1976) proposed the following procedure

$$G = \{\pi_i : \min_{j \neq i} (\bar{Y}_i - \bar{Y}_j + d' S(n_i^{-1} + n_j^{-1})^{1/2}) \geq 0\}$$

where $d' = \max_{1 \leq i \leq k} d^i_{k-1,\nu,\alpha}$ and $d^i_{k-1,\nu,\alpha}$ is the one-sided Dunnett critical value regarding treatment i as the control. When sample sizes are the same, $d^i_{k-1,\nu,\alpha}$ does not depend on i.

We have the following selection procedures:

- $G_C = \{\pi_i : \min_{j \neq i} (c_i^* - c_j^* + d_{k-1,\nu,\alpha}^i b^{-1} S (n_i^{-1} + n_j^{-1})^{1/2}) \geq 0\}$.

  where $c_i^* = N_{qk}^{-1} + b^{-1}(\bar{Y}_i - \bar{Y}_{qk}), i = 1, \ldots, k$.

- $G_C^o = \{\pi_i : \min_{j \neq i} (c_i^{*o} - c_j^{*o} + d_{k-1,\nu,\alpha}^i b^{o-1} S (n_i^{-1} + n_j^{-1})^{1/2}) \geq 0\}$, where $c_i^{*o} = N_{qk}^{-1} + b^{o-1}(\bar{Y}_i - \bar{Y}_{qk}), i = 1, \ldots, k$.

- $P_{\boldsymbol{\mu}}(\pi_{(k)} \in G_C)$ and $P_{\boldsymbol{\mu}}(\pi_{(k)} \in G_C^o)$ are at least $1 - \alpha$.

- The two subset selection procedures G_C and G_C^o are at least as good as Gupta and Huang's (1976) procedure.

5. A NUMERICAL EXAMPLE

Consider the data taken from Arman, Toygar, and Abuhijleh (2004). In this experiment, skeletal measurement increases in total anterior face height (N-Me) of chincup (CC), chincup plus bite plate (CC+P), and reverse headgear (RHg) therapies, labelled treatments 1, 2 and 3, respectively, with an untreated control group (C) were compared.

- $\bar{Y}_0 = 2.8, \bar{Y}_1 = 2.9, \bar{Y}_2 = 4.6, \bar{Y}_3 = 4.7$, $n_0 = 20, n_1 = 31, n_2 = 14$ and $n_3 = 14$. $S = 2.43$.

- In this study, N-Me increases in treated groups but orthodontists do not want a treatment to result in a very long face, as the final goal of any orthodontic treatment should be not only to obtain good function but also to improve facial attractiveness. Here the "best" treatment would be the treatment that results in the largest N-Me increase. It is of interest to know the difference between the "best" treatment mean and the control mean. If it is too large, a further study of these treatments is required.

5.1 Treatments at Least as Good as the Control

$H_0 : \mu_0 = \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_0 \leq \mu_i$ $(i = 1, 2, 3)$ with at least one strict inequality has a value of $T = 3.129$ and a p-value of 0.012 with $t_{3,75,.05} = 2.529$. One concludes that μ_{best} is significantly larger than μ_0. The $100(1-\alpha)\%$ lower confidence bound $L(\mu_{best} - \mu_0)$ can be computed as follows.

Step 0: $\hat{\mu} = 3.49, p_0 = l = 1, q_0 = 2, \beta_1 = 2.0, t_{3,75,\alpha_1} = 0.264 < t_{3,75,.05}$; stop.

The 95% lower confidence bound for $\mu_{best} - \mu_0$ is $L(\mu_{best} - \mu_0) = 0.35$, where $b = 26.062$ and the optimal coefficient in (6) is

$$\tilde{\mathbf{c}} = (-0.0219, -0.0181, 0.0338, 0.0376).$$

5.2 No Prior Knowledge

The test statistic T^o for testing $H_0' : \mu_i - \mu_0 \leq 0$ $(i = 1, 2, 3)$ versus $H_1' : \mu_i - \mu_0 > 0$ for some i has a value of $T^o = 2.603$ and a p-value of 0.019 with $t^o_{3,75,.05} = 2.178$, $r = 2$ and

$\boldsymbol{\mu^o} = (2.41, 3.49, 4.21, 4.31)$. One concludes that μ_{best} is significantly larger than μ_0. The $100(1-\alpha)\%$ lower confidence bound $L^o(\mu_{best} - \mu_0)$ can be computed as follows.

Step 0: $q_0 = r = 2, \beta_1 = 1.4, t^o_{3,75,\alpha_1} = 0.20 < t^o_{3,75,.05}$, stop.

The 95% lower confidence bound for $\mu_{best} - \mu_0$ is $L^o(\mu_{best} - \mu_0) = 0.30$, where $b^o = 18.055$ and the optimal coefficient in (11) is

$$\tilde{\mathbf{c}}^o = (-0.0500, 0.0000, 0.0329, 0.0385).$$

- Lower confidence bound by Dunnett's procedure $L^d(\mu_{best} - \mu_0) = 0.12$.

- $L(\mu_{best} - \mu_0) = 0.35$

- $L^o(\mu_{best} - \mu_0) = 0.30$

- The statistic T provides the sharpest lower confidence bound for $\mu_{best} - \mu_0$ with T^o a close second for this example.

- $G_C = G_C^o = G = \{2, 3\}$ at the 95% confidence level. For this data set, all three subset selection procedures indicate that either treatment 2 or treatment 3 is the "best" treatment.

- The new procedures have the advantage: the lower confidence bounds $L(\mu_{best} - \mu_0)$, $L^o(\mu_{best} - \mu_0)$, and $L^d(\mu_{best} - \mu_0)$ apply to the best treatment.

6. MEASURES OF PERFORMANCE

- Power Comparisons

  The behavior of the power functions of S_{01}, T, T^o, and D are studied. The Monte Carlo method is used with 10,000 iterations. For simplicity, we consider equal sample size case with $\nu = 20$.

- Expected Lower Confidence Bounds

  Sizes of expected lower confidence bounds at 95% level were computed by generating 10,000 normal sample of equal sample size one and $\sigma^2 = 1$ with $k = 2, 4, 6$ and $\mu_k - \mu_0 = 4.0$.

- Treatments at Least as Good as the Control $\Delta^2 = n\sum_{i=0}^{k}(\mu_i - \bar{\mu})^2/\sigma^2$ is the noncentrality parameter and $\bar{\mu} = \sum_{i=0}^{k}\mu_i/(k+1)$.

  - Case 1, the boundary direction $(-k/2, \ldots, -k/2, k/2+1, \ldots, k/2+1)$ with $k/2+1$ terms of $-k/2$ and $k/2$ terms of $k/2+1$, when one half of the treatments are effective while the remaining treatments are ineffective.

  - Case 2, the center direction $(-k, 1, \ldots, 1)$, when all treatments are effective and their effects are approximately the same.

  - Case 3, pairwise direction $(-1, 0, \ldots, 0, 1)$, when all treatments are effective but one treatment is more effective than the other treatments.

Direction	k	Δ	Test			
			S_{01}	T	T^o	D
Case 1	Boundary					
	2	1	16.39	16.23	15.10	15.33
		2	44.95	44.96	38.85	39.75
		3	78.42	78.77	69.85	70.97
		4	95.73	95.78	90.92	91.52
	4	1	12.78	12.57	12.05	12.32
		2	33.09	32.41	29.01	29.18
		3	64.19	64.04	52.61	52.59
		4	88.49	88.47	77.00	76.40
	6	1	10.93	10.71	11.23	10.92
		2	26.24	25.91	22.10	22.04
		3	54.26	53.65	42.13	41.05
		4	82.28	81.93	66.26	64.33
Case 2	Center					
	2	1	19.75	19.23	22.96	22.28
		2	50.83	49.96	56.62	55.14
		3	80.74	80.07	85.71	83.90
		4	96.53	96.35	98.03	97.31
	4	1	16.26	15.00	21.58	21.14
		2	41.74	39.57	55.58	53.07
		3	71.13	68.60	83.59	81.09
		4	91.53	90.36	96.89	95.64
	6	1	14.69	13.08	21.34	19.96
		2	35.14	31.70	52.73	49.79
		3	63.28	59.54	82.96	79.25
		4	86.29	83.69	96.48	95.01
Case 3	Pairwise					
	2	1	18.74	18.28	20.51	20.04
		2	50.25	49.42	53.41	53.24
		3	81.08	80.52	83.58	83.46
		4	96.29	96.15	97.11	97.07
	4	1	14.97	14.32	18.11	17.76
		2	40.80	38.82	48.31	47.62
		3	70.86	68.64	78.58	78.23
		4	91.38	90.37	95.49	95.41
	6	1	13.19	11.98	17.15	16.86
		2	33.13	30.17	43.76	42.29
		3	62.27	58.64	75.06	74.66
		4	85.86	83.56	93.92	94.02

- The LRT S_{01} has the highest power along the boundary direction among these four test statistics. Since statistic T has competitive power performance and it can provide lower confidence bound, statistic T is recommended for Case 1.

- The test statistic T^o is shown to be the most powerful one-sided test along the center direction when all treatments are better than the control. Therefore, we recommend T^o for Case 2.

- Statistic T^o or D is for Case 3, pending depending on whether there are at least two good treatments.

- No Prior Ordering About Treatments and the Control. Simulation study is also conducted for Cases 1, 2, 3 with one extra ineffective treatment, where the noncentrality parameter $\Delta^2 = n\sum_{\mu_i \geq \mu_0}(\mu_i - \bar{\mu})^2/\sigma^2$ and $\bar{\mu} = \sum_{\mu_i \geq \mu_0}\mu_i/(k+1)$. It applies to T^o and D only. The results are similar to those in Table 1. However, powers are somewhat lower than those in Table 1. This is due to larger critical values $t^o_{k+1,\nu,.05}$ and $d_{k+1,\nu,.05}$. The statistic T^o is more powerful than D for Case 2. However, the statistic D is slightly more powerful than T^o in Case 1 at $k = 2$ when there is only one effective treatment.

Table 2. Simulated Expected Lower Bounds for Three Procedures

		CI		
Direction	k	T	T^o	D
Case 1	Boundary			
	2	1.577	1.385	1.427
	4	1.850	1.563	1.573
	6	2.011	1.654	1.649
Case 2	Center			
	2	1.707	1.919	1.883
	4	1.490	2.046	1.983
	6	1.265	2.092	2.015
Case 3	Pairwise			
	2	1.282	1.439	1.456
	4	0.825	1.176	1.208
	6	0.591	1.054	1.091

- For Case 1, the expected lower bounds of T are larger than those of T^o and D.

- For Case 2, the expected lower bounds of T^o are larger than those of T and D.

- For Case 3, D yields the largest expected lower bounds and T^o yields the second largest.

7.DISCUSSION

- For H_0 versus H_1, the efficiencies of T, T^o and D depend on the number of good treatments and the number of ineffective treatments.
  - When there is at least one ineffective treatment, T is preferred.
  - When there are no ineffective treatments, T^o or D is recommended, depending on whether there are at least two good treatments.
- For H_0' versus H_1', only T^o or D applies.
  - When there are multiple candidates for the best treatment, procedure T^o is recommended.
  - when there is only one good treatment, Dunnett's procedure D is preferred.

- We focus on the duality of lower confidence bounds and the test statistics. The method can be applied to other optimization problems such as umbrella restrictions.

- The evaluation of the simultaneous confidence lower bound for the difference between the best treatment mean and the control mean is a concave programming problem subject to homogeneous linear inequality constraints. Utilizing the Kuhn-Tucker equivalence theorem is the key to the optimization problem.

Thanks!!!

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada for Lee and Peng.