A Complementary Design Theory for Doubling

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Based on

Chen, H. and Cheng, C.S. (2006) *Annals of Statistics*,

ongoing work with Hongquan Xu,

and some results from the literature of finite projective geometry (Davydov & Tombak, 1990; Bruen, Haddad and Wehlau, 1998; Bruen & Wehlau, 1999)

Objective: construction of two-level minimum aberration (*regular*) fractional factorial designs

$\boldsymbol{X}$: $N \times n$ matrix with entries 1 and -1
N : run size
n : # of factors

$N = 2^{n-p}$

2^{n-p} fractional factorial design

2^{5-2}:

A	B	C	AB D	ABC E
−1	−1	−1	1	−1
1	−1	−1	−1	1
−1	1	−1	−1	1
1	1	−1	1	−1
−1	−1	1	1	1
1	−1	1	−1	−1
−1	1	1	−1	−1
1	1	1	1	1

Defining relation:

$$I = ABD = ABCE = CDE$$

Resolution: length of the shortest word in the defining relation

Resolution III: no aliasing among main effects

Resolution IV: no aliasing of main effects with other main effects and two-factor interactions

Minimum aberration (Fries & Hunter, 1980): Sequentially minimize A_3, A_4, $\cdots$, where $A_k =$ number of words of length k in the defining relation.

Only resolution III+ (resolution III or higher) designs are considered: $A_1 = A_2 = 0$.

- Minimize the aliasing among lower order effects

$(A_3, A_4, \cdots)$: *Word length pattern*

Doubling

$$D(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{X} \\ \boldsymbol{X} & -\boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \boldsymbol{X}$$

Two facts:

(i) $\exists$ a two-level design of resolution III+
$\Rightarrow n \leq N - 1.$

(ii) $\exists$ a two-level design of resolution IV+
$\Rightarrow n \leq N/2.$

A two-level design of resolution III is called *saturated* if $n = N - 1.$

$\leftarrow$ Max. resolution $= 3$ $\rightarrow$

$n:\ N-1$ $N/2$

$\rightarrow$ Max resolution ≥ 4

Saturated regular designs of resolution III are unique (up to isomorphism).

For $N = 2^k$, a saturated design of resolution III can be obtained by deleting the first column of

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{k}.$$

A repeated double of $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Resolution IV designs with $n = N/2$ are also unique (up to isomorphism), and can be obtained by *folding over* saturated designs of resolution III.

If $\boldsymbol{X}$ is a 2^{n-p} design of resolution III, then

$\begin{bmatrix} \mathbf{1} & \boldsymbol{X} \\ -\mathbf{1} & -\boldsymbol{X} \end{bmatrix}$ is a $2^{(n+1)-p}$ design of resolution IV.

Resolution IV designs with $n = N/2$ can also be obtained by *repeatedly doubling the* 2^1 *design*.

Saturated designs of resolution III are *maximal* (the resolution is reduced to two whenever a factor is added), and they are the *only* maximal designs of resolution III.

Every fractional factorial design of resolution III+ can be obtained by choosing columns from a saturated design of resolution III (or we say that it's a *projection* of the saturated design of resolution III.)

Resolution IV designs with $n = N/2$ are *maximal* designs of resolution IV (the resolution is reduced to three whenever a factor is added). Unlike saturated designs of resolution III, they are *not* the only maximal designs of resolution IV.

Resolution IV designs with $n = N/2$ are *even* designs in the sense that $A_i = 0$ for all odd i (a consequence of foldover). They are the only even designs that are maximal. We shall call them the *maximal even designs*.

There is *exactly one* maximal design of resolution IV with $n = 5N/16$, and there is no maximal design of resolution IV with $5N/16 < n < N/2$.

The maximal design of resolution IV with $n = 5N/16$ can be constructed by repeatedly doubling the 2_V^{5-1} design defined by $I = ABCDE$.

There is *exactly one* maximal design of resolution IV with $n = 9N/32$, and there is no maximal design of resolution IV with $9N/32 < n < 5N/16$.

The maximal design of resolution IV with $n = 9N/32$ can be constructed by repeatedly doubling a 2^{9-4} design.

There are *five* maximal designs of resolution IV with $n = 17N/64$, and there is no maximal design of resolution IV with $17N/64 < n < 9N/32$.

Each maximal design of resolution IV with $n = 17N/64$ can be constructed by repeatedly doubling a 2^{17-11} design.

Block and Mee (2003)'s computer search found *five* 2^{17-11} maximal designs of resolution IV.

If $N = 2^k$, $k \geq 4$, then for $\frac{N}{4} + 1 \leq n \leq \frac{N}{2}$, a maximal regular design of resolution IV or higher exists *if and only if* n is an integer of the forms $\frac{N}{2}, \frac{5}{16}N, \frac{9}{32}N, \frac{17}{64}N, \frac{33}{128}N, \cdots$.

Such a design with $n = \frac{2^i+1}{2^{i+2}}N$ can be constructed by *repeatedly doubling* a maximal 2^{i+2}-run design with $2^i + 1$ factors.

$N - 1$ $\frac{N}{2}$ $\frac{5N}{16}$ $\frac{9N}{32}$ $\cdots$

2^{9-4}

2^{5-1}

$(I = ABCDE)$

2^{1-0}

The maximal 2^{5-1}, 2^{9-4} and 2^{17-11} designs of resolution IV are *partial* foldovers of the 2^{4-1}, 2^{8-4} and 2^{16-11} maximal even designs.

Let $\boldsymbol{X}$ be a maximal even design with $N/2$ runs and $N/4$ factors. Partition $\boldsymbol{X}$ so that $\boldsymbol{X} = [\boldsymbol{B}\ \ \boldsymbol{C}]$, and create the N-run design

$$\boldsymbol{S} = \begin{bmatrix} \mathbf{1} & \boldsymbol{B} & \boldsymbol{C} \\ -\mathbf{1} & -\boldsymbol{B} & \boldsymbol{C} \end{bmatrix}$$

with $N/4+1$ factors. Then $\boldsymbol{S}$ has resolution IV. It is maximal if $\boldsymbol{B}$ contains an *odd* number of columns. When $\boldsymbol{B}$ contains an *even* number of columns, $\boldsymbol{S}$ may or may not be maximal.

Complementary design theory

Every regular design $\boldsymbol{D}$ of resolution III+ is a *projection* of the saturated regular design of resolution III.

The factors that are not chosen form another design $\bar{\boldsymbol{D}}$, called the *complementary* design of $\boldsymbol{D}$.

Chen and Hedayat (1996 *AS*) and Tang and Wu (1996 *AS*) found a relationship between the wordlength patterns of $\boldsymbol{D}$ and $\bar{\boldsymbol{D}}$. The construction of $\boldsymbol{D}$ can be done via selecting $\bar{\boldsymbol{D}}$. This is useful when d is *nearly saturated*.

$A_k(\boldsymbol{D})$ can be determined by $\{A_i(\bar{\boldsymbol{D}})\}_{i\leq k}$.
MA $\Leftrightarrow$ sequentially minimizing $(-1)^k A_k(\bar{\boldsymbol{D}})$.

Not useful when $n < N/2$.

For $5N/16 < n < N/2$, a minimum aberration design is of resolution IV, and must be a projection of the maximal even design.

A complementary design theory was developed by Butler (2003).

Hegang Chen's talk tomorrow.

For an $N_0 \times n_0$ design $\boldsymbol{X}_0$, let $\boldsymbol{X}$ be obtained by doubling $\boldsymbol{X}_0$ t times.

Suppose $\boldsymbol{D}$ and $\bar{\boldsymbol{D}}$ are a pair of complementary subdesigns of $\boldsymbol{X}$.

There are identities relating the word length pattern of $\boldsymbol{D}$ to that of $\bar{\boldsymbol{D}}$, covering the results of Chen and Hedayat (1996), Tang & Wu (1996) and Butler (2003) as special cases.

$$A_k(\boldsymbol{D}) = (-1)^k A_k(\bar{\boldsymbol{D}}) + c_{k-1} A_{k-1}(\bar{\boldsymbol{D}}) + \cdots +$$
$$c_1 A_1(\bar{\boldsymbol{D}}) + c_0 + d_k \Delta_k(\boldsymbol{D},\, \bar{\boldsymbol{D}}) + \cdots + d_1 \Delta_1(\boldsymbol{D},\, \bar{\boldsymbol{D}})$$

where

$$\Delta_k(\boldsymbol{D},\, \bar{\boldsymbol{D}}) = \sum_{i=1}^{N_0} [W_i(\boldsymbol{D})^k - (n/2 - W_i(\bar{\boldsymbol{D}}))^k],\, n = n_0 2^t$$

$W_i(\boldsymbol{D})$ is the Hamming weight of the ith row of $\boldsymbol{D}$

Technical tool: Pless power moment identities

Each column of $\boldsymbol{X}_0$ generates 2^t columns of $\boldsymbol{X}$. Suppose $\boldsymbol{D}$ contains f_i columns that are generated by the ith column of $\boldsymbol{X}_0$. Then $\Delta_k(\boldsymbol{D},\, \bar{\boldsymbol{D}})$ depends on $\boldsymbol{X}_0$ and $f_1, \cdots, f_{n_0}$.

When applied to saturated designs of resolution III and maximal even designs, $\Delta_k(\boldsymbol{D},\, \bar{\boldsymbol{D}})$ does not depend on $f_1, \cdots, f_{n_0}$.

Theorem (Chen & Hedayat, 1996; Tang and Wu, 1996). Let $\boldsymbol{X}$ be the saturated design of resolution III. If $\boldsymbol{D}$ consists of u columns of $\boldsymbol{X}$, then $\boldsymbol{D}$ has minimum aberration among all possible u-factor designs if $-A_3(\bar{\boldsymbol{D}}),\ \ A_4(\bar{\boldsymbol{D}}),\ -A_5(\bar{\boldsymbol{D}}),\ A_6(\bar{\boldsymbol{D}}),\ -A_7(\bar{\boldsymbol{D}}),\ \cdots$ are sequentially minimized.

Theorem (Butler, 2003). Let $\boldsymbol{X}$ be a maximal even design. If $\boldsymbol{D}$ consists of u columns of $\boldsymbol{X}$ and $\bar{\boldsymbol{D}}$ is the complement of $\boldsymbol{D}$ in $\boldsymbol{X}$, then $\boldsymbol{D}$ has minimum aberration among all possible u-factor projections of $\boldsymbol{X}$ (and hence has minimum aberration among all u-factor designs) if $A_4(\bar{\boldsymbol{D}}),\ A_6(\bar{\boldsymbol{D}}),\ A_8(\bar{\boldsymbol{D}}),\ \cdots$ are sequentially minimized.

All projections of the maximal even design are *even* designs.

Theorem. Let $\boldsymbol{X}$ be the maximal design obtained by doubling the 2_V^{5-1} design t times. If $\boldsymbol{D}$ consists of certain u columns of $\boldsymbol{X}$ and $\bar{\boldsymbol{D}}$ is the complement of $\bar{\boldsymbol{D}}$ in $\boldsymbol{X}$, then for $u \leq 15 \cdot 2^{t-3}$, $\boldsymbol{D}$ has minimum aberration among all possible u-factor projections of $\boldsymbol{X}$ if

(i) $A_4(\bar{\boldsymbol{D}}), -A_5(\bar{\boldsymbol{D}}), A_6(\bar{\boldsymbol{D}}), -A_7(\bar{\boldsymbol{D}}), \cdots$ are sequentially minimized.

(ii) $|f_i - f_j| \leq 1$ for all $1 \leq i < j \leq 5$.

$N - 1$ $\frac{N}{2}$ $\frac{5N}{16}$ $\frac{9N}{32}$ $\cdots$

2^{9-4}

2^{5-1}
$(I = ABCDE)$

2^{1-0}

For $N/4+1 \leq n \leq 5N/16$, all projections of the maximal even design are worse than some projections of the maximal design with $n = 5N/16$.

For $N/4+1 \leq n \leq 5N/16$, all projections of the maximal design with $9N/32$ are also worse than some projections of the maximal design with $n = 5N/16$.

Theorem. For $17N/64 \le n \le 5N/16$, minimum aberration designs are projections of the maximal design obtained by repeatedly doubling the 2_V^{5-1} design.

Conjecture. The above theorem holds for $N/4 + 1 \le n \le \ 5N/16$.

Supported by Block and Mee's computer search of 128-run designs