On Topological String Theory

Samson L. Shatashvili

TCD & IHES

Based on joint work with Anton Gerasimov
hep-th/0409238, JHEP 0411:074, 2004

Related work: Dijkgraaf, Gukov, Neitzke, Vafa (hep-th/0411073)
Nekrasov (hep-th/0412021)

August 23, 2005
Nankai Institute of Mathematics

Topological strings in Mathematics and Physics

Mathematics

- Gromov-Witten Theory (A-model)
- Deformation theory, complex structures, non-commutative geometry (B-model)
- Mirror Symmetry (relation between A and B models)
- Donaldson-Witten invariants (4d Mirror formula) etc.

......

Physics

- Toy model for String Theory (most likely integrable)
- Gauge theories (brane construction in string theory)
- Black Holes
- Non-critical Strings
- Open/close correspondence (Gauge/Gravity relation)

.....

Physical picture

2d (SUSY) Sigma model - defined through classical action (and corresponding path integral): the functional on the space of maps Φ (and sections ψ)

$$\mathcal{S} = \frac{1}{f^2} \int_\Sigma \frac{1}{2} g_{IJ}(\Phi) \partial_z \phi^I \partial_{\bar{z}} \phi^J + + fermions$$

Σ - Riemann surface; $z, \bar{z}$ - local coordinates

V -Riemannian manifold of metric g; ϕ^I - local coordinates

Φ - a map: $\Sigma \to V$; $\phi^I(z, \bar{z})$ - in local coordinates

K ($\overline{K}$) - the canonical (anti-canonical) line bundles of Σ (the bundle of one forms of types $(1,0)$ ($(0,1)$))

TV - complexified tangent bundle of V.

to get supersymmetry $\Rightarrow$ add some set of Grassmann variables ψ_+^I, ψ_-^I - sections of $K^{1/2} \otimes \Phi^*(TV)$ and $\bar{K}^{1/2} \otimes \Phi^*(TV)$.

Suppose V - Kähler $\to$ $N = 2$ SUSY $\to$ extra term ($TV = T^{1,0}V + T^{0,1}V$ and $\phi^i, \phi^{\bar{i}} = \bar{\phi^i}, \psi^i, \psi^{\bar{i}}$):

$$\mathcal{S}' = i\theta \int_\Sigma \frac{1}{2} g_{i\bar{j}} \left(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} - \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} \right)$$

Topological **A** and **B** models are two different topological twists of same physical sigma model.

A-model

χ - section of $\Phi^*(TV)$

$\psi_+^{\bar{i}}$ - $(1,0)$ form on Σ with values in $\Phi^*(T^{0,1}V)$;

$\psi_+^{\bar{i}} = \psi_z^{\bar{i}}$.

ψ_-^i - $(0,1)$ form with values in $\Phi^*(T^{1,0}V)$; $\psi_-^i = \psi_{\bar{z}}^i$.

Topological transformation laws - one can define the action of certain operator δ acting on field space such that $\delta^2 = 0$.

Let $t = \theta + \frac{i}{f^2}$, where θ and f are some real numbers (couplings).

Action:

$$\mathcal{S} = \frac{1}{f^2} \int_\Sigma d^2 z \ \delta R + t \int_\Sigma \Phi^*(\omega)$$

$$R = g_{i\bar{j}} \left(\psi_z^{\bar{i}} \partial_{\bar{z}} \phi^j + \partial_z \phi^{\bar{i}} \psi_{\bar{z}}^j \right),$$

$$\int_\Sigma \Phi^*(\omega) = i \int_\Sigma d^2 z \ \left(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} g_{i\bar{j}} - \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} g_{i\bar{j}} \right)$$

– the integral of the pullback of the Kähler form $\omega = -i g_{i\bar{j}} dz^i dz^{\bar{j}}$.

$\int \Phi^*(\omega)$ - depends only on the cohomology class of ω and the homology class $\beta \in \mathrm{H}_2(V)$ of the image of the map Φ.

Definition. Observable $\{\mathcal{O}_i\}$ – a functional of the fields, s.t. $\delta\mathcal{O}_i = 0$.

Definition. Physical observable = a δ - cohomology class, $\mathcal{O}_i \sim \mathcal{O}_i + \delta\Psi_i$.

Definition. Correlator - path integral:

$$\langle\prod_a \mathcal{O}_a\rangle_\beta = e^{-2\pi t \int_\beta \omega} \int_{\mathcal{B}_\beta} D\phi\ D\chi\ D\psi\ \ e^{-\frac{1}{f^2}\delta\int R} \cdot \prod_a \mathcal{O}_a.$$

$\mathcal{B}_\beta$ - the component of the field space for maps of degree $\beta = [\Phi(\Sigma)] \in \mathrm{H}_2(V, \mathbf{Z})$, and $\langle\ \ \rangle_\beta$ - degree β contribution to the expectation value.

Correlators of the observables depend only on their δ-cohomology class, in particular — independent of the complex structure of Σ and V, and depend only on the cohomology class of the Kähler form ω.

Correlator is independent of f^2. If $f^2 \to \infty$ - Gaussian model.

Bosonic part of the *Action*

$$it\int \Phi^*(\omega) + \frac{1}{f^2}\int_\Sigma g_{i\bar{j}}(\phi)\partial_z\phi^{\bar{j}}\partial_{\bar{z}}\phi^i$$

for given β is minimized by holomorphic map:

$$\partial_{\bar{z}}\phi^i = \partial_z\phi^{\bar{i}} = 0.$$

The entire path integral, for maps of degree β, reduces to an integral over the space of degree β holomorphic maps $\mathcal{M}_\beta$.

Pick an n-form $W = W_{I_1 I_2 \dots I_n}(\phi) d\phi^{I_1} \wedge d\phi^{I_2} \wedge \dots \wedge d\phi^{I_n}$ on $V \Rightarrow$ a local functional

$$\mathcal{O}_W(P) = W_{I_1 I_2 \dots I_n}(\Phi(P)) \chi^{I_1} \dots \chi^{I_n}(P).$$

$$\delta \mathcal{O}_W = -\mathcal{O}_{dW},$$

d the exterior derivative on V.

$\Rightarrow W \mapsto \mathcal{O}_W$ - natural map from the de Rham cohomology of V to the space of physical observables of quantum field theory $A(V)$. For local operators - isomorphism.

Let d - be the DeRham differential on Σ. We have **descend equations**:

$\mathrm{d}\mathcal{O}_W = \delta \mathcal{O}_W^{(1)}$, $\oint_C \mathcal{O}_W^{(1)}$ - 1-observable. The physical observable depends on the homology class of C in $\mathrm{H}_1(\Sigma)$.

$\mathrm{d}\mathcal{O}_W^{(1)} = \delta \mathcal{O}_W^{(2)}$, $\int_\Sigma \mathcal{O}_W^{(2)}$ - 2-observable.

Deformations (preserving δ): change the action as follows:

$$\mathcal{S}_T = \mathcal{S} + T^a \int_\Sigma \mathcal{O}_{W_a}$$

T^a are the formal parameters (nilpotent). The path integral with the action $\mathcal{S}_T$ computes the *generating function* $\mathcal{F}_A(T)$ of the correlation functions of the two-observables:

$$\mathcal{F}_A(T) = \langle e^{-\int_\Sigma \mathcal{S}(T)} \rangle$$

$$\mathcal{S}(0) = \mathcal{S}, \qquad \frac{\partial \mathcal{S}}{\partial T^a}|_{T=0} = \int_\Sigma \mathcal{O}_{W_a}$$

B - model

$\psi_{\pm}^{\bar{i}}$ - sections of $\Phi^*(T^{0,1}\tilde{V})$

ψ_{+}^{i} - section of $K \otimes \Phi^*(T^{1,0}\tilde{V})$

ψ_{-}^{i} - section of $\overline{K} \otimes \Phi^*(T^{1,0}\tilde{V})$.

ρ - one form with values in $\Phi^*(T^{1,0}\tilde{V})$; $\rho_z^i = \psi_+^i$, $\rho_{\bar{z}}^i = \psi_-^i$

$\eta^{\bar{i}} = \psi_+^{\bar{i}} + \psi_-^{\bar{i}}$

$\theta_i = g_{i\bar{i}}\left(\psi_+^{\bar{i}} - \psi_-^{\bar{i}}\right)$

all fields above are valued in Grassmann algebra

Action:

$$\mathcal{S} = \frac{1}{f^2}\int_\Sigma d^2z\left(g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + i\eta^{\bar{i}}(D_z\rho_{\bar{z}}^i + D_{\bar{z}}\rho_z^i)g_{i\bar{i}}\right.$$
$$\left.+ i\theta_i(D_{\bar{z}}\rho_z{}^i - D_z\rho_{\bar{z}}{}^i) + R_{i\bar{i}j\bar{j}}\rho_z^i\rho_{\bar{z}}^j\eta^{\bar{i}}\theta_k g^{k\bar{j}}\right).$$

As in A model here also exists operation δ (which depends on complex structure) acting on fields such that $\delta^2 = 0$ and again one can rewrite the action using δ:

$$\mathcal{S} = \frac{1}{f^2}\int \delta U + \mathcal{S}_0$$

$$U = g_{i\bar{j}} \left(\rho^i_z \partial_{\bar{z}} \phi^{\bar{j}} + \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} \right)$$

$$\mathcal{S}_0 = \int_\Sigma \left(-\theta_i D\rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^{\bar{i}} \theta_k g^{k\bar{j}} \right).$$

B theory is independent of the complex structure of Σ and the Kähler metric of $\tilde{V}$. Change of complex structure of Σ or Kähler metric of $\tilde{V}$ - Action changes by irrelevant terms of the form $\delta(\ldots)$.

The theory depends on the complex structure of $\tilde{V}$, which enters δ

B model is independent of f^2; take limit $f^2 \to \infty$; In this limit, one expands around minima of the bosonic part of the Action = constant maps $\Phi : \Sigma \to \tilde{V}$:

$$\partial_z \phi^i = \partial_{\bar{z}} \phi^i = 0$$

The space of such constant maps is a copy of $\tilde{V}$; the path integral reduces to an integral over $\tilde{V}$.

Observables:

Consider $(0,p)$ forms on $\tilde{V}$ with values in $\wedge^q T^{1,0}\tilde{V}$, the q^{th} exterior power of the holomorphic tangent bundle of $\tilde{V}$.

$$W = d\bar{z}^{i_1} d\bar{z}^{i_2} \ldots d\bar{z}^{i_p} W_{\bar{i}_1\,\bar{i}_2 \ldots \bar{i}_p}{}^{j_1 j_2 \ldots j_q} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_q}}$$

W is antisymmetric in the j's as well as in the $\bar{i}$'s.

Form local operator:

$$\mathcal{O}_W = \eta^{\bar{i}_1} \ldots \eta^{\bar{i}_p} W_{\bar{i}_1 \ldots \bar{i}_p}{}^{j_1 \cdots j_q} \theta_{j_1} \ldots \theta_{j_q}.$$

$$\delta \mathcal{O}_W = -\mathcal{O}_{\bar{\partial} W},$$

$\mathcal{O}_W$ is δ-invariant iff $\bar{\partial} W = 0$ and δ-exact if $W = \bar{\partial} S$ for some S.

$W \mapsto \mathcal{O}_W$ - natural map from $\oplus_{p,q} H^p(V, \wedge^q T^{1,0} V)$ to the δ-cohomology of the B model. It is isomorphism for local operators.

The story of *Correlators in B model*, *Descend Equations*, *Deformation of the action by 2-observables*, *Generating function* $\mathcal{F}_B(T)$ is completely parallel.

$$S_T = S + T^a \int_\Sigma \mathcal{O}_a^{(2)}$$

$$\mathcal{F}_B(T) =< e^{-S(T)} >$$

• Interesting examples of the deformations:

$W = \mu_{\bar{i}}^{j} \frac{\partial}{\partial z^j} d\bar{z}^{\bar{j}}$ - deformation of the complex structure of $\tilde{V}$

$W = W(z)$ - holomorphic function (for non-compact $\tilde{V}$)- singularity (Landau-Ginzburg in physical terminology) theory

$W = \frac{1}{2}\pi^{ij}\frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j}$ - non-commutative deformation.

Type A sigma models: Gromov-Witten theory

Given a set of submanifolds $C_1, \ldots, C_k$, $C_i \subset V$, *compute the number* $N_{C_1,\ldots,C_k;\beta}$ *of rigid genus* g *holomorphic curves* $\Sigma \subset V$, $[\Sigma] = \beta \in \mathrm{H}_2(V; \mathbf{Z})$ *passing through them*

The cycles in $\mathrm{H}_*(V)$ represented by $C_1, \ldots, C_k$ are Poincare dual to some cohomology classes $\omega_1, \ldots, \omega_k \in \mathrm{H}^*(V)$.

Type B sigma models: Kodaira-Spencer theory

S - space of generalized (in the sense of Kontsevich-Witten) deformations of cmplx structures of variety $\tilde{V}$ (mirror to V).

The tangent space to S at some point s represented by a variety $\tilde{V}_s$ is given by:

$$T_s S = \bigoplus_{p,q} \mathrm{H}^p\left(\tilde{V}_s, \Lambda^q \mathcal{T}_{\tilde{V}_s}\right) \equiv \bigoplus_{p,q} \mathrm{H}^{-q,p}(\tilde{V}_s)$$

Let T denote special coordinates on this space.

Mirror formula

$$\sum_{n;\{k_1,\ldots,k_n\}} \frac{T^{k_1}\ldots T^{k_n}}{n!} \left\langle \mathcal{O}_a^{(0)}\mathcal{O}_b^{(0)}\mathcal{O}_c^{(0)} \int_\Sigma \mathcal{O}_{k_1}^{(2)} \cdots \int_\Sigma \mathcal{O}_{k_n}^{(2)} \right\rangle_A$$

$$= \frac{\partial^3 \mathcal{F}_B\,(T)}{\partial T^a \partial T^b \partial T^c}$$

Type A sigma model on V = Type B sigma model on $\tilde{V}$

For Kähler manifolds:

$$h^{p,q}(V) = h^{-p,q}(\tilde{V})$$

The concept of mirror symmetry extends to V symplectic and $\tilde{V}$ complex.

Mirror exchanges *kähler* (A) and *complex* (B) deformations.

Space-Time description - String Field Theory

Here we discuss the case of CY target space (3-complex dimensional with $c_1 = 0$; (3,0) form - uniquely defined up to the multiplication by the non-zero complex number)

A-model - world-sheet instantons $\rightarrow$ local field theory?

B-model - constant maps $\rightarrow$ one expects to have local theory.

- Let M be a compact Calabi-Yau (CY) manifold.

- Calibrated CY manifold - a pair (M, Ω_0) where M is a CY manifold supplied with a holomorphic $(3,0)$-form Ω_0.

Moduli space $\widehat{\mathcal{M}}_M$ of the calibrated CY manifolds is a $\mathbf{C}^*$-bundle over moduli space $\mathcal{M}_M$ of complex structures on M.

Fixing particular holomorphic $(3,0)$-form Ω_0 defines, locally, a section of the bundle.

Choice of complex structure:

$$D = D^{1,0} + D^{0,1} = \partial + \bar{\partial}$$

$\Omega^{-p,q}(M) \equiv \Omega^q(M, \wedge^p T)$, where T is a holomorphic tangent bundle.

- $(z^i, \bar{z}^{\bar{i}})$ be local coordinates on M

- $\overline{A} \in \Omega^{-1,1}(M)$, $(-1,1)$-differential written locally as:

$$\overline{A} = \sum \overline{A}^j_{\bar{i}} d\bar{z}^i \frac{\partial}{\partial z^j}$$

Deformation of the complex structure - deformation of the operator $D^{0,1} = \bar{\partial}$:

$$\bar{\partial} \to \bar{\partial}_{\overline{A}} = \bar{\partial} + \overline{A} = \sum d\bar{z}^i (\frac{\partial}{\partial \bar{z}^i} + \overline{A}^j_{\bar{i}} \frac{\partial}{\partial z^j})$$

subject to integrability condition - Kodaira-Spencer equation:

$$\bar{\partial}^2_{\overline{A}} = 0$$

$$\bar{\partial}\overline{A} + \frac{1}{2}[\overline{A}, \overline{A}] = 0$$

$$(\bar{\partial}_{\bar{i}} \overline{A}^k_{\bar{j}} + \overline{A}^l_{\bar{i}} \partial_l \overline{A}^k_{\bar{j}}) d\bar{z}^i \wedge d\bar{z}^j \frac{\partial}{\partial z^k} = 0$$

The moduli space of complex structures - space of the solutions of KS equation modulo the gauge transformations

$$\delta\overline{A} = \bar{\partial}_{\overline{A}}\epsilon = \bar{\partial}\epsilon + [\overline{A}, \epsilon]$$

where $\epsilon \in \Omega^{-1,0}(M)$.

Equivalently - solutions of the pair of the equations:

$$\partial\overline{A} + \frac{1}{2}[\overline{A}, \overline{A}] = 0$$

$$\partial A = 0$$

modulo the subgroup of the gauge transformations which leave the holomorphic three-form Ω_0 invariant.

Given a holomorphic $(3,0)$-form Ω_0 we could identify $\Omega^{-p,q}(M)$ with $(3-q,p)$-forms $\Omega^{3-p,q}(M)$:

$$\Omega^{-p,q}(M) \to \Omega^{3-p,q}(M)$$

$$A \to A^{\vee} = A \vdash \Omega_0$$

One can parametrise (2,1) form $A^{\vee}$ as:

$$\overline{A}^{\vee} = x + \partial b$$

where x is a ∂-closed $(2,1)$-form and $b \in \Omega^{1,1}(M)$.

Now the KS equation becomes the equation for b

$$\bar{\partial}\partial b + \frac{1}{2}\partial((x + \partial b) \circ (x + \partial b)) = 0$$

Operation $\circ$ is defined as:

$$A^{\vee} \circ B^{\vee} = (A \wedge B) \vdash \Omega_0,$$

This equation has a meaning of anti-holomorphicity for following (1,2)-current:

$$\bar{J}^{(1,2)} = \bar{\partial} b + \frac{1}{2}(x + \partial b) \circ (x + \partial b); \qquad \partial \bar{J}^{(1,2)} = 0$$

The action functional, String Field Theory Action, leading to the above equations of motion is:

$$S_{KS}(b|x) = \int_M (\frac{1}{2}\partial b \wedge \bar{\partial} b + \frac{1}{6} < (x+\partial b), (x+\partial b), (x+\partial b) >)$$

Here we use another notation:

$$< A^\vee, B^\vee, C^\vee >= A^\vee \wedge (B^\vee \circ C^\vee),$$

where $A^\vee = A \vdash \Omega_0$, $B^\vee = B \vdash \Omega_0$, $C^\vee = C \vdash \Omega_0$.

This action is equivalent to the action introduced by **BCOV** (Bershadsky, Cecotti, Ooguri and Vafa, 1993) but is written in slightly different variables.

- $Z_{KS}(T)$: Partition function of the space-time quantum field theory, **KS** theory

$$Z_{KS}(x) = \int [db] e^{-\frac{1}{\lambda} S_{KS}(b|x)}$$

λ - Plank constant in space-time field theory (coupling constant)

BCOV conjecture:

$$Z_{KS}(x) = \exp \sum_g \lambda^{2g-2} \mathcal{F}_g(T)$$

$\mathcal{F}_g(T)$ - $\mathcal{F}_B(T)$ for genus g

x - related to special coordinates T.

KS theory in terms of 7d quadratic field theory

Functional integral with the action **KS** action:

$$Z(x) = \int Db\ e^{-S_{KS}(b|x)}$$

integral representation of some particular wave function in the 7d field theory on $M \times \mathbf{R}$ with the action functional:

$$S(C) = \int_{M \times \mathbf{R}} C dC$$

C - real 3-form.

This "space-time" quantum field - higher dimensional generalization of the 3d abelian Chern-Simons but for 3-forms.

In $6+1$ notations for local coordinates (x^i, t) on $M \times \mathbf{R}$:

$$S = \int_{M \times \mathbf{R}} dt\, d^6x\, (\Omega \frac{\partial}{\partial t} \Omega + \omega_t d\Omega)$$

where $C = \Omega + \omega_t dt$;

Ω - a three-form component of C along M

$\omega_t\, dt$ - a two-form along M and one-form along $\mathbf{R}$.

We proceed in Hamiltonian formalism.

Consider the infinite-dimensional space of real 3-forms on M supplied with the symplectic structure:

$$\omega^{symp}(\delta_1\Omega, \delta_2\Omega) = \int_M \delta_1\Omega \wedge \delta_2\Omega$$

The phase space for 7d-theory is obtained - Hamiltonian reduction via first class constraint ($H^3(M, \mathbf{R})$):

$$d\Omega = 0$$

One needs to choose the polarization (before Hamiltonian reduction) for symplectic form and impose the constraint on wave function in given polarization.

Linear Polarization

Here we first intriduce the complex structure on CY which induces the complex structure on H^3.

If Ω_C denotes complex 3-form we choose the complex structure on M and:

$$\Omega_{\mathbf{C}} = \Omega^{3,0} \oplus \Omega^{2,1} \oplus \Omega^{1,2} \oplus \Omega^{0,3}$$

The real forms are singled out by the reality condition:

$$\Omega^{0,3} = \overline{\Omega^{0,3}}, \qquad \Omega^{1,2} = \overline{\Omega^{2,1}}$$

The subspace:

$$\Omega^{3,0} \oplus \Omega^{2,1}$$

defines the complex Lagrangian (linear) sub-manifold in the space of complex 3-forms $\rightarrow$ the complex polarization of the space of real three-forms. Parameterize this subspace as:

$$\Omega_c = \rho(\Omega_0 + \overline{A} \vdash \Omega_0)$$

Ω_0 - fixed normalized holomorphic $(3,0)$-form in a reference complex structure

ρ - function on M

$\overline{A}$ - $(-1,1)$-differential.

Non-linear polarization (following Hitchin)

Here we first introduce the complex structure on H^3 which defines the complex structure on $\mathbf{CY}$.

Special properties of 3-forms in six dimensions $\rightarrow$ possibility to decompose any generic real three-form as a sum of two (possibly complex conjugate) decomposable 3-forms:

$$\Omega = \Omega_+ + \Omega_- = E^1 \wedge E^2 \wedge E^3 + E^4 \wedge E^5 \wedge E^6$$

E^i's - some 1-forms. Generically $\Omega_- \wedge \Omega_+ \neq 0$ and E^i's produce the frame in the complexified cotangent bundle $T^*_{\mathbf{C}}M$ to M.

This representation follows from the existence of the open orbit of the group $GL(V)$ acting on $\wedge^3 V^*$ where V a 6d vector space.

One could explicitly reconstruct the decomposable forms $\Omega_\pm$ as follows:

Let $\mathcal{A}^p_M$ be the space of the real p-forms on M and $Vect_M$ be the space of vector fields. Given a three-form Ω, consider the operator K_Ω:

$$K_\Omega : Vect_M \rightarrow \mathcal{A}^5_M \simeq Vect_M \otimes \mathcal{A}^6_M,$$

defined as:

$$v \rightarrow (v \vdash \Omega) \wedge \Omega.$$

For instance, given the decomposition with real E^i's, the action of K_Ω on the dual real frame E^*_i is:

$$K_\Omega : E^*_i \to E^*_i, \ \ i = 1, 2, 3,$$

$$K_\Omega : E^*_i \to -E^*_i, \ \ i = 4, 5, 6.$$

Let K^*_Ω be the group action of K_Ω on $\mathcal{A}^p(M)$. Then:

$$2\Omega_+ = \Omega + \lambda(\Omega)^{-3/2} K^*_\Omega \Omega,$$

$$2\Omega_- = \Omega - \lambda(\Omega)^{-3/2} K^*_\Omega \Omega,$$

where:

$$\lambda(\Omega) = \frac{1}{6} tr\, K^2_\Omega \in (\mathcal{A}^6_M)^{\otimes 2}$$

Decomposition is non-degenerate (i.e $\Omega_+ \wedge \Omega_-$ has no zeros) if the form $\lambda(\Omega)$ has no zeros.

The sign of $\lambda(\Omega)$ defines whether $\Omega_\pm$ are real ($\lambda(\Omega) > 0$) or complex conjugate to each other $\Omega_- = \overline{\Omega}_+$ ($\lambda(\Omega) < 0$). Denote the corresponding subspaces in the space of real three-forms by U_+ and U_-.

In the case $\Omega \in U_-$ the operator I

$$I_\Omega = (-\lambda(\Omega))^{-1/2} K_\Omega,$$

defines the (pseudo)complex structure. The condition of the integrability of this complex structure:

$$d\Omega = d\widehat{\Omega} = 0,$$

where $\widehat{\Omega} \equiv \Omega_+ - \Omega_-$. Note that $\Omega + i\widehat{\Omega}$ is a holomorphic $(3,0)$-form without zeros in this complex structure.

Condition of integrability of this complex structure - condition for the critical points of the functional $\Phi(\Omega)$ written in terms of the closed three-form Ω, $d\Omega = 0$

$$\Phi(\Omega) = \int_M \sqrt{|\lambda(\Omega)|}$$

The variation of this functional is given by:

$$\delta\Phi(\Omega) = -\int_M \widehat{\Omega} \wedge \delta\Omega.$$

The following relation holds

$$\Omega_+ \wedge \Omega_- = \frac{1}{2}\Omega \wedge \widehat{\Omega} = (\lambda(\Omega))^{1/2}.$$

Let us restrict the space of three-forms Ω by the condition: $d\Omega = 0$, and fix the class of Ω in $H^3(M, \mathbf{R})$. Such three-forms may be parameterized by two-form ϕ as: $\Omega = x + d\phi$, where x is some fixed closed three-form $[\Omega - x] = 0$ in $H^3(M, \mathbf{R})$.

The critical points of the functional $\Phi(\Omega)$ under the variation $\delta\Omega = d\delta\varphi$ are given by the solutions of the equation $d\widehat{\Omega} = 0$.

Given the cohomology class of the real closed 3-form on M, and using the critical point condition one could reconstruct unambiguously the holomorphic structure and the holomorphic non-degenerate $(3,0)$-form $\Omega + i\widehat{\Omega}$ on M; thus M has a trivial canonical class.

One can show (**Hitchin**) that up to the action of the diffeomorphisms the critical point is isolated, so it defines the map of the subspace of $H^3(M, \mathbf{R})$ (such that the corresponding critical value Ω is in U_-) into the extended moduli space of complex structures $\widehat{\mathcal{M}}_M$.

Consider the 3-forms in U_-:

$$\Omega = \Omega_+ + \Omega_- = E^1 \wedge E^2 \wedge E^3 + \overline{E}^1 \wedge \overline{E}^2 \wedge \overline{E}^3$$

with E^i being complex one-forms.

The subspace of the decomposable forms defines the Lagrangian family:

$$\omega^{symp}(\delta_1\Omega_+, \delta_2\Omega_+) = \int \delta_1(E^1 \wedge E^2 \wedge E^3) \wedge \delta_2(E^1 \wedge E^2 \wedge E^3) = 0$$

Locally the decomposable three-form Ω_+ may be parameterized as:

$$\Omega_+ = \frac{1}{6}\epsilon_{ijk}\varrho(dz^i + \mu^i_{\bar{i}} d\bar{z}^{\bar{i}})(dz^j + \mu^j_{\bar{j}} d\bar{z}^{\bar{j}})(dz^k + \mu^i_{\bar{k}} d\bar{z}^{\bar{k}}) =$$

$$= \varrho(\Omega_0 + \mu \vdash \Omega_0 + \frac{1}{2}\mu^2 \vdash \Omega_0 + \frac{1}{6}\mu^3 \vdash \Omega_0) = \varrho e^{\mu\vdash}\Omega_0$$

here $\mu \in \Omega^{-1,1}(M)$ and ϱ is a function on M and we use the notations $\mu^n \vdash := (\mu \vdash)^n$.

Wave-function in linear polarization

Wave-functions - functions of $\Omega^{3,0}, \Omega^{2,1}$ subject to constraints (reduction). In this polarization the constraint equations $d\Omega = 0$:

$$\begin{aligned} \bar{\partial}\Omega^{3,0} + \partial\Omega^{2,1} &= 0, \\ \bar{\partial}\Omega^{2,1} + \partial\Omega^{1,2} &= 0, \\ \bar{\partial}\Omega^{1,2} + \partial\Omega^{0,3} &= 0, \end{aligned}$$

Quantum mechanically take the form of the following equations on the wave function $\Psi(\Omega^{3,0}, \Omega^{2,1})$

$$(\bar{\partial}\Omega^{3,0} + \partial\Omega^{2,1})\Psi = 0$$

$$(\bar{\partial}\Omega^{2,1} + \partial\frac{\delta}{\delta\Omega^{2,1}})\Psi = 0$$

$$(\bar{\partial}\frac{\delta}{\delta\Omega^{2,1}} + \partial\frac{\delta}{\delta\Omega^{3,0}})\Psi = 0$$

The formal solution can be written in terms of the path integral. Given an arbitrary function $\Psi_0(\Omega^{3,0}, \Omega^{2,1})$, one can construct the formal solution representing it in the form:

$$\Psi(\Omega^{3,0}, \Omega^{2,1}) = (\Pi\Psi_0)(\Omega^{3,0}, \Omega^{2,1})$$

where the projection operator Π is given by:

$$\Pi = \int D\Lambda\, D\sigma\, Db\; e^{\int_M \Lambda(\bar{\partial}\Omega^{3,0} + \partial\Omega^{2,1})} e^{\int_M \sigma(\bar{\partial}\frac{\delta}{\delta\Omega^{2,1}} + \partial\frac{\delta}{\delta\Omega^{3,0}})} \times$$

$$\times e^{\int_M b(\bar{\partial}\Omega^{2,1} + \partial\frac{\delta}{\delta\Omega^{2,1}})}$$

The only restriction on Ψ_0 is the convergence of the integrals.

Use parametrization of 3-forms in terms of function ρ, (2,0)-form χ and (1,1)-form λ:

$$\Omega^{3,0} = \rho_0 \Omega_0 + \partial \chi$$

$$\Omega^{2,1} = x + \partial \lambda - \bar{\partial} \chi$$

where x is a ∂-closed (2,1)-form.

In these variables the projected wave-function is:

$$\Psi(\Omega^{3,0}, \Omega^{2,1}) = \Psi(\rho_0, \chi, x, \lambda) = e^{-\int_M (\frac{1}{2} \partial \lambda \bar{\partial} \lambda)} \times$$

$$\int D\sigma \, Db \; e^{-\int_M (\frac{1}{2} \partial b \bar{\partial} b + \bar{\partial} b \wedge \partial \lambda)} \Psi_0(\rho_0 \Omega_0 - \partial \sigma, x - \partial b - \bar{\partial} \sigma)$$

The scalar product of the wave functions in given by:

$$< \Psi_1 | \Psi_2 >= \int D\Omega^{3,0} \, D\Omega^{2,1} \, D\Omega^{1,2} \, D\Omega^{0,3} \; e^{\int_M (\Omega^{3,0} \wedge \Omega^{0,3} + \Omega^{2,1} \wedge \Omega^{1,2})} \times$$

$$\times \overline{\Psi}_1(\Omega^{0,3}, \Omega^{1,2}) \Psi_2(\Omega^{3,0}, \Omega^{2,1})$$

where integration is over the real subspace $\overline{\Omega^{0,3}} = \Omega^{3,0}$, $\overline{\Omega^{1,2}} = \Omega^{2,1}$.

Wave-functions in non-linear polarization

In this polarization wave-functions depend on subspace defined by Ω_+.

$$\Omega_+ = \varrho(\Omega_0 + \mu \vdash \Omega_0 + \frac{1}{2}\mu^2 \vdash \Omega_0 + \frac{1}{6}\mu^3 \vdash \Omega_0)$$

$$\Omega_- = \overline{\Omega}_+ = \overline{\varrho}(\overline{\Omega}_0 + \overline{\mu} \vdash \overline{\Omega}_0 + \frac{1}{2}\overline{\mu}^2 \vdash \overline{\Omega}_0 + \frac{1}{6}\overline{\mu}^3 \vdash \overline{\Omega}_0)$$

Scalar product - defined via canonical transformation from variables μ, ρ to $\bar{\mu}, \bar{\rho}$:

$$\sum_i P_i \delta Q^i - \sum_i p_i \delta q^i = \delta S(Q, q)$$

We find:

$$S(\rho, \mu; \bar{\rho}, \bar{\mu}) = \Omega_-(\overline{\varrho}, \overline{\mu}) \wedge \Omega_+(\varrho, \mu)$$

and:

$$< \Psi_1 | \Psi_2 >= \int \mathcal{D}(\mu, \overline{\mu}, \varrho, \overline{\varrho}) e^{\int_M \Omega_-(\overline{\varrho}, \overline{\mu}) \wedge \Omega_+(\varrho, \mu)} \overline{\Psi}_1(\overline{\varrho}, \overline{\mu}) \Psi_2(\varrho, \mu)$$

Note - the exponential factor is Hitchin functional

$$\Omega_+(\Omega) \wedge \Omega_-(\Omega) = -\sqrt{\lambda(\Omega)}$$

Imposing the constraints in this polarization - quite trivial procedure.

The constraints generate the gauge transformations:

$$\Omega = \Omega_+ + \Omega_- \to \Omega + d\phi =$$

$$= (\Omega_+ + \delta\Omega_+(\Omega_\pm, \phi)) + (\Omega_- + \delta\Omega_-(\Omega_\pm, \phi))$$

and the transformations of $\Omega_\pm$ is highly non-linear in terms of the initial $\Omega_\pm$ - in this polarization the gauge transformation mixes "coordinates" and "momenta" in a complicated way and the constraints are given by rather complex differential operators acting on the wave function.

We use the following strategy:

1. start with the simple unconstrained wave-function in the non-linear polarization

2. transform this wave function into the corresponding wave function in the linear polarization

3. impose the constraints.

This gives us the constraint wave function in the linear polarization whose particular form reflects the simplicity of the initial wave function in the non-linear polarization.

We construct the generating function $S(A,\bar{\rho}|\mu,\varrho)$ from the "old coordinates" $\bar{\rho}, A$ to new coordinates ϱ,μ using two definitions of Ω given above:

$$\rho\Omega_0 = \varrho\Omega_0 + \frac{1}{6}\overline{\varrho}\,\overline{\mu}^3 \vdash \overline{\Omega}_0$$

$$\rho\overline{A} \vdash \Omega_0 = \varrho\mu \vdash \Omega_0 + \frac{1}{2}\overline{\varrho}\,\overline{\mu}^2 \vdash \overline{\Omega}_0$$

$$\overline{\rho}A \vdash \overline{\Omega}_0 = \overline{\varrho}\,\overline{\mu} \vdash \overline{\Omega}_0 + \varrho\frac{1}{2}\mu^2 \vdash \Omega_0$$

$$\overline{\rho}\,\overline{\Omega}_0 = \overline{\varrho}\,\overline{\Omega}_0 + \frac{1}{6}\varrho\mu^3 \vdash \Omega_0$$

Define:

$$< \overline{\mu}^3 > \overline{\Omega}_0 \wedge \Omega_0 = \frac{1}{6}\overline{\Omega}_0 \wedge (\overline{\mu}^3 \vdash \overline{\Omega}_0)$$

$$< \mu^3 > \Omega_0 \wedge \overline{\Omega}_0 = \frac{1}{6}\Omega_0 \wedge (\mu^3 \vdash \Omega_0),$$

$$\overline{\mu}^{\vee} \vdash \overline{\Omega}_0 = \frac{1}{2}\mu^2 \vdash \Omega_0.$$

Answer for S is:

$$S(A,\overline{\rho}|\mu,\varrho) = \int_M ((\overline{\rho}\varrho + \varrho^2 < \mu^3 > + \frac{< (A\overline{\rho} - \varrho\overline{\mu}^{\vee})^3 >}{(\overline{\rho} - \varrho < \mu^3 >)})\Omega_0 \wedge \overline{\Omega}_0 +$$

$$+\overline{\rho}\varrho(\mu \vdash \Omega_0) \wedge (A \vdash \overline{\Omega}_0))$$

Semi-classical approximation (classical and one-loop) for quantum-mechanical wave function is given by path integral:

$$\Psi(\varrho,\mu) = \int D\overline{\rho}\, DA\ e^{S(A,\overline{\rho},\mu,\varrho)}\Psi(\overline{\rho},A)$$

$$\Psi(\overline{\rho},A) = \int D\overline{\rho}\, DA\ e^{-S(A,\overline{\rho},\mu,\varrho)}\Psi(\varrho,\mu)$$

Beyond one-loop - see below.

Now we can project to physical state in linear polarization:

$$\Psi(\Omega^{3,0},\Omega^{2,1}) =< \Omega^{2,1},\Omega^{3,0}|\Pi|\psi >,$$

Claim: the following choice leads to desired result -

$$\psi(\overline{\varrho},\overline{\mu}) = \delta(\overline{\mu})\exp\int_M \overline{\varrho}$$

The reasoning for such choice: this wave function in the (ϱ,μ)-polarization is given by:

$$\psi(\varrho,\mu) = \delta(\varrho - 1)$$

This corresponds to the fixing of the holomorphic volume form (stated differently - to the choice of the closed string coupling constant).

Thus we have:

$$\Psi(\Omega^{3,0},\Omega^{2,1}) = \int D\overline{\mu} D\overline{\varrho}\ e^{-S(\Omega^{3,0},\Omega^{2,1},\overline{\mu},\overline{\varrho})}\delta(\overline{\mu})\exp\int_M \overline{\varrho} =$$

$$= \delta(\Omega^{3,0}-\Omega_0)\exp(-\int_M \frac{1}{6} < \Omega^{2,1},\Omega^{2,1},\Omega^{2,1} >).$$

Final answer after the action of projector:

$$\Psi(\Omega^{3,0},\Omega^{2,1}) = \Psi(\rho_0,\chi,x,\lambda,\widetilde{\lambda}) =$$

$$= const \int Db\ e^{-\int(\frac{1}{2}\partial b\bar{\partial} b+\frac{1}{6}<(x+\partial b),(x+\partial b),(x+\partial b)>)}$$

Consider the following formal path integral:

$$Z = \int_{(x+d\phi)\in U_-} d\phi \int Dk \; e^{\int_M \frac{1}{\sqrt{\frac{1}{6}trk^2}}(x+d\phi)k(x+d\phi)}$$

x - some fixed element of $H^3(M, \mathbf{R})$,
ϕ - a two-form and $k \in End(T^*M)$ acts on arbitrary differential form as an element of the Lie algebra.

The equations of motion for k are algebraic and its solution is

$$k = \rho K_\Omega$$

for $\Omega = (x+d\phi)$ with ρ being an arbitrary non-zero function. Substituting this solution into the action in action one finds that (in the classical approximation over k) the theory is equivalent to:

$$Z = \int_{U_-} D\phi \; e^{\sqrt{-\lambda(x+d\phi)}}$$

Note that latter does not depend on ρ.

This should be compared with well-known procedure in two dimensions. Mainly - start with Polyakov formulation of the string moving in d dimension:

$$Z = \int \left(\prod_{a=1}^{p} d\phi^a\right) Dg_{ij} \; e^{\int_M \sqrt{g} g^{ij} \sum_{a=1}^{p} \partial_i \phi^a \partial_j \phi^a}$$

In two dimensions the analog of k can be explicitly described in terms of the metric as:

$$k_i^j = |g|\epsilon_{ik}g^{kj}$$

One has $tr\, k^2 = 2|g|$.
Then the action in 2d is given by

$$S = \int_M \frac{1}{\sqrt{\frac{1}{2}trk^2}} \sum_{a=1}^{p} d\phi^a \wedge (kd\phi^a).$$

One can get rid of k using its equations of motion, so result is a Nambu-Goto action:

$$S = \int_M \sqrt{\det_{i,j=1,2}(\sum_{a=1}^{p} \partial_i\phi^a\partial_j\phi^a)}$$

Note that the proper generalization of the metric in two dimensions in this context is given by the non-normalized operator of the complex structure k.

In 2d if one integrates out ϕ^a instead:

$$\mathcal{S} = p\int\int \sqrt{g}R\frac{1}{\Delta(g)}\sqrt{g}R = \Gamma(\mu) + \Gamma(\bar{\mu}) + S_{Liouv}(\rho,\mu,\bar{\mu})$$

where we parametrized the metric:

$$ds^2 = g_{ij}d\zeta^i d\zeta^j = e^\rho|dz + \mu d\bar{z}|^2$$

Here we explained that KS action is analog of $\Gamma(\mu)$ and Hitchin action is analog of Liouville action.

7d CS theory of 3-forms is an analog of $SL(2,R)$ CS action describing 2d gravity for 3-manifold $\Sigma \times R$.

Q1: In 2d - theory defined via gravitational WZW action $\Gamma(\mu)$ is equivalent to Liouville theory. Is this true in 6d?

Q2: Can one describe KS theory as a simple field theory similar to what we know in 2d?

Yes! To do this we need to consider 4 complex dimensional non-compact CY manifold N with original 3d compact CY M being a divisor in N. Theory of free fermions in 8d interacting with three forms on divisor M (complex boundary for N) gives full answer for $Z(\lambda, t)$.

Above statement is a conjecture for 3 complex dimensional CY, but it is a theorem for 0-dimensional CY - point. Latter - $c < 1$ noncritical strings where exact answers are known due to Kontsevich.