EXOTIC GALILEAN SYMMETRY NON-COMMUTATIVITY & THE HALL EFFECT

P. Horvathy (Tours, FR)

based on joint work with
C. Duval (Marseille, FR)
Z. Horvath (Budapest, HU)
L. Martina (Lecce, IT)
M. Plyushchay (Santiago, CHILE)
P. Stichel (Bielefeld, GER)

August 21, 2005

- Group Theory 1995 – 2000:

models with “exotic” galilean symmetry

- Condensed Matter Physics 1995 – 2000:

semiclassical Bloch electron in crystal

- Field Theory:

noncommutative Chern-Simons theory

2000-2004

EXOTIC MECHANICS IN THE PLANE

Lévy-Lebond '1972: "exotic" central extension of planar Galilei group. Boosts don't commute:

$$[\mathcal{G}_1, \mathcal{G}_2] = \kappa. \tag{1}$$

Duval, Grigore '1995: model by Souriau's orbit method. [Alternative: Lukierski et al. '1996].

- free motions as usual; exotic contribution only in conserved quantities

$$\begin{aligned} j &= \mathbf{x} \times \mathbf{p} + (\theta/2)\mathbf{p}^2 && \text{ang. mom.} \\ K_i &= m x_i - p_i t + m\theta\epsilon_{ij}p_j && \text{boosts} \end{aligned} \tag{2}$$

Coupled to abelian gauge field:

$$\begin{aligned} m^* \dot{x}_i &= p_i - em\theta\,\varepsilon_{ij}E_j, \\ \dot{p}_i &= eE_i + eB\,\varepsilon_{ij}\dot{x}_j, \end{aligned} \tag{3}$$

where $m^* = m(1 - e\theta B)$ effective mass, $\theta = k/m^2$ noncommutative parameter.

- Velocity, $\dot{x}_i$, and momentum, p_i, differ in anomalous velocity term $em\theta\,\varepsilon_{ij}E_j$.

(3) comes from “exotic” Lagrangian

$$\int (\mathbf{p} - \mathbf{A}) \cdot d\mathbf{x} - \frac{p^2}{2} dt + \frac{\theta}{2} \mathbf{p} \times d\mathbf{p} . \qquad (4)$$

For $m^* \neq 0$ (3) also Hamiltonian, $\dot{\xi} = \{h, \xi^\alpha\}$, $\xi = (p_i, x^j), h = \mathbf{p}^2$ and Poisson brackets

$$\begin{aligned} \{x_1, x_2\} &= (m/m^*)\,\theta, \\ \{x_i, p_j\} &= (m/m^*)\,\delta_{ij}, \\ \{p_1, p_2\} &= (m/m^*)\,eB. \end{aligned} \qquad (5)$$

N.B (5) similar to naive relations

$$\begin{aligned} \{x_1, x_2\} &= \theta, \\ \{x_i, p_j\} &= \delta_{ij}, \\ \{p_1, p_2\} &= eB. \end{aligned} \qquad (6)$$

posited (later !) by Nair & Polychronakos. Latter is

- equivalent to exotic model if B constant s. t. $m^* \neq 0$;

- violates Jacobi identity if B not constant;

- makes no useful physical prediction when $m^* = 0$.

Critical case

For vanishing effective mass $m^* = 0$ i.e. for

$$B = \frac{1}{e\theta} \tag{7}$$

exotic system becomes singular. Faddeev-Jackiw reduction yields 2-dimensional, simple system, similar to Chern-Simons mechanics of Dunne, Jackiw, Trugenberger (alias planar vortex dynamics Onsager '1949 $\Longrightarrow$ Kirchhoff '1883 !)

Reduced model $\sim$ guiding center coordinates

$$Q_i = x_i + \epsilon_{ij}\frac{p_j}{eB} = x_i - \frac{mE_i}{eB^2}. \tag{8}$$

Reduced Poisson bracket & energy

$$\{Q_1, Q_2\}_{red} = \frac{1}{eB}, \tag{9}$$

$$H_{red} = eV(Q_1, Q_2) + m\vec{E}^2/2B^2. \tag{10}$$

- All motions follow (generalized) Hall law

$$\dot{Q} = \epsilon_{ij}\frac{E_j}{B}. \tag{11}$$

"condensation into collective ground state"
(Laughlin '1983 for FQHE !)

• Quantization of reduced model yields Laughlin wave functions:

$$\psi(z, \bar{z}) = e^{-z\bar{z}/4} f(z), \tag{12}$$

where $z = \sqrt{eB}(Q_1 + iQ_2)$, $f(z)$ holomorphic. Ground states Fractional Quantum Hall Effect.

$$\widehat{z}(f) = zf, \qquad \widehat{\bar{z}}(f) = 2\frac{\partial f}{\partial z}, \tag{13}$$

$$H_{red} = e\widehat{V}(z, \bar{z}) + \frac{\theta^2 e^2 m}{2}\widehat{\mathbf{E}}^2(z, \bar{z}) \tag{14}$$

with

$$\left[\widehat{\bar{z}}, \widehat{z}\right] = 2. \tag{15}$$

Peierls'1933, DJT'1991.

N. B. Ordering problem.

• e. g. oscillator $\mathbf{E} = -m\omega^2\mathbf{x}$. In critical case $e\theta B = 1$: red. energy

$$H_{red} = \frac{m\omega^2}{2}\Big(1 + em^2\omega^2\theta^2\Big)Q^2. \tag{16}$$

$\propto$ red. ang. momentum $\propto \frac{eB}{2}Q^2$. Spectrum :

$$E_n = \frac{em\omega^2\theta}{1 + em^2\theta^2\omega^2}\Big(\tfrac{1}{2} + n\Big) \qquad n = 0, 1, \ldots \tag{17}$$

SEMICLASSICAL BLOCH ELECTRON

Around same time (1995-2000), independently, similar theory in solid state physics. Niu et al. Berry-phase argument to Bloch electron yields semiclassical model. Equations of motion in n^{th} band

$$\dot{\mathbf{x}} = \frac{\partial \epsilon_n(\mathbf{p})}{\partial \mathbf{p}} - \dot{\mathbf{p}} \times \vec{\Omega}(\mathbf{p}), \quad (18)$$

$$\dot{\mathbf{p}} = -e\mathbf{E} - e\dot{\mathbf{x}} \times \mathbf{B}(\mathbf{x}), \quad (19)$$

$\mathbf{x}$ electron's intracell position, $\mathbf{p}$ quasimomentum, $\epsilon_n(\mathbf{p})$ band energy. Purely momentum dependent $\vec{\Omega}$ is Berry curvature of Bloch states $\Omega_i(\mathbf{p}) = \epsilon_{ijl}\partial_{\mathbf{p}_j}\mathcal{A}_l(\mathbf{p})$. $\mathcal{A}$ Berry connection.

Berry-phase term $\vec{\Omega}$ in (18) instrumental in describing:

- Anomalous Hall Effect in ferromagnetics Niu et al.

- Spin Hall Effect Murakami, Nagaosa, S. Zhang, S.-C. Zhang, Y.-S. Wu, etc.

- optical Hall effect Onoda, Murakami, Nagaosa

(18-19) derive from Niu Lagrangian

$$\Big(p_i - eA_i(\mathbf{x},t)\Big)\dot{x}^i - \Big(\epsilon_n(\mathbf{p}) - eV(\mathbf{x},t)\Big) + \mathcal{A}^i(\mathbf{p})\dot{p}_i \tag{20}$$

• Also consistent with Hamiltonian structure

$$\{x^i, x^j\}^{Bloch} = \frac{\varepsilon^{ijk}\Omega_k}{1+e\mathbf{B}\cdot\vec{\Omega}}, \tag{21}$$

$$\{x^i, p_j\}^{Bloch} = \frac{\delta^i{}_j + eB^i\Omega_j}{1+e\mathbf{B}\cdot\vec{\Omega}}, \tag{22}$$

$$\{p_i, p_j\}^{Bloch} = -\frac{\varepsilon_{ijk}eB^k}{1+e\mathbf{B}\cdot\vec{\Omega}} \tag{23}$$

and Hamiltonian $h = \epsilon_n - eV$.

PB satisfies Jacobi identity !

• Restricted to plane, for $\epsilon_n(\mathbf{p}) = \mathbf{p}^2/2m$ and $\mathcal{A}_i = -(\theta/2)\epsilon_{ij}p_j$, eqns reduce to exotic equations (3). Niu Lagrangian (20) becomes "exotic" expression (4).

Exotic galilean symmetry lost if θ not constant. Critical case only possible for $\theta(\mathbf{p}) = \theta$ constant.

NON-COMMUTATIVE CHERN-SIMONS THEORY

Laughlin '1983: Fractional Quantum Hall Effect explained by motion of charged vortices. Such vortices arise as exact solutions in Chern-Simons field theory of matter coupled to abelian gauge field A_ν. NR Lagrangian

$$i\bar{\psi}D_t\psi - \frac{1}{2}\Big|\mathbf{D}\psi\Big|^2 + \mu\boxed{\Big(\frac{1}{2}\epsilon_{ij}\partial_t A_i A_j + A_t B\Big)} \tag{24}$$

where $D_\nu = \partial_\nu - ieA_\nu$, $\nu = t, i$.

Infinitesimal galilean boosts, implemented conventionally as

$$\delta^0\psi = i\mathbf{b}\cdot\mathbf{x}\,\psi - t\mathbf{b}\cdot\vec{\nabla}\psi, \tag{25}$$

$$\delta^0 A_i = -t\mathbf{b}\cdot\vec{\nabla}A_i, \tag{26}$$

$$\delta^0 A_t = -\mathbf{b}\cdot\mathbf{A} - t\mathbf{b}\cdot\vec{\nabla}A_t, \tag{27}$$

are generated by constants of the motion

$$\mathcal{G}_i^0 = t\mathcal{P}_i - \int x_i\,|\psi|^2\,d^2\mathbf{x}, \tag{28}$$

$$\mathcal{P}_i = \int\frac{1}{2i}\Big(\bar{\psi}\partial_i\psi - (\overline{\partial_i\psi})\psi\Big)d^2\mathbf{x} - \frac{\mu}{2}\int\epsilon_{jk}A_k\partial_i A_j d^2\mathbf{x}.$$

No "exotic" galilean symmetry: $\{\mathcal{G}_1^0, \mathcal{G}_2^0\} = 0$.

Non-commutative version constructed by replacing ordinary products with Moyal product

$$\Big(f \star g\Big)(x_1, x_2) =$$

$$\exp\left(i\frac{\theta}{2}\Big(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1}\Big)\right) f(x_1, x_2) g(y_1, y_2)\Big|_{\mathbf{x}=\vec{y}}$$

Classical Lagrangian formally still (24), but with

$$D_\mu \psi = \partial_\mu - ieA_\mu \star \psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie(A_\mu \star A_\nu) - A_\nu \star A_\mu)$$

+ non-Abelian Chern-Simons term

$$\frac{\mu}{2}\epsilon_{\mu\nu\sigma}\left(A_\mu \star \partial_\nu A_\sigma - \frac{2ie}{3} A_\mu \star A_\nu \star A_\sigma\right).$$

Variational equations

$$iD_t\psi + \frac{1}{2}\mathbf{D}^2\psi = 0, \quad (29)$$

$$\kappa E_i - e\epsilon_{ik} j^l{}_k = 0, \quad (30)$$

$$\kappa B + e\rho^l = 0, \quad (31)$$

where ρ^l and $\jmath^l$ *left density* and *left current*,

$$\rho^l = \psi \star \bar{\psi}, \qquad \jmath^l = \frac{1}{2i}\left(\mathbf{D}\psi \star \bar{\psi} - \psi \star (\overline{\mathbf{D}\psi})\right).$$

Schaposnik et al., Bak et al.'2000 : exact vortex solutions similar to Jackiw-Pi '1990.

Modified theory *not* invariant w. r. t. "ordinary" boosts. Galilean invariance restored by implementing as

$$\delta\psi = \psi \star (i\mathbf{b}\cdot\mathbf{x}) - t\mathbf{b}\cdot\vec{\nabla}\psi \tag{32}$$
$$= (i\mathbf{b}\cdot\mathbf{x})\psi - t\mathbf{b}\cdot\vec{\nabla}\psi + \frac{\theta}{2}\mathbf{b}\times\vec{\nabla}\psi$$

supplemented by (26)-(27). Then generators

$$\mathcal{G}_i = t\mathcal{P}_i - \int x_i\bar{\psi}\star\psi\, d^2\mathbf{x} \tag{33}$$

do satisfy the "exotic" relation (1) with

$$\kappa = -\theta \int |\psi|^2\, d^2\mathbf{x}. \tag{34}$$

- Scaling symmetry (Jackiw Pi '1991) broken !

CONCLUSION

Similar structures arose, independetly and around the same time (1995 – 2000) in Mathematical Physics ("exotic" galilean symmetry) and in Condensed Matter Physics (semiclassical Bloch electron). Both theories contain the crucial anomalous velocity term, first considered by Luttinger in 1954 and used to explain the Anomalous/Spin/Optical Hall effects.

This may well be the mathematical structure behind all Hall- type effects.

Q: Does exotic galilean symmetry physically exist ?

Q: can one derive exotic Galilean structure as Berry-phase ?