Generalized complex geometry and topological sigma-models

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Outline

- Review of $N = 2$ sigma-models and their topologically twisted versions
- $N = 2$ sigma-models with H-flux and generalized complex geometry
- Topological sigma-models with H-flux
- Discussion

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N=1 sigma-models

Let M be a manifold, g be a Riemannian metric on M, and B be a B-field.
Roughly speaking, a B-field is a 2-form on M. Let $H = dB$.
More precisely, a B-field is a pair $(\mathcal{B}, H)$, where $\mathcal{B}$ is a homomorphism $Z_2(M, \mathbb{Z}) \to U(1)$, and H is a closed 3-form on M such that on 2-boundaries one has

$$\mathcal{B}(\partial\alpha) = \exp\left(i \int_\alpha H\right)$$

To the triple (M, g, B) one can associate a classical 2d field theory with $N = (1, 1)$ SUSY (the sigma-model).

N=2 sigma-models

If $H = dB = 0$, and (M, g) is a Kähler manifold with complex structure I, then the sigma-model has $N = (2, 2)$ SUSY.

Suppose $c_1(M) = 0$. Then the quantized sigma-model is an $N = (2, 2)$ Super-Conformal Field Theory (SCFT).

M is called a Calabi-Yau manifold if K_M is holomorphically trivial. CY $\implies c_1(M) = 0$.

The CY condition is required to define the topologically twisted version of the theory (B-model).

Topological twist

If M is Kähler, one can twist the SCFT to a topological field theory (TFT) known as the A-model. The A-model is an invariant of the symplectic structure on M.

If M is also a Calabi-Yau, there is another twist, which gives another 2d TFT, known as the B-model. It is an invariant of the complex structure on M.

Roughly speaking, twisting amounts to truncating the space of states and operators to a cohomology of a certain operator (the BRST operator).

Topological Field Theory

TFT is essentially the same as a supercommutative Frobenius algebra, i.e. a unital supercommutative algebra $\mathcal{A}$ with an invariant metric:

$$\langle a, bc\rangle = \langle ab, c\rangle$$

Instead of specifying the metric, it is convenient to specify the trace function on $\mathcal{A}$:

$$\mathrm{tr} : \mathcal{A} \to \mathbb{C}, \quad a \mapsto \langle 1, a\rangle$$

The A-model

For the A-model, $\mathcal{A}$ is the quantum cohomology ring of M, which is a deformation of the De Rham cohomology ring

$$\oplus_p H^p(M, \mathbb{C}).$$

The deformed product depends on the symplectic structure $\omega = gI$. The trace is given by

$$\mathrm{tr} : a \mapsto \int_M a, \quad \forall a \in H^\bullet(M).$$

The B-model

For the B-model, $\mathcal{A}$ is the sum of Dolbeault cohomology groups:

$$\mathcal{A} = \oplus_{p,q} H^p(\Lambda^q TX^{1,0}).$$

Let Ω be a holomorphic trivialization of the canonical line bundle of M. The trace function is given by

$$\mathrm{tr} : a \mapsto \int_M \Omega \wedge \iota_a \Omega, \quad \forall a \in H^\bullet(\Lambda^\bullet TX).$$

It is nonvanishing only for $p = q = n$.

N=2 sigma-models with H-flux

If $H \neq 0$, M does not have to be Kähler for the sigma-model to have $N = 2$ SUSY.

Necessary and sufficient conditions for $N = 2$ SUSY (Gates, Hull, Rocek, 1984):

- Two complex structures I_+, I_- such that g is Hermitian with respect to both.
- $\nabla_\pm I_\pm = 0$, where

$$\nabla_\pm = \nabla_{LC} \pm \frac{1}{2} g^{-1} H.$$

Formulation of the problem

- What is the analogue of the Calabi-Yau condition?
- What is the geometric meaning of the corresponding TFTs?
- What is the geometric significance of topological D-branes?

Generalized CY condition I

It is easy to find the generalization of the weak form of the CY condition. For the A-model (resp. B-model) it is the condition of the vanishing of the anomaly in the vector (resp. axial) R-current.

Anomaly in the left-moving (resp. right-moving) R-current is $c_1(M, I_+)$ (resp. $-c_1(M, I_-)$).

$$c_1(I_+) - c_1(I_-) = 0 \implies \mathrm{A-twist\ exists}$$

$$c_1(I_+) + c_1(I_-) = 0 \implies \mathrm{B-twist\ exists}$$

The BRST cohomology

The theory has four supercharges: $Q_+, Q_-, \bar{Q}_+, \bar{Q}_-$.

A-model BRST operator:

$$Q_A = Q_+ + \bar{Q}_-.$$

B-model BRST operator:

$$Q_B = Q_+ + \bar{Q}_+.$$

The BRST cohomology of operators:

$$\mathcal{A} = \mathrm{ker}[Q, \bullet]/\mathrm{im}[Q, \bullet].$$

The BRST cohomology of any twisted N=2 theory is a supercommutative ring.

The Dorfman-Courant bracket

If M is a smooth manifold, one defines a bilinear operation on the sections of $TM \oplus T^*M$:

$$(X + \xi) \circ (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi.$$

This operation, known as the Dorfman bracket, is not skew-symmetric, but satisfies a sort of Jacobi identity. Its skew-symmetrization is known as the Courant bracket.

On $TM \oplus T^*M$ there is also a metric of signature (n, n):

$$q(X + \xi, Y + \eta) = X(\eta) + Y(\xi).$$

Twisted Dorfman bracket

Let H be a closed 3-form on M. One can define a twisted version of the Dorfman bracket:

$$(X+\xi)\circ_H(Y+\eta)=[X,Y]+\mathcal{L}_X\eta-\iota_Y d\xi+\iota_X\iota_Y H.$$

The (twisted) Dorfman bracket is invariant under automorphisms of $TM\oplus T^*M$ induced by closed 2-forms:

$$X+\xi\mapsto X+\xi+\iota_X B,\quad B\in\Omega^2_{cl}(M).$$

This is called a B-field transformation. It is a gauge symmetry of the classical sigma-model.

Generalized complex manifolds

A (twisted) generalized complex structure on M is an endomorphism $\mathcal{I}$ of $TM \oplus T^*M$ such that

- $\mathcal{I}^2 = -1$.
- $q(\mathcal{I}x, \mathcal{I}y) = q(x, y)$, i.e. q is of type $(1, 1)$.
- Let E be the eigenbundle of $\mathcal{I}$ with eigenvalue $-i$. E must be closed with respect to the (twisted) Dorfman bracket.

The last condition is the integrability condition for the GC structure $\mathcal{I}$.

This notion was introduced by N. Hitchin (2003).

Examples of GC structures

- If I is a complex structure on M, then

$$\mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & -I^{\vee} \end{pmatrix}$$

  is a GC structure.

- If ω is a symplectic form on M, then

$$\mathcal{I} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

  is a GC structure.

Generalized Kähler manifolds

A generalized Kähler (GK) structure on M is a pair of commuting GC structures $\mathcal{I}, \mathcal{J}$ such that the symmetric bilinear form

$$\mathcal{G} : (TM \oplus T^*M) \times (TM \oplus T^*M) \mapsto \mathbb{R},$$
$$\mathcal{G}(x, y) = q(\mathcal{I}x, \mathcal{J}y),$$

is positive-definite.

Rationale: if $\mathcal{I}$ is made from a complex structure I, and $\mathcal{J}$ is made from a symplectic form ω, then $(\mathcal{I}, \mathcal{J})$ define a GK structure iff (I, ω) define a Kähler structure.

The Gualtieri theorem (2003)

Let g, I_+, I_-, H be the geometric data on a manifold M satisfying the constraints imposed by N=2 SUSY. Let

$$\mathcal{I} = \frac{1}{2}\begin{pmatrix} I_+ + I_- & -\omega_+^{-1} + \omega_-^{-1} \\ \omega_+ - \omega_- & -I_+^\vee - I_-^\vee \end{pmatrix}$$

be an endomorphism of $TM \oplus T^*M$. Let $\mathcal{J}$ be another such tensor obtained from $\mathcal{I}$ by $I_- \to -I_-$, $\omega_- \to -\omega_-$. Then $\mathcal{I}$ and $\mathcal{J}$ define a twisted GK structure on M. Moreover, any GK structure on M arises in this way, up to a B-field transform.

Lie algebroids I

From a GC structure we get a complex Lie algebroid.

Definition 1 *A complex Lie algebroid over* M *is a triple* $(E, [\bullet, \bullet], a)$*, where* E *is a complex vector bundle over* M, $[\bullet, \bullet]$ *is a Lie bracket on* $\Gamma(E)$*, and* $a : E \longrightarrow TM_{\mathbb{C}}$ *is a Lie algebra homomorphism such that* $\forall s_1, s_2 \in \Gamma(E), \forall f \in C^\infty(M)$

$$[fs_1, s_2] = f[s_1, s_2] - a(s_2)(f) \cdot s_1.$$

A complex Lie algebroid can be thought of as a "generalized $TM_{\mathbb{C}}$".

Given a GC-structure $\mathcal{I}$ on M, let E be $\ker(\mathcal{I} + i)$. Then E is a complex Lie algebroid.

Lie algebroids II

An equivalent definition (A. Vaintrob, 1997):

Definition 2 *A complex Lie algebroid over M is a pair (E, Q), where E is a complex vector bundle over M, and Q is a degree-1 vector field on the graded supermanifold ΠE such that $\{Q, Q\} = 0$.*

Relation between definitions: let e_α be a local trivialization of E, $[e_\alpha, e_\beta] = c^\gamma_{\alpha\beta} e_\gamma$, and $a(e_\alpha) = a^i_\alpha \partial_i$. Let θ^α be fermionic coordinates on ΠE. Let

$$Q = a^i_\alpha \theta^\alpha \partial_i + c^\gamma_{\alpha\beta} \theta^\alpha \theta^\beta \frac{\partial}{\partial \theta^\gamma}.$$

Then E is a Lie algebroid iff $\{Q, Q\} = 0$.

Lie algebroid cohomology

Since $C^\infty(\Pi E) \simeq \Gamma(\Lambda^\bullet E^\vee)$, we can associate to any Lie algebroid a complex

$$Q : \Gamma(\Lambda^p E^\vee) \to \Gamma(\Lambda^{p+1} E^\vee).$$

Its cohomology will be called Lie algebroid cohomology.

Examples: if $E = TM_{\mathbb{C}}$, then this is the usual De Rham complex. If M is complex and $E = TM^{0,1}$, then this is the complex $(\Omega^{0,\bullet}(M), \bar\partial)$.

This illustrates a general principle: any natural construction on TM makes sense for arbitrary Lie algebroids.

Generalized chiral ring

Computing the BRST-cohomology of operators for the generalized B-model, one gets the following result (A.K. and Yi Li, 2004):

Theorem 1 *On the classical level, the algebra of the generalized B-model is isomorphic to the Lie algebroid cohomology of* $E = \ker(\mathcal{I} + i)$*. For the generalized A-model, one has to replace* $\mathcal{I}$ *with* $\mathcal{J}$*.*

This suggests that the generalized B-model (resp. A-model) depends only on the GC structure $\mathcal{I}$ (resp. $\mathcal{J}$). This will be confirmed by the computation of the Frobenius trace (see below).

Differential forms as spinors

Recall that $TM \oplus T^*M$ has a natural metric q, so it generates a bundle of Clifford algebras. Using the natural polarization, its spinor bundle can be identified with

$$\oplus_p \Lambda^p T^*M.$$

An element $X + \xi \in TM \oplus T^*M$ acts on a form α as follows:

$$\alpha \mapsto \iota_X \alpha + \xi \wedge \alpha$$

Forms on a GC manifold

On a GC manifold, $TM_{\mathbb{C}} \oplus T^*M_{\mathbb{C}}$ has an isotropic subbundle E. Let U_0 be a subbundle of $\Lambda^\bullet(T^*M_{\mathbb{C}})$ defined by the condition that it is annihilated by any element of E.

We can think of elements of $E^\vee$ as "creation operators", and elements of E as "annihilation operators." Then U_0 is the "vacuum bundle", so it must be a line bundle. We can decompose

$$\Omega^\bullet(M) = U_0 \oplus U_1 \oplus \ldots \oplus U_{2n},$$

where U_k consists of forms obtained from U_0 by the action of k elements of $E^\vee$. The new grading is not compatible with the wedge product of forms, in general.

Generalized Dolbeault complex

Let π_k be a projection to U_k. Let

$$\bar{\partial}_E = \pi_{k+1} \circ (d - H) \circ \pi_k, \quad \partial_E = \pi_{k-1} \circ (d - H) \circ \pi_k$$

M. Gualtieri proved

Theorem 2 *The integrability of $\mathcal{I}$ is equivalent to* $d - H = \partial_E + \bar{\partial}_E$.

Thus $\Omega^\bullet(M)$ equipped with the new grading and the differential $\bar{\partial}_E$ becomes a complex.

If $\mathcal{I}$ came from a complex structure I, then

$$U_k = \oplus_{p-q=k} \Omega^{p,q}(M),$$

and $\bar{\partial}_E$ is the usual $\bar{\partial}$.

Generalized CY condition II

Cancellation of R-anomaly requires

$$c_1(I_+) + c_1(I_-) = -c_1(E) = 0.$$

Equivalently, this condition says that U_0 is topologically trivial.

Definition 3 *(Hitchin/Gualtieri) The GC manifold is called a generalized Calabi-Yau if there exists a trivialization Ω of the line bundle U_0 satisfying*

$$\bar{\partial}_E \Omega = 0.$$

Equivalently, one may require $(d - H)\Omega = 0$.

Generalized B-model I

From the physical viewpoint, gen. CY condition for $\mathcal{I}$ ensures the existence of a BRST-invariant measure in the path-integral for the generalized B-model. If it is satisfied, we can write down a formula for the Frobenius trace on the Lie algebroid cohomology.

Let Ω be the "generalized-holomorphic" section of U_0, as above. Consider the following automorphism of $TM \oplus T^*M$:

$$p : (X, \xi) \mapsto (X, -\xi), \quad \forall X \in \Gamma(TM), \forall \xi \in \Gamma(T^*M)$$

It takes the Dorfman bracket twisted by H to the Dorfman bracket twisted by $-H$. It also takes the H-twisted GC structure $\mathcal{I}$ and the corresponding form Ω to $-H$-twisted $\mathcal{I}', \Omega'$.

Generalized B-model II

Let ρ be a section of $\Lambda^\bullet(E^\vee)$ annihilated by Q. It can be shown (A.K. and Yi Li, 2004) that its Frobenius trace is given by

$$\mathrm{tr}(\rho) = \int_M \Omega' \wedge \rho \cdot \Omega,$$

where ρ acts on forms via the spinor representation. If $I_+ = I_-$, $H = 0$, then

- $E^\vee = T^*M^{0,1} \oplus TM^{1,0}, Q = \bar{\partial},$
- U_0 is the canonical line bundle of M,

so the above formula reduces to the usual one for the B-model.

Generalized B-model III

If $I_+ = -I_-$, $H = 0$, then the generalized B-model is equivalent to the usual A-model. One has

$$\Omega = e^{i\omega}, \quad \Omega' = e^{-i\omega}.$$

The Lie algebroid E has the form

$$E = \{X - i \cdot \iota_X \omega | X \in TM\}$$

It is isomorphic to the standard Lie algebroid $TM_{\mathbb{C}}$ via

$$X - i \cdot \iota_X \omega \mapsto X.$$

The same map takes $\bar{\partial}_E$ to d, and Ω and Ω' to 1, giving the usual formulas for the A-model.

Discussion

- To any GK manifold $(M, \mathcal{I}, \mathcal{J})$ satisfying the generalized CY condition with respect to $\mathcal{I}$ one can associate a TFT whose space of states $\mathcal{A}$ is the Lie algebroid cohomology of $\mathcal{I}$.
- The ring structure on $\mathcal{A}$ coming from physics may differ from the "classical" one by corrections coming from "generalized holomorphic instantons."
- There is evidence that this TFT does not depend on $\mathcal{J}$, but this has not been proved yet.
- Presumably, this TFT is the Hochschild cohomology of some category of D-branes associated to any GC manifold.