Orthogonal Arrays Obtained By Generalized Kronecker Product

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Abstract

In this paper, we propose a new general approach to construct asymmetrical orthogonal arrays, namely generalized Kronecker product. The operation is not usual Kronecker product in the theory of matrices but it is interesting since the interaction of two columns of asymmetrical orthogonal arrays can be often written out by the generalized Kronecker product. As an application of the method, some new mixed-level orthogonal arrays of run sizes 72 and 96 are constructed.
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1. Introduction

An $n \times m$ matrix A, having k_i columns with p_i levels, $i = 1, \ldots, t, t$ is an integer, $m = \sum_{i=1}^{t} k_i, p_i \neq p_j$, for $i \neq j$, is called an orthogonal array (OA) of strength d and size n if each $n \times d$ submatrix of A contains all possible $1 \times d$ row vectors with the same frequency. Unless stated otherwise, we consider an orthogonal array of strength 2, using the notation $L_n(p_1^{k_1} \cdots p_t^{k_t})$ for such an array. An orthogonal array is said to be mixed level (or asymmetrical) if $t \geq 2$.The proceeding definition also includes the case $t = 1$, and the array is usually called a symmetrical orthogonal array, denoted by $L_n(p^m)$. For simplicity, the symmetrical and asymmetrical will only be used when needed.

An essential concept for the construction of asymmetrical orthogonal arrays is that of difference matrices. Using the notation for additive (or Abelian) groups, a difference matrix(or difference scheme) with level p is an $\lambda p \times m$ matrix with the entries from a finite additive group G of order p such that the vector differences of any two columns of the array, say $d_i - d_j$ if $i \neq j$, contain every element of G exactly λ times. We will denote such an array by $D(\lambda p, m; p)$, although this notation suppresses the relevance of the group G. In most of our examples, G will correspond to the additive group associated with a Galois field $GF(p)$. The difference matrix $D(\lambda p, m; p)$ is called a **generalized Hadamard matrix** if $\lambda p = m$. In particular, $D(\lambda 2, \lambda 2; 2)$ is the usual Hadamard matrix.

If a $D(\lambda p, m; p)$ exists, it can always be constructed so that only one of its rows and one of its columns contain the zero element of G. Deleting this column from $D(\lambda p, m; p)$, we obtain a difference matrix, denoted by $D^0(\lambda p, m-1; p)$, called **an atom of difference**

matrix $D(\lambda p, m; p)$ or **an atomic difference matrix**. Without loss of generality, the matrix $D(\lambda p, m; p)$ can be written as

$$D(\lambda p, m; p) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = (0 \quad D^0(\lambda p, m-1; p)).$$

The property is important for the following discussions.

For two matrices $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{s\times t}$ both with the entries from group G, define their **Kronecker sum** (Shrikhande 1964) to be

$$A \oplus B = (a_{ij} \oplus B)_{1\le i\le n, 1\le j\le m},$$

where each sub-matrix $a_{ij} \oplus B$ of $A \oplus B$ is obtained by adding a_{ij} to each entry of B. Shrikhande (1964) showed that $A \oplus B$ is a difference matrix if both A and B are difference matrices. And, Zhang(1993) showed that A is a difference matrix if both $A \oplus B$ and B are difference matrices.

Nowadays orthogonal arrays have many important application in statistics, and they play important roles in coding theory and cryptography.The current emphasis is in the area of asymmetrical factorial design, or namely asymmetrical orthogonal arrays. Many new construction methods on the asymmetrical orthogonal arrays have been proposed, but most of these methods are only or mainly from the construction of symmetrical orthogonal arrays. Refer to the book by Hedayat et al (1999) for further references.

A new theory or procedure of constructing asymmetrical orthogonal arrays by using the orthogonal decompositions of projection matrices has been given by Zhang, Lu and Pang (1999). The first who used it with this objective was Zhang(1989), (1990a), (1990b), (1991a), (1991b), (1992)and (1993). The idea comes from the theory of multilateral matrices – a mathematical technique to solve the problems of system with complexity. In general, the procedure of constructing asymmetrical orthogonal arrays in our theory has been partitioned mainly into five parties: orthogonal-array addition, subtraction, multiplication, division and replacement. The technique, namely generalized Kronecker product (Definition 1) which belongs to the orthogonal-array multiplication class, has also been proposed for the construction of asymmetrical orthogonal arrays by Zhang(1993) in the theory of multilateral matrices. In this paper the technique will be further explained and extended to construct some new asymmetrical (or mixed-level) orthogonal arrays by using the orthogonal decompositions of projection matrices.

Section 2 contains the basic concepts and main theorems while in Section 3 we describe the method of constructions. Some new mixed level OA's with run sizes 72 and 96 are constructed in Section 4.

2. Basic Concepts and Main Theorems

In our procedure, an important idea is to find the relationship among difference matrices, projection matrices and permutation matrices. The following notations are used.

Let 1_r be the $r \times 1$ vector of 1's, 0_r the $r \times 1$ vector of 0's, I_r the identity matrix of order r and $J_{r,s}$ the $r \times s$ matrix of 1's, also $J_r = J_{r,r}$. Of course, the two matrices $P_r = (1/r)1_r 1_r^T = (1/r)J_r$ and $\tau_r = I_r - P_r$ are projection matrices.

Define

$$(r) = (0, \dots, r-1)^T_{1\times r}, e_i(r) = (0 \cdots 0 \overset{i}{1} 0 \cdots 0)^T_{1\times r},$$

where $e_i(r)$ is the base vector of R^r (r-dim vector space) for any i. We can construct two permutation matrices as follows:

$$N_r = e_1(r)e_2^T(r) + \cdots + e_{r-1}(r)e_r^T(r) + e_r(r)e_1^T(r)$$

and

$$K(p,q) = \sum_{i=1}^{p}\sum_{j=1}^{q} e_i(p)e_j^T(q) \otimes e_j(q)e_i^T(p), \tag{1}$$

where $\otimes$ is the usual Kronecker product in the theory of matrices. The permutation matrices N_r and $K(p,q)$ have the following properties:

$$N_r(r) = 1 \oplus (r), \ \bmod r, \ \text{and}\ K(p,\lambda p)((\lambda p) \oplus (p)) = (p) \oplus (\lambda p).$$

Let $D = (d_{ij})_{\lambda p \times m}$ be a matrix over an additive group G of order p. Then for any given $d_{ij} \in G$ there exists a permutation matrix $\sigma(d_{ij})$ such that

$$\sigma(d_{ij})(p) = d_{ij} \oplus (p),$$

where the vector (p) is with entries from G. Define $H(\lambda p, m; p) = (\sigma(d_{ij}))_{\lambda p^2 \times mp}$, where each entry or submatrix $\sigma(d_{ij})$ of $H(\lambda p, m; p)$ is a $p \times p$ permutation matrix. And Zhang (1993) has proved that the matrix $D = (d_{ij})_{\lambda p \times m}$ over the additive group G is a difference matrix $D(\lambda p, m; p)$ if and only if

$$H^T(\lambda p, m; p)H(\lambda p, m; p) = \lambda p(I_m \otimes \tau_p + J_m \otimes P_p).$$

On the other hand, the permutation matrices $\sigma(d_{ij})$ are often obtained by the permutation matrices N_r and $K(p,q)$. Furthermore, by the permutation matrices $\sigma(d_{ij})$ and $K(\lambda p, p)$, the Kronecker sum (Shrikhande 1964) of difference matrices can be written as

$$\begin{aligned}(p) \oplus D(\lambda p, m; p) = \ & K(p,\lambda p)[D(\lambda p, m; p) \oplus (p)] \\ = \ & K(p,\lambda p)(\sigma(d_{ij})(p))_{\lambda p^2 \times m} \\ = \ & K(p,\lambda p)(S_1(0_{\lambda p} \oplus (p)), \ldots, S_m(0_{\lambda p} \oplus (p))) \\ = \ & (Q_1((p) \oplus 0_{\lambda p}), \ldots, Q_m((p) \oplus 0_{\lambda p})),\end{aligned}$$

where

$$Q_j = K(p,\lambda p)S_jK(p,\lambda p)^T, S_j = \ \text{diag}(\sigma(d_{1j}), \ldots, \sigma(d_{rj})), (r = \lambda p), \tag{2}$$

are permutation matrices for any $j = 1, \ldots, m$ and where $0_{\lambda p} \oplus (p) = 1_{\lambda p} \otimes (p)$ holds for the additive group associated with Galois Field GF(p). Therefore, both the projection matrices P_r and τ_r and the permutation matrices $N_r, K(p,q), Q_j$ and S_j(defined in (1) and (2)) are often used to construct the asymmetrical orthogonal arrays in our procedure.

Definition 1. Let $k(x,y)$ be a map from $\Omega_1 \times \Omega_2$ to V, where $\Omega_1 \times \Omega_2 = \{(x,y) : x \in \Omega_1, y \in \Omega_2\}$ and Ω_1, Ω_2, V are some sets. For two matrices $A = (a_{ij})_{n \times m}$ with entries from Ω_1 and $B = (b_{uv})_{s \times t}$ with entries from Ω_2, define their **generalized Kronecker product**, denoted by $\overset{k}{\otimes}$, as follows

$$A \overset{k}{\otimes} B = (k(a_{ij}, b_{uv}))_{ns \times mt} = (k(a_{ij}, B))_{1 \le i \le n, 1 \le j \le m},$$

where each submatrix $k(a_{ij}, B) = (k(a_{ij}, b_{uv}))_{s\times t}$ of $A \overset{k}{\otimes} B$ is obtained by operating a_{ij} to each entry of B under the map $k(x, y)$.

Unless stated otherwise, we consider the sets Ω_1 and Ω_2 to be finite, using the vector notations $(p) = (0, 1, \cdots, p-1)^T$ and $(q) = (0, 1, \cdots, q-1)^T$ for two example sets. When V is a row-vector space of m-dimensions, the map $k(i, j)$ can be represented by a $pq \times m$ matrix D, i.e.,

$$k : (p) \overset{k}{\otimes} (q) = D = (d_{(1)}, \ldots, d_{(pq)})^T,$$

with $k(i, j) = d^T_{(iq+j+1)}$(or $k(i, j)$ is the $(iq+j+1)$th row of D). For this case in the following discussions, the generalized Kronecker product $\overset{k}{\otimes}$ will only be defined as $(p) \overset{k}{\otimes} (q) = D$.

Let $\Omega_1 = \Omega_2 = V = G$ (a finite multiplicative group) and $k(i, j) = ij$. Then the generalized Kronecker product $\overset{k}{\otimes}$ is really the usual Kronecker product in the theory of matrices, denoted by $\otimes$. Using the notation for a finite additive (or Abelian) group G, i.e., let $\Omega_1 = \Omega_2 = V = G$ (a finite additive group) and $k(i, j) = i + j$, the generalized Kronecker product $\overset{k}{\otimes}$ will be the usual Kronecker sum (Shrikhande (1964)), denoted by $\oplus$.

Furthermore, if the Ω_1, Ω_2 and V are additive (or abelian) groups G_1, G_2 of order λp, p and a row-vector space of m-dimensions respectively, and if $k(i, j)$ is the $(ip + j + 1)$th row of $D^0(\lambda p, m-1; p) \oplus (p)$(i.e., the usual Kronecker sum $\oplus$ of $D^0(\lambda p, m-1; p)$ and (p) (Shrikhande 1964)), the generalized Kronecker product $\overset{k}{\otimes}$ is really denoted by $(\lambda p) \overset{k}{\otimes} (p) = D^0(\lambda p, m-1; p) \oplus (p)$, namely **normal Kronecker sum**.

In general, if the Ω_1, Ω_2 and V are additive (or abelian) groups G_1, G_2 of order p, q and a row-vector space of m-dimensions respectively, and if $k(i, j)$ is the $(iq + j + 1)$th row of L(an orthogonal array) for any i, j, the generalized Kronecker product $\overset{k}{\otimes}$ can be only defined as $(p) \overset{k}{\otimes} (q) = L$, namely an **orthogonal-array product**.

The generalized Kronecker product $\overset{k}{\otimes}$ has many properties similar to the usual Kronecker product $\otimes$ and the Kronecker sum $\oplus$(Shrihande 1964). Such as

$$K(p, q) \cdot (p) \overset{k}{\otimes} (q) = (q) \overset{k}{\otimes} (p), (\text{ if } k(i, j) = k(j, i) \text{ is a row vector}),$$

$$(0_q \oplus a) \overset{k}{\otimes} (p) = 0_q \oplus [a \overset{k}{\otimes} (p)],$$

$$(a, b) \overset{k}{\otimes} (p) = [a \overset{k}{\otimes} (p), b \overset{k}{\otimes} (p)].$$

The notations $\otimes, \oplus, \overset{k}{\otimes}$ are very useful for the construction of asymmetrical orthogonal arrays and many other designs.

For example, if define

$$(2) \overset{k}{\otimes} (2) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad (4) \overset{k}{\otimes} (2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus (2),$$

and

$$(3) \overset{k}{\otimes} (3) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \oplus (3), \quad (6) \overset{k}{\otimes} (3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 & 1 \end{pmatrix} \oplus (3),$$

then the following arrays

$$((2) \oplus 0_4, 0_2 \oplus (4)) \overset{k}{\otimes} (2) = (((2) \oplus 0_2) \overset{k}{\otimes} (2), (0_2 \oplus (4)) \overset{k}{\otimes} (2)),$$

$$((3) \oplus 0_6, 0_3 \oplus (6)) \overset{k}{\otimes} (3) = (((3) \oplus 0_6) \overset{k}{\otimes} (3), (0_3 \oplus (6)) \overset{k}{\otimes} (3)),$$

are all orthogonal arrays (Theorem 5).

The array product is an essential operation of the generalized Kronecker product for constructing asymmetrical arrays.

Definition 2. Let A be an orthogonal array of strength 1, i.e.,

$$A = (a_1, \dots, a_m) = (T_1(0_{r_1} \oplus (p_1)), \dots, T_m(0_{r_m} \oplus (p_m))),$$

where $r_i p_i = n, T_i$ is a permutation matrix for any $i = 1, \dots, m$. The following projection matrix,

$$A_j = T_j(P_{r_j} \otimes \tau_{p_j})T_j^T, \tag{3}$$

is called the **matrix image** (MI) of the jth column a_j of A, denoted by $m(a_j) = A_j$ for $j = 1, \dots, m$. In general,the MI of a subarray of A is defined as the sum of the MI's of all its columns. In particular, we denote the MI of A by $m(A)$.

In Definition 2, for a given column $a_j = T_j(0_{r_j} \oplus (p_j))$, the matrices A_j defined in equation (3) are unique though the permutation matrix T_j introduced here is not unique.

If a design is an orthogonal array, then the MI's of its columns has some interesting properties which can be used to construct orthogonal arrays. For example, by the definition, we have

$$m(0_r) = P_r \text{ and } m((r)) = \tau_r.$$

Theorem 1. For any permutation matrix T and any orthogonal array L with strength at least one, we have

$$m(T(L \oplus 0_r)) = T(m(L) \otimes P_r)T^T \text{ and } m(T(0_r \oplus L)) = T(P_r \otimes m(L))T^T.$$

Theorem 2. Let the array A be an orthogonal array of strength 1, i.e.,

$$A = (a_1, \dots, a_m) = (T_1(0_{r_1} \oplus (p_1)), \dots, T_m(0_{r_m} \oplus (p_m))),$$

where $r_i p_i = n, T_i$ is a permutation matrix, for $i = 1, \dots, m$.

The following statements are equivalent.

(1). A is an orthogonal array of strength 2.

(2). The MI of A is a projection matrix.

(3). The MI's of any two columns of A are orthogonal, i.e $m(a_i)m(a_j) = 0 (i \neq j)$.

(4). The projection matrix τ_n can be decomposed as

$$\tau_n = m(a_1) + \ldots + m(a_m) + \triangle,$$

where $rk(\triangle) = n - 1 - \sum_{j=1}^{m} (p_j - 1)$ is the rank of the matrix $\triangle$.

Definition 3. An orthogonal array A is said to be saturated if $\sum_{j=1}^{m} (p_j - 1) = n - 1$ (or,equivalently, $m(A) = \tau_n$).

Corollary 1. Let (L,H) and K be orthogonal arrays of run size n. Then (K,H) is an orthogonal array if $m(K) \leq m(L)$, where $m(K) \leq m(L)$ means that the difference $m(L) - m(K)$ is nonnegative definite.

Corollary 2. Suppose L and H are orthogonal arrays. Then $K = (L,H)$ is also an orthogonal array if $m(L)$ and $m(H)$ are orthogonal, i.e., $m(L)m(H) = 0$. In this case $m(K) = m(L) + m(H)$.

By Corollaries 1 and 2, in order to construct an orthogonal array L_n of run size n, we should decompose the projection matrix τ_n into $C_1 + \cdots + C_k$ such that $C_iC_j = 0$ for $i \neq j$ and find orthogonal arrays H_j such that $m(H_j) \leq C_j$ for $j = 1, 2, \cdots, k$, because the array $L_n = (H_1, \cdots, H_k)$ is an orthogonal array of run size n. The method of constructing orthogonal arrays by using the orthogonal decompositions of projection matrices is also called **orthogonal-array addition** (Zhang, Lu and Pang 1999).

Definition 4. An orthogonal array L_n is called satisfactory if there doesn't exist any orthogonal array K such that (L_n, K) is an orthogonal array.

Theorem 3. (Optimality) Let p, q and r be integers satisfying $p, q \geq 2, n = pqr$ and $(p,q) = 1$ where $(p,q) = 1$ means the maximal common divisor of p and q is 1. Then there is no any orthogonal array K of run size n such that $m(K) \leq \tau_p \otimes I_r \otimes \tau_q$.

These theorems and corollaries can be found in Zhang et al (1991b,1992,1993,1999,2001) and Pang, Liu and Zhang (2002).

The following Theorem is a main result in our procedure of generalized Kronecker product for constructing the asymmetrical orthogonal arrays.

Theorem 4. Let $D^0(\lambda p, m-1; p)$ be an atom of difference matrix $D(\lambda p, m; p)$. Then $D^0(\lambda p, m-1; p) \oplus (p)$ is an orthogonal array whose MI (defined in (3)) is less than or equal to $\tau_{\lambda p} \otimes \tau_p$.

Proof. From (3),Theorems 1 and 2 and the construction of Bose and Bush(1952), we have that

$$m[(0_r \oplus (p), D^0(r, m-1; p) \oplus (p)] = P_r \otimes \tau_p + m(D^0(r, m-1; p) \oplus (p))$$

$$= P_r \otimes \tau_p + S_1(P_r \otimes \tau_p)S_1^T + \ldots + S_m(P_r \otimes \tau_p)S_m^T$$

is a projection matrix where the atomic difference matrix $D^0(r, m-1; p) = (d_{ij})_{r\times(m-1)}, S_j = \mathrm{diag}(\sigma(d_{1j}), \ldots, \sigma(d_{rj})), r = \lambda p$, is defined in (2) and the MI's of any two columns of $(0_r \oplus (p), D^0(r, m-1; p) \oplus (p))$ are orthogonal,i.e.,

$$(P_r \otimes \tau_p)S_j(P_r \otimes \tau_p)S_j^T = 0 \text{ and } S_i(P_r \otimes \tau_p)S_i^T S_j(P_r \otimes \tau_p)S_j^T = 0, \quad (i \neq j).$$

Thus we now only need to prove that

$$S_j(P_r \otimes \tau_p)S_j^T \leq I_r \otimes \tau_p,$$

since $m(D^0(r, m-1; p) \oplus (p)) \leq I_r \otimes \tau_p - P_r \otimes \tau_p = \tau_r \otimes \tau_p$. In fact,by the matrix properties $P_r \otimes \tau_p \leq I_r \otimes \tau_p$, we have

$$\begin{aligned} S_j(P_r \otimes \tau_p)S_j^T \leq & \; S_j(I_r \otimes \tau_p)S_j^T \\ = & \; \mathrm{diag}(\sigma(d_{1j})\tau_p\sigma(d_{1j})^T, \ldots, \sigma(d_{rj})\tau_p\sigma(d_{rj})^T) \\ = & \; \mathrm{diag}(\tau_p, \ldots, \tau_p) = I_r \otimes \tau_p. \end{aligned}$$

This completes the proof.

Definition 5. Let $L_{n_1} = [L_{n_1}(p_1^{x_1}), \ldots, L_{n_1}(p_s^{x_s})]$ and $L_{n_2} = [L_{n_2}(q_1^{y_1}), \ldots, L_{n_2}(q_t^{y_t})]$ be two orthogonal arrays. If for given i, j the map $k_{ij}(s,t)$ of generalized Kronecker product $\overset{k_{ij}}{\otimes}$ is $k_{ij} : (p_i) \overset{k_{ij}}{\otimes} (q_j) = H_{ij}$ (an orthogonal array) such that $m(H_{ij}) \leq \tau_{p_i} \otimes \tau_{q_j}$, then we define the **orthogonal-array product** of L_{n_1} and L_{n_2} as

$$L_{n_1} \overset{K}{\otimes} L_{n_2} = [\ldots, L_{n_1}(p_i^{x_i}) \overset{k_{ij}}{\otimes} L_{n_2}(q_j^{y_j}), \ldots],$$

where $K = \{k_{ij}; i = 1, 2, \ldots, s, j = 1, 2, \ldots, t\}$.

Theorem 5. Suppose that

$$L_{n_1} = [L_{n_1}(p_1^{x_1}), \ldots, L_{n_1}(p_s^{x_s})]$$

and

$$L_{n_2} = [L_{n_2}(q_1^{y_1}), \ldots, L_{n_2}(q_t^{y_t})]$$

are two orthogonal arrays. Then the array product of L_{n_1} and L_{n_2} , i.e., $L_{n_1} \overset{K}{\otimes} L_{n_2}$, is also orthogonal array whose MI is less than or equal to $m(L_{n_1}) \otimes m(L_{n_2})$.

Proof. Without loss of generality, the orthogonal arrays L_{n_1} and L_{n_2} can be written as

$$L_{n_1} = [S_1(0_{r_1} \oplus (p_1)), \ldots, S_{m_1}(0_{r_{m_1}} \oplus (p_{m_1}))]$$

and

$$L_{n_2} = [Q_1((q_1) \oplus 0_{t_1}), \ldots, Q_{m_2}((q_{m_2}) \oplus 0_{t_{m_2}})]$$

where $r_i p_i = n_1, t_j q_j = n_2$, and S_j, Q_j are permutation matrices for any i, j. By Theorems 1 and 2, we have

$$m(L_{n_1}) \otimes m(L_{n_2}) = [\sum_{i=1}^{m_1} S_i(P_{r_i} \otimes \tau_{p_i}) S_i^T] \otimes [\sum_{j=1}^{m_2} Q_j(\tau_{q_j} \otimes P_{t_j}) Q_j^T]$$

$$= \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} (S_i \otimes Q_j)(P_{r_i} \otimes \tau_{p_i} \otimes \tau_{q_j} \otimes P_{t_j})(S_i \otimes Q_j)^T$$

is an orthogonal decomposition of projection matrix $m(L_{n_1}) \otimes m(L_{n_2})$. If there exists an orthogonal array H_{ij} such that $m(H_{ij}) \leq \tau_{p_i} \otimes \tau_{q_j}$,i.e, $k_{ij} : (p_i) \overset{k_{ij}}{\otimes} (q_j) = H_{ij}$ for any i, j, then we have

$$(\ldots, (S_i \otimes Q_j)(0_{r_i} \oplus (p_i) \overset{k_{ij}}{\otimes} (q_j) \oplus 0_{t_j}), \ldots)$$

is an orthogonal array by Theorems 1 and 2 ,Corollary 1 and Definition 5. The proof is completed.

By Theorem 4, the orthogonal arrays H_{ij} in Definition 5 can be taken into

$$D^0(p_i, u_i; q_j) \oplus (q_j) \text{ or } (p_i) \oplus D^0(q_j, v_j; p_i),$$

For any i, j.

For example, in Definition 1, we can define

$$(2) \overset{k}{\otimes} (2) = D^0(2, 1; 2) \oplus (2), (4) \overset{k}{\otimes} (2) = D^0(4, 3; 2) \oplus (2),$$

$$(3) \overset{k}{\otimes} (3) = D^0(3, 2; 3) \oplus (3), (6) \overset{k}{\otimes} (3) = D^0(6, 5; 3) \oplus (3), \ldots,$$

where each of the above maps $k(i,j)$'s can be defined by the corresponding formula.

By Theorem 5, finding all orthogonal arrays H such that $m(H) \leq \tau_p \otimes \tau_q$ is also an essential operation of the generalized Kronecker product for constructing asymmetrical orthogonal arrays.If there exists an orthogonal array H such that $m(H) = \tau_p \otimes \tau_q$, then the orthogonal array H is called the **interaction** of two columns $(p \oplus 0_q)$ and $0_p \oplus (q)$. Thus the operation of finding the generalized Kronecker products is similar to that of finding the interactions in experiment designs.

3. General Methods for Constructing OA's by Generalized Kronecker Product

Our procedure of constructing mixed-level orthogonal arrays by using the generalized Kronecker product based on the orthogonal decomposition of the projection matrix τ_n consists of the following three steps:

Step 1. Orthogonally decompose the projection matrix τ_n :

$$\tau_n = T_1(A_1 \otimes B_1)T_1^T + \ldots + T_{k_1}(A_{k_1} \otimes B_{k_1})T_{k_1}^T + C_1 + \ldots + C_{k_2} + \triangle$$

where all $A_i, B_j, C_s, \triangle$ are projection matrices and all T_t are permutation matrices.

Step 2. Find orthogonal arrays H_i^1, H_j^2 and H_s from some known orthogonal arrays such that

$$m(H_i^1) \leq A_i, m(H_j^2) \leq B_j \text{ and } m(H_s) \leq C_s.$$

Step 3. Lay out the new orthogonal array L by Theorem 5,Corollaries 1 and 2:

$$L = (T_1(H_1^1 \overset{K_1}{\otimes} H_1^2), \ldots, T_{k_1}(H_{k_1}^1 \overset{K_{k_1}}{\otimes} H_{k_1}^2), H_1, \ldots, H_{k_2}),$$

where all $\overset{K_1}{\otimes}, \ldots, \overset{K_{k_1}}{\otimes}$ are orthogonal-array products.

In applying Step 1, the following orthogonal decomposition of τ_n is very useful,

$$\tau_{pq} = I_p \otimes \tau_q + \tau_p \otimes P_q = \tau_p \otimes P_q + P_p \otimes \tau_q + \tau_p \otimes \tau_q = \tau_p \otimes I_q + P_p \otimes \tau_q,$$

$$\tau_{prq} = \tau_p \otimes I_r \otimes P_q + P_p \otimes \tau_{rq} + \tau_p \otimes I_r \otimes \tau_q. \quad (4)$$

These equations are easy to verify from $\tau_p = I_p - P_p, P_{pq} = P_p \otimes P_q$ and $I_{pq} = I_p \otimes I_q$.

The following properties play very useful role in the procedure:

Corollary 3. (Two-factor method) Let L_p^1, L_p^2, L_q^1 and L_q^2 be orthogonal arrays.Then

$$(L_p^1 \oplus 0_q, 0_p \oplus L_q^1, L_p^2 \overset{K}{\otimes} L_q^2)$$

is an orthogonal array.

Proof. The proof follows from Theorem 5 and the orthogonal decomposition of τ_{pq}(in (4)):

$$\tau_{pq} = \tau_p \otimes P_q + P_p \otimes \tau_q + \tau_p \otimes \tau_q.$$

Corollary 4. (Three-factor method) Let $n = prq$ and let L_{pr}, L_{rq} and L_q be orthogonal arrays of run sizes pr, rq and q, respectively. If there exist orthogonal arrays $L_{pr}^{(-)}$,$L_{pr}^{(=)}$ and $L_{rq}^{(-)}$ such that $m(L_{pr}^{(-)}), m(L_{pr}^{(=)}) \leq \tau_p \otimes I_r$ and $m(L_{rq}^{(-)}) \leq I_r \otimes \tau_q$, then

$$[L_{pr} \oplus 0_q, 0_p \oplus L_{rq}^{(-)}, L_{pr}^{(=)} \overset{K}{\otimes} L_q]$$

and

$$[L_{pr}^{(-)} \oplus 0_q, 0_p \oplus L_{rq}, L_{pr}^{(=)} \overset{K}{\otimes} L_q]$$

are orthogonal arrays.

Proof. The proof follows from Theorem 5 and the orthogonal decompositions of τ_{prq}(in (4)):

$$\tau_{prq} = \tau_{pr} \otimes P_q + P_p \otimes [I_r \otimes \tau_q] + [\tau_p \otimes I_r] \otimes \tau_q$$

and

$$\tau_{prq} = [\tau_p \otimes I_r] \otimes P_q + P_p \otimes \tau_{rq} + [\tau_p \otimes I_r] \otimes \tau_q.$$

On Corollary 4 (Three factor method), it is useful to pay attention to that the two orthogonal arrays $L_{pr}^{(-)}$ and $L_{pr}^{(=)}$ in the second constructed array are not necessarily the same.

4. Constructions of OA's with Run Sizes 72 and 96

4.1. Construction of OA $L_{72}(2^{61}3^1 4^1)$

Since $72 = 18 \times 2 \times 2$, by Corollary 4 (Three-factor method), we have

$$[L_{36}^{(-)} \oplus 0_2, 0_{18} \oplus (4), L_{36}^{(=)}(2^{34}) \overset{k}{\otimes} (2)]$$

is an orthogonal array for any orthogonal arrays $L_{36}^{(-)}$ and $L_{36}^{(=)}(2^{34})$ such that $m(L_{36}^{(-)}) \leq \tau_{18} \otimes I_2$ and $m(L_{36}^{(=)}(2^{34})) = \tau_{18} \otimes I_2$.

Now we want to find an OA $L_{36}^{(=)}(2^{34})$ whose MI is equal to $\tau_{18} \otimes I_2$. Many forms of orthogonal array $L_{36}(2^{35})$ can be constructed such as Plackett and Burman (1946). Without loss of generality, the first column can always be supposed to be $0_{18} \oplus (2)$. Deleting the column $0_{18} \oplus (2)$ from $L_{36}(2^{35})$, we obtain an orthogonal array in Table 2, denoted by $L_{36}^{(=)}(2^{34})$, whose MI is equal to $\tau_{18} \otimes I_2$ since $\tau_{18} \otimes I_2 = \tau_{36} - P_{18} \otimes \tau_2$.

By Theorem 4, there exists a generalized Kronecker product $(2) \overset{k}{\otimes} (2) = (0\ 1\ 1\ 0)^T = (2) \oplus (2)$, mod 2, i.e., the Kronecker sum (Shrikhande 1964). Therefore by Theorem 5 the Kronecker sum $L_{36}^{(=)}(2^{34}) \oplus (2)$ is an OA whose MI is $\tau_{18} \otimes I_2 \otimes \tau_2$.

On the other hand, a new satisfactory orthogonal array $L_{36}(2^{28}3^1)$ which has a 2-level column $0_{18} \oplus (2)$ (in Table 2) can be obtained by an approach similar to that by Zhang et al (1999) through complicated computing. Deleting the column $0_{18} \oplus (2)$ from $L_{36}(2^{28}3^1)$, we obtain an orthogonal array, denoted by $L_{36}^{(-)}(2^{27}3^1)$, whose MI is less than $\tau_{18} \otimes I_2$ since $\tau_{18} \otimes I_2 = \tau_{36} - P_{18} \otimes \tau_2$.

By Corollary 4 (Three-factor method), we obtain an orthogonal array $L_{72}(2^{61}3^1 4^1)$ as follows:

$$L_{72}(2^{61}3^1 4^1) = [L_{36}^{(-)}(2^{27}3^1) \oplus 0_2, 0_{18} \oplus (4), L_{32}^{(=)}(2^{34}) \oplus (2)],$$

which is satisfactory since $\triangle = \tau_{72} - m(L_{72}(2^{61}3^1)) \leq (\tau_{36} - m(L_{36}(2^{28}3^1))) \otimes P_2$. This orthogonal array is new, which is not included in Hedayat et al (1999) and http://sas.techsup/technote/ts723.html maintained by Warren F.Kuhfeld yet.

Furthermore, replacing the orthogonal array $L_{36}(2^{28}3^1)$ by any one of orthogonal arrays $L_{36}(2^x \cdots)$ which has at least a 2-level column, we will able to construct an orthogonal array for this family. There are at least 3 new orthogonal arrays for this family which are included in Table 1.

4.2. Construction of OA $L_{72}(2^{28}3^{11}6^{1}12^{1})$

Since $72 = 12 \times 3 \times 2$, by Corollary 4 (Three - factor method), we have

$$[L_{36}^{(-)} \oplus 0_2, 0_{12} \oplus (6), L_{36}^{(=)}(2^{28}) \overset{k}{\otimes} (2)]$$

is an orthogonal array for any orthogonal arrays $L_{36}^{(-)}$ and $L_{36}^{(=)}(2^{28})$ such that $m(L_{36}^{(-)}) \leq \tau_{12} \otimes I_3$ and $m(L_{36}^{(=)}(2^{28})) \leq \tau_{12} \otimes I_3$.

Similarly to Section 4.1, we can find an OA $L_{36}^{(=)}(2^{28})$ from $L_{36}(3^{1}2^{28})$ (in Table 2) such that $m(L_{36}^{(=)}(2^{28})) \leq \tau_{12} \otimes I_3$. Thus the Kronecker sum $L_{36}^{(=)}(2^{28}) \oplus (2)$ is an OA whose MI is $\tau_{12} \otimes I_3 \otimes \tau_2$.

On the other hand, there is a saturated orthogonal array $L_{36}(3^{12}12^{1})$ which has a 3-level column $0_{12} \oplus (3)$ (Zhang et al 2001). Deleting the column $0_{12} \oplus (3)$ from $L_{36}(3^{12}12^{1})$, we obtain an orthogonal array, denoted by $L_{36}^{(-)}(3^{11}12^{1})$, whose MI is equal to $\tau_{12} \otimes I_3$ since $\tau_{12} \otimes I_3 = \tau_{36} - P_{12} \otimes \tau_3$.

By Corollary 4 (Three-factor method), we obtain an orthogonal array $L_{72}(2^{28}3^{11}6^{1}12^{1})$ as follows:

$$L_{72}(2^{28}3^{11}6^{1}12^{1}) = [L_{36}^{(-)}(3^{11}12^{1}) \oplus 0_2, 0_{12} \oplus (6), L_{36}^{(=)}(2^{28}) \oplus (2)],$$

which is satisfactory since $\triangle = \tau_{72} - m(L_{72}(3^{11}6^{1}12^{1})) \leq (\tau_{36} - m(L_{36}(2^{28}3^{1}))) \otimes \tau_2$. This orthogonal array is new, which is not included in Hedayat et al (1999) and http://sas.techsup/technote/ts723.html maintained by Warren F.Kuhfeld yet.

Furthermore, replacing the orthogonal array $L_{36}(3^{12}12^{1})$ by any one of orthogonal arrays $L_{36}(3^{x} \cdots)$ which has at least a 3-level column, we will also able to construct an orthogonal array for this family. There are at least 11 new orthogonal arrays of run size 72 for this family which are included in Table 1.

Similarly, by using orthogonal arrays $L_{36}(2^{18}6^{1}) = [0_6 \oplus (6), L_{36}^{(=)}(2^{18})]$ (in Table 2) and $L_{36}(6^{x} \cdots) = [0_6 \oplus (6), L_{36}^{(-)}(\cdots)]$, we can construct the following orthogonal arrays

$$[L_{36}^{(-)}(\cdots) \oplus 0_2, 0_{12} \oplus (12), L_{36}^{(=)}(2^{18}) \overset{k}{\otimes} (2)].$$

There are at least 7 new orthogonal arrays of run size 72 for this family which are included in Table 1.

There are lots of asymmetrical orthogonal arrays with moderate run sizes (of course run size 72) which can be obtained by only using the simple procedures of both the two factor method and the three factor method. The generalized Kronecker product (or orthogonal-array product) is more powerful for constructing larger arrays from lesser ones.

4.3. Construction of OA $L_{96}(2^{12}4^{20}24^{1})$

Consider the three-step procedure of generalized Kronecker product in Section 3. The following is a recipe for constructing the new orthogonal array $L_{96}(2^{12}4^{20}24^{1})$ by using the three-step procedure of generalized Kronecker product for case $k_1 = 3$, $k_2 = 1$ and $\triangle = 0$.

Step 1. Orthogonally decompose the projection matrix τ_{96}. From (4), we have

$$\tau_{96} = I_{24} \otimes \tau_4 + \tau_{24} \otimes P_4. \tag{5}$$

Based on the Abelian group $G = \{0, 1, 2, 4\}$ of order 4 with the addition table: $(4) \oplus (4)^T = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$, consider the particular form of difference matrix $D(12, 12; 4)$ (Zhang 1993,or Zhang,Duan,Lu and Zheng 2002) as follows

$$D(12,12;4) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 0 & 0 & 2 & 2 & 2 & 3 & 3 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 3 & 3 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 \\ 2 & 3 & 1 & 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\ 2 & 3 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 \\ 3 & 1 & 2 & 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \\ 3 & 1 & 2 & 2 & 3 & 1 & 1 & 2 & 3 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 & 1 & 2 & 2 & 3 & 1 & 1 & 2 & 3 \end{pmatrix},$$

By Definition 5, we obtain

$$D(12,12;4) \overset{k}{\otimes} (4) = [((4) \oplus 0_3) \overset{k}{\otimes} (4), T_1(((4) \oplus 0_3) \overset{k}{\otimes} (4)),$$

$$T_2(((4) \oplus 0_3) \overset{k}{\otimes} (4)), T_3(((4) \oplus 0_3) \overset{k}{\otimes} (4))\ ],$$

where the map $k(i, j)$ of generalized Kronecker product $(4) \overset{k}{\otimes} (4)$ over above Abelian group G of order 4 satisfies

$$k : (4) \overset{k}{\otimes} (4) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \oplus (4)$$

$$= \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 \end{pmatrix}^T,$$

and the permutation matrices T_1, T_2 and T_3 are defined as

$$\begin{aligned} T_1 = &\ \operatorname{diag}(\sigma(1), \sigma(2), \sigma(3), K(3,3) \otimes I_4), \\ T_2 = &\ \operatorname{diag}(\sigma(2), \sigma(3), \sigma(1), [\operatorname{diag}(I_3, N_3, N_3^2)K(3,3)] \otimes I_4), \\ T_3 = &\ \operatorname{diag}(\sigma(3), \sigma(1), \sigma(2), [\operatorname{diag}(I_3, N_3^2, N_3)K(3,3)] \otimes I_4), \end{aligned}$$

in which the permutation matrices N_3 and $K(3,3)$ are defined in (1) and $\sigma(j)(4) = j \oplus (4)$ over above Abelian group G of order 4 for $j = 0, 1, 2, 3$. For example, by the notations of the permutation matrices I_2 and N_2 in (1), we can take

$$\sigma(0) = I_4, \sigma(1) = I_2 \otimes N_2, \sigma(2) = N_2 \otimes I_2, \sigma(3) = N_2 \otimes N_2.$$

By Theorems 5 and 2, we obtain

$$I_{12} \otimes \tau_4 = \sum_{i=0}^{3} T_i(\tau_4 \otimes P_3 \otimes \tau_4)T_i^T, \tag{6}$$

where $T_0 = I_{48}$. By Theorem 5 and (6), an orthogonal decomposition of projection matrix $I_{24} \otimes \tau_4$ can be obtained as follows:

$$\begin{aligned}
I_{24} \otimes \tau_4 = &\ I_2 \otimes [I_{12} \otimes \tau_4] \\
= &\ I_2 \otimes \left(\sum_{i=0}^{3} T_i (\tau_4 \otimes P_3 \otimes \tau_4) T_i^T \right) \\
= &\ \sum_{i=0}^{3} (I_2 \otimes T_i)(I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4)(I_2 \otimes T_i)^T, \\
= &\ \sum_{i=0}^{3} S_i (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4) S_i^T,
\end{aligned}$$

where $S_0 = I_{96}$ and $S_i = I_2 \otimes T_i,\ i = 1, 2, 3$.

Denoted that $M_i = S_i K(8, 12)$ for $i = 0, 1, 2, 3$. Since $K(8, 12)(P_3 \otimes \tau_4 \otimes I_2 \otimes \tau_4) K(8, 12)^T = I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4$, from (5) and above equation we obtain an orthogonal decomposition of projection matrix τ_{96} as follows:

$$\tau_{96} = I_{24} \otimes \tau_4 + \tau_{24} \otimes P_8 = \sum_{i=1}^{3} M_i (P_3 \otimes \tau_4 \otimes I_2 \otimes \tau_4) M_i^T + (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4 + \tau_{24} \otimes P_4). \quad (7)$$

The above decompositions are orthogonal because of the orthogonality in each step.

Step 2. First, we now want to find an orthogonal array $L_{32}(2^3 4^5)$ such that its MI is $\tau_4 \otimes I_2 \otimes \tau_4$. From (4) and some operations of matrices , we have the following orthogonal decomposition of the projection matrix $\tau_4 \otimes I_2 \otimes \tau_4$:

$$\begin{aligned}
&\ \tau_4 \otimes I_2 \otimes \tau_4 \\
= &\ (\tau_2 \otimes P_4 \otimes \tau_2 \otimes P_2 + P_2 \otimes \tau_2 \otimes P_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2) \\
+ &\ (\tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 + P_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2) \\
+ &\ (\tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2) \\
+ &\ (\tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes \tau_2 \otimes P_2 \otimes P_2 \otimes \tau_2 + P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2) \\
+ &\ (P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes P_4 \otimes \tau_2 \otimes \tau_2) \\
+ &\ P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes P_8 \otimes \tau_2 + \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2.
\end{aligned} \quad (8)$$

By Theorem 4 we have $m((2) \oplus (2)) = \tau_2 \otimes \tau_2$. If define the general Kronecker product $(2) \overset{k}{\otimes} (2) = (2) \oplus (2)$, then we can construct an orthogonal array $L_{32}(2^{18})$ whose MI is equal to $\tau_4 \otimes I_2 \otimes \tau_4$ as follows

$$\begin{aligned}
L_{32}(2^{18}) = &\ [(4) \oplus 0_2, (4) \overset{k}{\otimes} (2)] \overset{k}{\otimes} (4), \\
= &\ [((2) \oplus 0_4 \oplus (2) \oplus 0_2, 0_2 \oplus (2) \oplus 0_2 \oplus 0_2 \oplus (2), (2) \oplus (2) \oplus 0_2 \oplus (2) \oplus (2)), \\
&\ ((2) \oplus (2) \oplus 0_2 \oplus (2) \oplus 0_2, 0_2 \oplus (2) \oplus (2) \oplus 0_2 \oplus (2), (2) \oplus 0_2 \oplus (2) \oplus (2) \oplus (2)), \\
&\ ((2) \oplus (2) \oplus (2) \oplus (2) \oplus 0_2, (2) \oplus 0_2 \oplus (2) \oplus 0_2 \oplus (2), 0_2 \oplus (2) \oplus 0_2 \oplus (2) \oplus (2)), \\
&\ ((2) \oplus 0_2 \oplus (2) \oplus (2) \oplus 0_2, (2) \oplus (2) \oplus 0_2 \oplus 0_2 \oplus (2), 0_2 \oplus (2) \oplus (2) \oplus (2) \oplus (2)), \\
&\ (0_2 \oplus (2) \oplus (2) \oplus (2) \oplus 0_2, (2) \oplus (2) \oplus (2) \oplus 0_2 \oplus (2), (2) \oplus 0_4 \oplus (2) \oplus (2)), \\
&\ 0_2 \oplus (2) \oplus 0_2 \oplus (2) \oplus 0_2, (2) \oplus 0_8 \oplus (2), (2) \oplus (2) \oplus (2) \oplus (2) \oplus (2)],
\end{aligned} \quad (9)$$

where $0_2 = (0, 0)^T, (2) = (0, 1)^T, \oplus, \ldots$ are corresponding to $P_2, \tau_2, \otimes, \ldots$, respectively.

By the usual Hadamard product $\circ$ in matrix theory, we find the each of items in (8) corresponding to the each of items in (9) having the forms

$$(A + B + 32A \circ B) \text{ and } (a, b, a + b), \text{ respectively,}$$

where $A = m(a), B = m(b)$ and the addition $'+'$ of $a + b$ is the usual modulo 2. From the method of generalized Hadamard product $\overset{h}{\circ} = \diamond$ where $h(i, j) = 2i + j$ (Zhang, Pang and Wang 2001) , each of the items $(a, b, a + b)$ can be replaced by a 4-level column whose form is $a \diamond b$ where $[(2) \oplus 0_2] \diamond [0_2 \oplus (2)] = (4)$. Thus we obtain an orthogonal array $L_{32}(2^3 4^5)$ whose MI is equal to $\tau_4 \otimes I_2 \otimes \tau_4$ and whose form is

$$L_{32}(2^3 4^5) = [0_2 \oplus (2) \oplus 0_2 \oplus (2) \oplus 0_2, (2) \oplus 0_8 \oplus (2),$$

$$(2) \oplus (2) \oplus (2) \oplus (2) \oplus (2), D(8, 5; 4) \oplus (4)],$$

in which the structure of difference matrix $D(8, 5; 4)$ can be obtained by using the definition of the generalized Hadamard product above $\diamond$ as follows:

$$D(8, 5; 4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 2 \\ 1 & 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 1 & 3 \\ 2 & 2 & 1 & 3 & 3 \\ 2 & 3 & 2 & 0 & 1 \\ 3 & 1 & 2 & 1 & 2 \\ 3 & 0 & 1 & 2 & 0 \end{pmatrix}.$$

Secondly, we now want to find orthogonal arrays $L_{96}(\cdots)$ such that $m(L_{96}(\cdots)) \leq (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4 + \tau_{24} \otimes P_4)$. Of course, $L_{96}(4^5 24^1) = [M_0(0_3 \oplus L_{32}(4^5 2^3)), (24)^1 \oplus 0_4]$ satisfies the condition since $m(M_0(0_3 \oplus L_{32}(4^5 2^3))) \leq I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4$ where $M_0 = K(8, 12)$ and $m((24) \oplus 0_4) = \tau_{24} \otimes P_4$.

On the other hand, there are also lots of other orthogonal arrays satisfying the condition, which can be obtained by the generalized Kronecker product similar to above operation, such as

1.

$$L_{96}^{(-)}(2^9 4^7 12^1) = [K(8, 12)(0_3 \oplus (8) \overset{k}{\otimes} (4)), L_{24}^{(-)}(2^9 12^1) \oplus 0_4]$$

$$L_{96}^{(-)}(2^{17} 4^8) = [K(8, 12)(0_3 \oplus (8) \overset{k}{\otimes} (4)), L_{24}^{(-)}(2^{17} 4^1) \oplus 0_4]$$

$$L_{96}^{(-)}(2^{10} 3^1 4^8) = [K(8, 12)(0_3 \oplus (8) \overset{k}{\otimes} (4)), L_{24}^{(-)}(2^{10} 3^1 4^1) \oplus 0_4]$$

$$L_{96}^{(-)}(2^8 6^1 4^8) = [K(8, 12)(0_3 \oplus (8) \overset{k}{\otimes} (4)), L_{24}^{(-)}(2^8 4^1 6^1) \oplus 0_4]$$

where

$$L_{24}(2^{12} 12^1) = [(2) \oplus L_4(2^3) \oplus 0_3, (L_{24}^{(-)}(2^9 12^1)],$$

$$L_{24}(2^{20} 4^1)) = [(2) \oplus L_4(2^3) \oplus 0_3, L_{24}^{(-)}(2^{17} 4^1)],$$

$$L_{24}(2^{13} 3^1 4^1)) = [(2) \oplus L_4(2^3) \oplus 0_3, L_{24}^{(-)}(2^{10} 3^1 4^1)],$$

$$L_{24}(2^{11} 6^1 4^1)) = [(2) \oplus L_4(2^3) \oplus 0_3, L_{24}^{(-)}(2^8 4^1 6^1)]$$

and

$$(8) \overset{k}{\otimes} (4) = D^0(8,7;4) \oplus (4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 & 3 & 0 & 2 \\ 1 & 2 & 3 & 2 & 1 & 3 & 0 \\ 2 & 2 & 0 & 3 & 3 & 1 & 1 \\ 2 & 3 & 1 & 0 & 1 & 2 & 3 \\ 3 & 1 & 2 & 2 & 0 & 1 & 3 \\ 3 & 0 & 3 & 1 & 2 & 2 & 1 \end{pmatrix} \oplus (4).$$

2.

$$L_{96}^{(-)}(2^{11}4^4 8^1 12^1) = [L_{96}^{(=)}(2^2 4^4 8^1), L_{24}^{(=)}(2^9 12^1) \oplus 0_4]$$

$$L_{96}^{(-)}(2^{19}4^5 8^1) = [L_{96}^{(=)}(2^2 4^4 8^1), L_{24}^{(=)}(2^{17} 4^1) \oplus 0_4]$$

$$L_{96}^{(-)}(2^{12}3^1 4^5 8^1) = [L_{96}^{(=)}(2^2 4^4 8^1), L_{24}^{(=)}(2^{10} 3^1 4^1) \oplus 0_4]$$

$$L_{96}^{(-)}(2^{10}4^5 6^1 8^1) = [L_{96}^{(=)}(2^2 4^4 8^1), L_{24}^{(=)}(2^8 4^1 6^1) \oplus 0_4]$$

where

$$L_{96}^{(=)}(2^2 4^4 8^1) = [((2) \oplus 0_{48}) \diamond (0_2 \oplus (2) \oplus 0_6 \oplus (2) \oplus 0_2) \diamond ((2) \oplus 0_2 \oplus (2) \oplus 0_6 \oplus (2)),$$

$$((2) \oplus (2) \oplus 0_{24}) \diamond (0_4 \oplus (2) \oplus 0_3 \oplus (2) \oplus 0_2),$$

$$((2) \oplus 0_2 \oplus (2) \oplus 0_{12}) \diamond (0_2 \oplus (2) \oplus 0_{12} \oplus (2)),$$

$$((2) \oplus (2) \oplus 0_6 \oplus (2) \oplus (2)) \diamond (0_2 \oplus (2) \oplus (2) \oplus 0_6 \oplus (2)),$$

$$((2) \oplus (2) \oplus 0_{12} \oplus (2)) \diamond (0_2 \oplus (2) \oplus (2) \oplus 0_3 \oplus (2) \oplus 0_2),$$

$$0_2 \oplus (2) \oplus 0_6 \oplus (2) \oplus (2),\ \ 0_4 \oplus (2) \oplus 0_3 \oplus (2) \oplus (2)],$$

and

$$L_{24}(2^{12}12^1) = [(2) \oplus [0_4, (2) \oplus 0_2, 0_2 \oplus (2)] \oplus 0_3, L_{24}^{(=)}(2^9 12^1)],$$

$$L_{24}(2^{20}4^1)) = [(2) \oplus [0_4, (2) \oplus 0_2, 0_2 \oplus (2)] \oplus 0_3, L_{24}^{(=)}(2^{17} 4^1)],$$

$$L_{24}(2^{13}3^1 4^1)) = [(2) \oplus [0_4, (2) \oplus 0_2, 0_2 \oplus (2)] \oplus 0_3, L_{24}^{(=)}(2^{10} 3^1 4^1)],$$

$$L_{24}(2^{11}6^1 4^1)) = [(2) \oplus [0_4, (2) \oplus 0_2, 0_2 \oplus (2)] \oplus 0_3, L_{24}^{(=)}(2^8 4^1 6^1)].$$

The existence of particular forms $L_{24}^{(-)}(\cdots)$ and $L_{24}^{(=)}(\cdots)$ dues to Table 5 in Zhang et al (2001).

Step 3. By Corollaries 1,2 and (7), we lay out the new orthogonal array

$$L_{96}(2^{12}4^{20}24^1) = [M_1(0_3 \oplus L_{32}(2^3 4^5)), M_2(0_3 \oplus L_{32}(2^3 4^5)), M_3(0_3 \oplus L_{32}(2^3 4^5)),$$

$$M_0(0_3 \oplus L_{32}(2^3 4^5)), (24) \oplus 0_4].$$

By the definition of permutation matrices M_0, M_1, M_2, M_3 and the form of orthogonal array $L_{32}(2^3 4^5)$, we can change the orthogonal array $L_{96}(2^{12}4^{20}24^1)$ into the form

$$L_{96}(2^{12}4^{20}24^1) = [D_1(12,4;2) \oplus 0_2 \oplus (2) \oplus 0_2, D_2(12,4;2) \oplus 0_4 \oplus (2),$$

$$D_3(12,4;2) \oplus (2) \oplus (2) \oplus (2), D(24,20;4) \oplus (4), (24) \oplus 0_4],$$

or the form

$$L_{96}(2^{12}4^{20}24^1) = [(2) \oplus 0_2 \oplus D_1(12,4;2) \oplus 0_2, 0_2 \oplus (2) \oplus D_2(12,4;2) \oplus 0_2,$$

$$(2) \oplus (2) \oplus D_3(12,4;2) \oplus (2), (4) \oplus D(24,20;4), 0_4 \oplus (24)].$$

Thus a new difference matrix $D(24,20;4)$ and a normal mixed difference matrix

$$[D(24,20;4), D_1(12,4;2) \oplus 0_2, D_2(12,4;2) \oplus 0_2, D_3(12,4;2) \oplus (2)]$$

also can be drawn out from the orthogonal array over above Abelian group $G = \{0,1,2,3\}$ which was observed by Pang, Zhang and Liu (2004).

Furthermore, replacing the sub-array $[M_0(0_3 \oplus L_{32}(2^3 4^5)), (24) \oplus 0_4]$ in $L_{96}(2^{12}4^{20}24^1)$ by above 96-run orthogonal arrays $L_{96}^{(-)}(\cdots)$, we can construct new mixed-level orthogonal arrays as follows:

$$L_{96}(2^{18}4^{22}12^1), L_{96}(2^{26}4^{23}), L_{96}(2^{19}3^1 4^{23}), L_{96}(2^{17}4^{23}6^1),$$

$$L_{96}(2^{20}4^{19}8^1 12^1), L_{96}(2^{28}4^{20}8^1), L_{96}(2^{21}3^1 4^{20}8^1), L_{96}(2^{19}4^{20}6^1 8^1), \cdots$$

respectively. These orthogonal arrays are new, which are not included in Hedayat et al (1999) and http://sas.techsup/technote/ts723.html maintained by Warren F.Kuhfeld yet (except for the orthogonal array $L_{96}(2^{12}4^{20}24^1)$ which is exhibited by author).Many new orthogonal arrays can also be obtained from above orthogonal arrays by using generalized Hadamard products (Zhang et al 1991) through complicated computing, such as

$$L_{96}(2^{12}4^{24}12^1), L_{96}(2^{20}4^{25}), L_{96}(2^{13}3^1 4^{25}), L_{96}(2^{11}4^{25}6^1),$$

$$L_{96}(2^{14}4^{21}8^1 12^1), L_{96}(2^{22}4^{22}8^1), L_{96}(2^{15}3^1 4^{22}8^1), L_{96}(2^{13}4^{22}6^1 8^1), \cdots.$$

By this method, we conjecture that there also exists an orthogonal array $L_{96}(4^{28}12^1)$.

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Table 1. Orthogonal arrays $L_{72}(\cdots)$ Obtained in Section 4.1 and 4.2

No.	Krown OAs $L_{36}(\cdots)$	Obtained OAs $L_{72}(\cdots 4^1)$	Obtained OAs $L_{72}(\cdots 6^x)$	Obtained OAs $L_{72}(\cdots 12^1)$
1	$L_{36}(2^{35})$	$L_{72}(2^{68}4^1)$	–	–
2	$L_{36}(2^{28}3^1)$(new)	$L_{72}(2^{61}3^1 4^1)$(new)	$L_{72}(2^{56}6^1)$ (new)	–
3	$L_{36}(2^{20}3^2)$	$L_{72}(2^{53}3^2 4^1)$	$L_{72}(2^{48}3^1 6^1)$	–
4	$L_{36}(2^{18}3^1 6^1)$	$L_{72}(2^{51}3^1 6^1 4^1)$	$L_{72}(2^{46}6^2)$	$L_{72}(2^{36}3^1 12^1)$
5	$L_{36}(2^{16}3^4)$	$L_{72}(2^{49}3^4 4^1)$	$L_{72}(2^{44}3^3 6^1)$	–
6	$L_{36}(2^{16}9^1)$	$L_{72}(2^{49}9^1 4^1)$	–	–
7	$L_{36}(2^{13}6^2)$	$L_{72}(2^{46}6^2 4^1)$	–	$L_{72}(2^{31}6^1 12^1)$ (new)
8	$L_{36}(2^{12}3^5)$(new)	$L_{72}(2^{45}3^5 4^1)$(new)	$L_{72}(2^{40}3^4 6^1)$	–
9	$L_{36}(2^{11}3^{12})$	$L_{72}(2^{44}3^{12}4^1)$	$L_{72}(2^{39}3^{11}6^1)$ (new)	–
10	$L_{36}(2^{11}3^4 6^1)$(new)	$L_{72}(2^{44}3^4 6^1 4^1)$(new)	$L_{72}(2^{39}3^3 6^2)$	$L_{72}(2^{29}3^4 12^1)$
11	$L_{36}(2^{10}3^8 6^1)$	$L_{72}(2^{43}3^8 6^1 4^1)$	$L_{72}(2^{38}3^7 6^2)$ (new)	$L_{72}(2^{28}3^8 12^1)$
12	$L_{36}(2^{10}3^1 6^2)$	$L_{72}(2^{43}3^1 6^2 4^1)$	$L_{72}(2^{38}6^3)$	$L_{72}(2^{28}3^1 6^1 12^1)$ (new)
13	$L_{36}(2^9 3^4 6^2)$	$L_{72}(2^{42}3^4 6^2 4^1)$	$L_{72}(2^{37}3^3 6^3)$	$L_{72}(2^{27}3^4 6^1 12^1)$
14	$L_{36}(2^8 6^3)$	$L_{72}(2^{41}6^3 4^1)$	–	$L_{72}(2^{26}6^2 12^1)$ (new)
15	$L_{36}(2^4 3^{13})$	$L_{72}(2^{37}3^{13}4^1)$	$L_{72}(2^{32}3^{12}6^1)$	–
16	$L_{36}(2^4 3^1 6^3)$	$L_{72}(2^{37}3^1 6^3 4^1)$	$L_{72}(2^{32}6^4)$ (new)	$L_{72}(2^{22}3^1 6^2 12^1)$ (new)
17	$L_{36}(2^3 3^9 6^1)$	$L_{72}(2^{36}3^9 6^1 4^1)$	$L_{72}(2^{31}3^8 6^2)$	$L_{72}(2^{21}3^9 12^1)$
18	$L_{36}(2^3 3^2 6^3)$	$L_{72}(2^{36}3^2 6^3 4^1)$	$L_{72}(2^{31}3^1 6^4)$ (new)	$L_{72}(2^{21}3^2 6^2 12^1)$ (new)
19	$L_{36}(2^2 3^{12}6^1)$	$L_{72}(2^{35}3^{12}6^1 4^1)$	$L_{72}(2^{30}3^{11}6^2)$(new)	$L_{72}(2^{20}3^{12}12^1)$
20	$L_{36}(2^2 3^5 6^2)$	$L_{72}(2^{35}3^5 6^2 4^1)$	$L_{72}(2^{30}3^4 6^3)$(new)	$L_{72}(2^{20}3^5 6^1 12^1)$
21	$L_{36}(2^2 18^1)$	$L_{72}(2^{35}18^1 4^1)$	–	–
22	$L_{36}(2^1 3^8 6^2)$	$L_{72}(2^{34}3^8 6^2 4^1)$	$L_{72}(2^{29}3^7 6^3)$(new)	$L_{72}(2^{19}3^8 6^1 12^1)$
23	$L_{36}(2^1 3^3 6^3)$	$L_{72}(2^{34}3^3 6^3 4^1)$	$L_{72}(2^{29}3^2 6^4)$(new)	$L_{72}(2^{19}3^3 6^2 12^1)$ (new)
24	$L_{36}(3^{13}4^1)$	–	$L_{72}(2^{28}3^{12}6^1 4^1)$	–
25	$L_{36}(3^{12}12^1)$	–	$L_{72}(2^{28}3^{11}6^1 12^1)$(new)	–
26	$L_{36}(3^7 6^3)$	–	$L_{72}(2^{28}3^6 6^4)$(new)	$L_{72}(2^{18}3^7 6^2 12^1)$ (new)
27	$L_{36}(4^1 9^1)$	–	–	–
	...	...	...	...

Table 2. Orthogonal arrays $L_{36}(\cdots)$ used in Section 4.1 and 4.2

No.	$B_1 - B_8$	$B_9 - B_{17}$	$B_{18} - B_{26}$	$B_{27} - B_{35}$	CF
1	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0
2	1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	0 1 0 0 1 1 1 1 1	1 1
3	0 1 1 1 0 0 1 1	1 0 0 0 1 0 1 0 1	1 1 1 0 1 0 0 1 0	1 0 0 0 0 1 1 1 0	2 2
4	1 1 0 0 1 0 0 1	0 1 1 1 1 0 0 0 1	1 1 1 0 0 1 1 0 0	0 0 1 0 0 1 0 1 1	0 3
5	0 0 0 1 1 0 1 0	1 1 1 1 1 1 1 1 1	1 0 1 1 0 0 0 0 0	0 1 0 0 1 1 0 0 0	1 4
6	1 1 1 0 1 0 1 0	1 0 0 0 0 1 1 1 0	1 0 0 0 0 1 1 1 0	0 1 1 1 0 1 0 1 0	2 5
7	0 0 1 0 1 1 0 1	0 0 0 1 0 1 1 0 1	0 1 1 1 0 1 0 1 0	1 1 1 0 1 0 0 1 0	0 0
8	1 1 0 1 0 1 0 0	0 0 0 1 0 1 1 0 1	1 0 1 0 1 1 0 0 1	1 1 0 1 0 1 0 0 1	1 1
9	0 1 1 1 0 0 1 1	0 0 1 0 0 1 0 1 1	1 1 0 1 0 1 0 0 1	0 1 1 0 0 0 1 0 1	2 2
10	1 1 0 0 1 0 0 1	1 0 0 1 1 1 0 1 0	0 1 1 1 1 0 0 0 1	0 1 0 1 0 0 1 1 0	0 3
11	0 1 1 1 1 1 0 0	1 1 1 0 1 0 0 1 0	0 0 1 0 0 1 0 1 1	1 1 0 0 0 0 0 1 1	1 4
12	1 1 1 0 1 0 1 0	0 1 1 0 0 0 1 0 1	0 1 1 0 0 0 1 0 1	1 1 0 1 1 0 1 0 0	2 5
13	0 0 0 0 0 0 0 0	0 1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 0
14	1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	1 1
15	0 1 1 1 0 0 1 1	0 1 0 1 0 0 1 1 0	0 0 1 1 1 1 1 0 0	0 0 0 1 1 0 0 1 1	2 2
16	1 1 0 0 1 0 0 1	1 1 1 0 0 1 1 0 0	1 0 0 1 1 1 0 1 0	1 0 0 0 1 0 1 0 1	0 3
17	0 0 0 1 1 0 1 0	1 0 1 1 0 0 0 0 0	0 1 0 0 1 1 0 0 0	1 1 1 1 1 1 1 1 1	1 4
18	1 1 1 0 1 0 1 0	0 0 0 1 1 0 0 1 1	0 0 0 1 1 0 0 1 1	1 0 1 0 1 1 0 0 1	2 5
19	0 0 1 0 1 1 0 1	0 0 1 0 1 0 1 1 0	1 1 0 1 1 0 1 0 0	1 1 0 1 0 1 0 0 1	0 0
20	1 1 0 1 0 1 0 0	0 0 1 0 1 0 1 1 0	0 1 1 1 0 1 0 1 0	0 0 1 1 1 1 1 0 0	1 1
21	0 1 0 0 0 1 1 1	0 1 1 1 0 1 0 1 0	1 1 0 0 0 0 0 1 1	1 0 0 1 1 1 0 1 0	2 2
22	1 0 1 1 0 0 0 1	0 1 1 1 0 1 0 1 0	0 0 1 0 1 0 1 1 0	1 1 1 0 0 1 1 0 0	0 3
23	0 1 1 1 1 1 0 0	1 1 0 1 0 1 0 0 1	0 1 0 1 0 0 1 1 0	0 0 0 1 0 1 1 0 1	1 4
24	1 0 1 0 0 1 1 0	0 1 1 1 1 0 0 0 1	1 0 0 1 1 1 0 1 0	0 1 0 1 0 0 1 1 0	2 5
25	0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	1 0 1 1 0 0 1 1 1	1 0 1 1 0 0 1 1 1	0 0
26	1 0 0 1 1 1 1 1	1 0 1 1 0 0 1 1 1	0 1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 1
27	0 1 0 0 0 1 1 1	1 1 0 1 1 0 1 0 0	0 0 0 1 0 1 1 0 1	1 1 1 0 0 1 1 0 0	2 2
28	1 0 1 1 0 0 0 1	1 1 0 1 1 0 1 0 0	1 1 0 0 0 0 0 1 1	0 1 1 1 1 0 0 0 1	0 3
29	0 0 0 1 1 0 1 0	0 1 0 0 1 1 0 0 0	1 1 1 1 1 1 1 1 1	1 0 1 1 0 0 0 0 0	1 4
30	1 0 1 0 0 1 1 0	1 0 0 1 1 1 0 1 0	1 1 1 0 0 1 1 0 0	1 0 0 0 1 0 1 0 1	2 5
31	0 0 1 0 1 1 0 1	1 1 0 0 0 0 0 1 1	1 0 1 0 1 1 0 0 1	0 0 1 1 1 1 1 0 0	0 0
32	1 1 0 1 0 1 0 0	1 1 0 0 0 0 0 1 1	1 1 0 1 1 0 1 0 0	1 1 1 0 1 0 0 1 0	1 1
33	0 1 0 0 0 1 1 1	1 0 1 0 1 1 0 0 1	0 0 1 0 1 0 1 1 0	0 1 1 1 1 0 0 0 1	2 2
34	1 0 1 1 0 0 0 1	1 0 1 0 1 1 0 0 1	0 0 0 1 0 1 1 0 1	1 0 0 1 1 1 0 1 0	0 3
35	0 1 1 1 1 1 0 0	0 0 1 1 1 1 1 0 0	1 0 0 0 1 0 1 0 1	0 0 1 0 1 0 1 1 0	1 4
36	1 0 1 0 0 1 1 0	1 1 1 0 0 1 1 0 0	0 1 1 1 1 0 0 0 1	0 0 1 0 0 1 0 1 1	2 5

$$L_{36}(2^{35}) = (B_1 - B_{35}) = [0_{18} \oplus (2), L_{36}^{(=)}(2^{34})]$$

$$L_{36}(3^1 2^{28}) = (CB_1 B_9 - B_{35}) = [0_{12} \oplus (3), L_{36}^{(=)}(2^{28})]$$

$$L_{36}(6^1 2^{18}) = (FB_{18} - B_{35}) = [0_6 \oplus (6), L_{36}^{(=)}(2^{18})]$$

$No.$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	f	c_1	c_2	c_3	c_4	c_5
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2
2	1	1	1	1	0	0	0	1	0	1	0	1	1	0	1	1	2	2
3	0	1	0	0	1	1	1	1	0	0	0	1	0	0	2	2	1	1
4	1	0	1	1	1	1	0	1	0	0	1	0	1	0	2	2	1	1
5	0	0	0	0	1	0	0	1	1	1	1	1	2	1	1	2	1	2
6	1	0	1	0	0	1	0	0	1	0	1	1	3	1	1	2	1	2
7	0	0	1	1	0	0	1	1	1	0	1	0	2	1	2	1	2	1
8	1	0	0	1	1	0	1	0	1	0	0	1	3	1	2	1	2	1
9	0	1	1	1	0	1	1	0	0	1	1	1	4	2	1	2	2	1
10	1	1	0	0	1	0	1	0	0	1	1	0	5	2	1	2	2	1
11	0	1	1	1	1	1	0	0	1	1	0	0	4	2	2	1	1	2
12	1	1	0	0	0	1	1	1	1	1	0	0	5	2	2	1	1	2
13	0	1	1	0	0	0	0	0	0	0	0	0	2	1	2	2	0	0
14	1	1	0	1	0	0	0	1	0	1	0	1	3	1	2	2	0	0
15	0	1	1	0	1	1	1	1	0	0	0	1	2	1	0	0	2	2
16	1	1	0	1	1	1	0	1	0	0	1	0	3	1	0	0	2	2
17	0	0	0	0	1	0	0	1	1	1	1	1	4	2	2	0	2	0
18	1	1	1	0	0	1	0	0	1	0	1	1	5	2	2	0	2	0
19	0	1	0	1	0	0	1	1	1	0	1	0	4	2	0	2	0	2
20	1	0	1	1	1	0	1	0	1	0	0	1	5	2	0	2	0	2
21	0	0	0	1	0	1	1	0	0	1	1	1	0	0	2	0	0	2
22	1	0	1	0	1	0	1	0	0	1	1	0	1	0	2	0	0	2
23	0	0	1	1	1	1	0	0	1	1	0	0	0	0	0	2	2	0
24	1	0	0	0	0	1	1	1	1	1	0	0	1	0	0	2	2	0
25	0	0	0	0	0	0	0	0	0	0	0	0	4	2	0	0	1	1
26	1	0	1	1	0	0	0	1	0	1	0	1	5	2	0	0	1	1
27	0	0	1	0	1	1	1	1	0	0	0	1	4	2	1	1	0	0
28	1	0	0	1	1	1	0	1	0	0	1	0	5	2	1	1	0	0
29	0	1	1	0	1	0	0	1	1	1	1	1	0	0	0	1	0	1
30	1	1	0	0	0	1	0	0	1	0	1	1	1	0	0	1	0	1
31	0	1	1	1	0	0	1	1	1	0	1	0	0	0	1	0	1	0
32	1	1	0	1	1	0	1	0	1	0	0	1	1	0	1	0	1	0
33	0	0	0	1	0	1	1	0	0	1	1	1	2	1	0	1	1	0
34	1	1	1	0	1	0	1	0	0	1	1	0	3	1	0	1	1	0
35	0	1	0	1	1	1	0	0	1	1	0	0	2	1	1	0	0	1
36	1	0	1	0	0	1	1	1	1	1	0	0	3	1	1	0	0	1

$$L_{36}(2^{12}3^5) = [b_1 - b_{12}, c_1 - c_5]$$

$$L_{36}(2^{11}3^4 6^1) = [b_2 - b_{12}, c_2 - c_5, f]$$