MODIFIED SECOND ORDER SLOPE ROTATABLE DESIGNS USING BIBD

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ABSTRACT

In this paper, a new method of modified second order slope rotatable designs (SOSRD) using balanced incomplete block designs (BIBD) for $4 \leq v \leq 16$ is presented. In this method the number of design points required is in some cases less than the number required in Victorbabu (2005) modified slope rotatable central composite designs. Further, a new method of construction of three level modified SOSRD using BIBD is presented. The modified SOSRD can be viewed as SOSRD constructed with the technique of augmentation of second order rotatable design (SORD) using BIBD to SOSRD. These designs are useful in parts to estimate responses and slopes with spherical variance functions.

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1. Introduction

Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces. Das and Narasimham (1962) constructed rotatable designs through balanced incomplete block designs (BIBD). The study of rotatable designs mainly emphasized on the estimation of absolute response. Estimation of differences in response at two different points in the factor space will often be of great importance. If differences at two points close together, estimation of local slope (rate of change) of the response is of interest. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in an animal etc., (c.f. Park, 1987).

Hader and Park (1978) introduced slope rotatable central composite designs (SRCCD). Park (1987) introduced a class of multifactor designs for estimating the slope of response surfaces. Victorbabu and Narasimham (1991) constructed second order slope rotatable designs (SOSRD) using BIBD. Victorbabu and Narasimham (1993) constructed three level SOSRD using BIBD. Victorbabu (2002) suggested a note on the construction of four and six level SOSRD. Victorbabu (2005) studied modified SRCCD. In this paper, a new method of modified SOSRD using BIBD for $4 \le v \le 16$ is presented. It is found that in some cases this method leads to modified SOSRD with less number of design points compared to modified SRCCD. Specifically for 7, 9 factors these new designs need 128 and 162 points whereas corresponding modified SRCCD need 144 and 200 design points respectively. It may also be noted that for 4,13 and 16 factors, the new method leads to modified SOSRD in same number of design points as in modified SRCCD. Further, a new method of construction of three level modified SOSRD using BIBD is presented. The modified SOSRD can be viewed as SOSRD constructed with the technique of augmentation of second order rotatable design (SORD) using BIBD to SOSRD. These designs are useful in parts to estimate responses and slopes with spherical variance functions.

2. Conditions for modified SOSRD

A second order response surface design $D = ((x_{iu}))$ for fitting,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j}\sum b_{ij} x_{iu} x_{ju} + e_u \tag{2.1}$$

where x_{iu} denotes the level of the i^{th} factor ($i = 1,2,...,v$) in the u^{th} run (u =1,2,…,N) of the experiment, e_u's are uncorrelated random errors with mean zero and variance σ^2. A second order response surface design D is said to be a SOSRD, if the design points satisfy the following conditions (cf. Hader and Park (1978), Victorbabu and Narasimham (1991)).

$$\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \le 4 \tag{2.2}$$

$$\text{(i)} \sum_{u=1}^{N} x_{iu}^2 = \text{constant} = N\lambda_2$$

$$\text{(ii)} \sum_{u=1}^{N} x_{iu}^4 = \text{constant} = cN\lambda_4 \text{, for all i} \tag{2.3}$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4 \text{, for } i \ne j \tag{2.4}$$

$$(c + v - 1)\lambda_4 > v\lambda_2^2 \tag{2.5}$$

$$\lambda_4[v(5-c) - (c-3)^2] + \lambda_2^2[v(c-5) + 4] = 0 \tag{2.6}$$

where c, λ_2 and λ_4 are constants and the summation is over the design points.

The usual method of construction of SOSRD is to take combinations with unknown constants, associate a 2^{v} factorial combinations or a suitable fraction of it with factors each at ± 1 levels to make the level codes equidistant. All such combinations form a design. Generally SOSRD need at least five levels (suitably coded) at $0, \pm 1, \pm a$ for all factors (($0, 0, ..., 0$) – chosen center of the design, unknown level 'a' to be chosen suitably to satisfy slope rotatability). Generation of design points this way ensures satisfaction of all the conditions even though the design points contain unknown levels.

Alternatively by putting some restrictions indicating some relation among $\sum x_{iu}^{2}$, $\sum x_{iu}^{4}$ and $\sum x_{iu}^{2} x_{ju}^{2}$ some equations involving the unknowns are obtained and their solution gives the unknown levels. In SOSRD the restriction used is $V(b_{ij}) = 4V(b_{ii})$ viz. equation (2.6). Other restrictions are also possible though, it seems, not exploited well. We shall investigate the restriction $(\sum x_{iu}^{2})^2 = N \sum x_{iu}^{2} x_{ju}^{2}$ i.e., $(N\lambda_2)^2 = N(N\lambda_4)$ i.e., $\lambda_2^2 = \lambda_4$ to get modified SOSRD. By applying the new restriction in equation (2.6), we get c=1 or c=5. The non-singularity condition (2.5) leads to c=5. It may be noted $\lambda_2^2 = \lambda_4$ and c=5 are equivalent conditions. The variances and co-variances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{(v+4)\sigma^2}{4N}$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\sqrt{\lambda_4}}$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4}$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{4N\lambda_4}$$

$\mathrm{Cov}\,(\hat{b}_0, \hat{b}_{ii}) = \dfrac{-\sigma^2}{4N\sqrt{\lambda_4}}$ and other co-variances are zero.

$$V\left(\frac{\partial \hat{Y}}{\partial x_i}\right) = \left[\frac{\sqrt{\lambda_4} + d^2}{N\lambda_4}\right]\sigma^2 .$$

3. CONSTRUCTION OF MODIFIED SOSRD USING BIBD

Balanced incomplete block design (BIBD): A BIBD denoted by (v, b, r, k, λ) is an arrangement of v-treatments in b-blocks each containing $k(< v)$ treatments, if (i) every treatment occurs at most once in a block, (ii) every treatment occurs in exactly r-blocks and (iii) every pair of treatments occurs together in λ blocks.

Let (v, b, r, k, λ) be a BIBD, $2^{t(k)}$ denote a fractional replicate of 2^k in ± 1 levels in which no interaction with less than five factors is confounded. $[1-(v, b, r, k, \lambda)]$ denote the design points generated from the transpose of the incidence matrix of BIBD. $[1-(v, b, r, k, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from BIBD by "multiplication" (see Raghavarao, 1971, pp.298-300). Let $(a,0,0,...,0)2^1$ denote the design points generated from $(a,0,0,...,0)$ point set. Repeat this set of additional design points say 'n_a' times when r<5λ. Let $(a, a,..., a)2^{t(v)}$ denote the design points generated from $(a, a,..., a)$ point set. Repeat this set of additional design points say 'n_a' times when r>5λ. Let n_0 be the number of central points in modified SOSRD.

Theorem (3.1): Case (i): If r<5λ, then the design points,

$[1-(v,b,r,k,\lambda)]\, 2^{t(k)} \cup n_a\,(a,0,0,...,0)2^1 \cup n_0$ give a v-dimensional modified SOSRD in $N = \dfrac{\left(r2^{t(k)} + 2n_a a^2 \right)^2}{\lambda 2^{t(k)}}$ design points if,

$$a^4 = \frac{(5\lambda - r)2^{t(k)-1}}{n_a}, \tag{3.1}$$

$$n_0 = \frac{\left(r2^{t(k)} + 2n_a a^2 \right)^2}{\lambda 2^{t(k)}} - \left[b2^{t(k)} + 2n_a v\right]. \tag{3.2}$$

Case (ii): If r = 5λ, then the design points, $[1-(v,b,r,k,\lambda)]\, 2^{t(k)} \cup n_0$ give a three level v-dimensional modified SOSRD in $N = \dfrac{\left(r2^{t(k)}\right)^2}{\lambda 2^{t(k)}}$ design points if,

$$n_0 = \frac{\left(r2^{t(k)}\right)^2}{\lambda 2^{t(k)}} - b2^{t(k)}. \tag{3.3}$$

Case (iii): If r>5λ, then the design points,

$[1-(v,b,r,k,\lambda)]\ 2^{t(k)} \cup n_a\ (a,a,a,...,a)2^{t(v)} \cup n_0$ give a v-dimensional modified

SOSRD in $N = \dfrac{\left(r2^{t(k)} + n_a 2^{t(v)} a^2\right)^2}{\lambda 2^{t(k)} + n_a 2^{t(v)} a^4}$ design points if,

$$a^4 = \frac{(r-5\lambda)2^{t(k)-t(v)-2}}{n_a}, \tag{3.4}$$

$$n_0 = \frac{\left(r2^{t(k)} + n_a 2^{t(v)} a^2\right)^2}{\lambda 2^{t(k)} + n_a 2^{t(v)} a^4} - \left(b2^{t(k)} + n_a 2^{t(v)}\right). \tag{3.5}$$

Proof: Case (i): Let r<5λ, from conditions of modified SOSRD, we have

$$\sum x_{iu}^2 = r2^{t(k)} + 2n_a a^2 = N\lambda_2 \tag{3.6}$$

$$\sum x_{iu}^4 = r2^{t(k)} + 2n_a a^4 = 5N\lambda_4 \tag{3.7}$$

$$\sum x_{iu}^2\, x_{ju}^2 = \lambda 2^{t(k)} = N\lambda_4 \tag{3.8}$$

The modified condition $\lambda_2^2 = \lambda_4$, leads to N (Alternatively N may be obtained directly as $N = b2^{t(k)} + 2vn_a + n_0$, where n_0 is given in equation (3.2)). Equations (3.7) and (3.8) leads to a^4 given in equation (3.1).

Case (ii): Let r = 5λ, from conditions of modified SOSRD, we have,

$$\sum x_{iu}^2 = r2^{t(k)} = N\lambda_2 \tag{3.9}$$

$$\sum x_{iu}^4 = r2^{t(k)} = 5N\lambda_4 \tag{3.10}$$

$$\sum x_{iu}^2\, x_{ju}^2 = \lambda 2^{t(k)} = N\lambda_4 \tag{3.11}$$

The modified condition $\lambda_2^2 = \lambda_4$, leads to N (Alternatively N may be obtained directly as $N = b2^{t(k)} + n_0$, where n_0 is given in equation (3.3)). Equations (3.10) and (3.11), we have $r2^{t(k)} = 5\lambda 2^{t(k)}$ implies r = 5λ.

Case (iii): Let r > 5λ, from conditions of modified SOSRD, we have,

$$\sum x_{iu}^2 = r2^{t(k)} + n_a 2^{t(v)} a^2 = N\lambda_2 \tag{3.12}$$

$$\sum x_{iu}^4 = r2^{t(k)} + n_a 2^{t(v)} a^4 = 5N\lambda_4 \tag{3.13}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} + n_a 2^{t(v)} a^4 = N\lambda_4 \tag{3.14}$$

The modified condition $\lambda_2^2 = \lambda_4$, leads to N (Alternatively N may be obtained directly as $N = b2^{t(k)} + n_a 2^{t(v)} + n_0$, where n_0 is given in equation (3.5)). Equations (3.13) and (3.14), leads to a^4 given in equation (3.4). We note that for the existence of the modified SOSRD using BIBD, 'n_a' should be chosen such that 'a^2' is an integer.

Example 3.1: We illustrate the construction of modified SOSRD for 7-factors with the help of a BIBD $(v = 7, b = 7, r = 3, k = 3, \lambda = 1)$. The design points, $[1-(v = 7, b = 7, r = 3, k = 3, \lambda = 1)]\, 2^3 \cup n_a (a,0,0,\ldots,0)2^1 \cup n_0$ give a five level modified SOSRD in N=128 design points for 7-factors. Here equations (3.6), (3.7) and (3.8) are

$$\sum x_{iu}^2 = 24 + 2n_a a^2 = N\lambda_2 \tag{3.15}$$

$$\sum x_{iu}^4 = 24 + 2n_a a^4 = 5N\lambda_4 \tag{3.16}$$

$$\sum x_{iu}^2 x_{ju}^2 = 8 = N\lambda_4 \tag{3.17}$$

Equations (3.16) and (3.17) leads to $n_a a^4 = 8$, which implies $a^2 = 2.00$ for $n_a = 2$. From equations (3.15), (3.17) using the modified condition ($\lambda_2^2 = \lambda_4$), with $a^2 = 2.00$ and $n_a = 2$, we get N=128. Equation (3.2) leads to n_0=44.

Here we may point out that the modified SOSRD using BIBD for 7-factors has only 128 design points, where as the corresponding modified SRCCD obtained by Victorbabu (2005) needs 144 design points. Thus the new method leads to a 7-factor modified SOSRD in less number of design points than the corresponding modified SRCCD.

Example 3.2: Here we construct a modified SOSRD for v=9 factors with the help of a BIBD $(v=9, b=12, r=4, k=3, \lambda=1)$. The design points,

$[1-(v=9, b=12, r=4, k=3, \lambda=1)]\, 2^3 \cup n_a (a,0,0,...,0)2^1 \cup n_0$ give a five level modified SOSRD in N=162 design points for 7-factors. Here equations (3.6), (3.7) and (3.8) are

$$\sum x_{iu}^2 = 32 + 2n_a a^2 = N\lambda_2 \tag{3.18}$$

$$\sum x_{iu}^4 = 32 + 2n_a a^4 = 5N\lambda_4 \tag{3.19}$$

$$\sum x_{iu}^2 x_{ju}^2 = 8 = N\lambda_4 \tag{3.20}$$

Equations (3.19) and (3.20) lead to $n_a a^4 = 4$, which implies $a^2 = 2.00$ for $n_a = 1$. From equations (3.18), (3.20) using the modified condition ($\lambda_2^2 = \lambda_4$), with $a^2 = 2.00$ and $n_a = 1$, we get N=162. Equation (3.2) leads to n_0=48. Here we may point out that the modified SOSRD using BIBD for 9-factors has only 162 design points, where as the corresponding modified SRCCD obtained by Victorbabu (2005) needs 200 design points. Thus the new method leads to a 9-factor modified SOSRD in less number of design points than the corresponding modified SRCCD.

A list of modified SOSRD using BIBD for 4≤v≤16 is given in Table 3.1.

4. THREE LEVEL MODIFIED SOSRD USING BIBD

Let (v, b, r, k, λ) be a BIBD, $2^{t(k)}$ denote a fractional replicate of 2^k in ±1 levels in which no interaction with less than five factors is confounded. $[a-(v,b,r,k,\lambda)]$ denote the design points generated from the transpose of the incidence matrix of BIBD. $[a-(v,b,r,k,\lambda)]2^{t(k)}$ denote the $b2^{t(k)}$ design points generated from BIBD by "multiplication". Let n_0 be the number of central points in modified SOSRD.

Case (i): Modified SOSRD can be constructed as follows with three level factors using BIBD when $r < 5\lambda$. Choose the additional unknown combinations $(a,0,0,...,0)$ by permuting over the different factors and multiply them with 2^1-associate combinations to obtain the additional design points. Repeat this set of additional design points say 'n_a' times.

Theorem (4.1): The design points, $\left[a-(v, b, r, k, \lambda)\right] 2^{t(k)} \cup n_a (a,0,0,...,0)2^1 \cup n_0$ give a three level v-dimensional modified SOSRD in $N = 25\lambda 2^{t(k)}$ design points if,

$$n_a = (5\lambda - r)2^{t(k)-1}, \tag{4.1}$$

$$n_0 = 25\lambda 2^{t(k)} - \left[b2^{t(k)} + 2n_a v\right] \tag{4.2}$$

Proof: From conditions of modified SOSRD, we have

$$\sum x_{iu}^2 = r2^{t(k)}a^2 + 2n_a a^2 = N\lambda_2 \tag{4.3}$$

$$\sum x_{iu}^4 = r2^{t(k)}a^4 + 2n_a a^4 = 5N\lambda_4 \tag{4.4}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)}a^4 = N\lambda_4 \tag{4.5}$$

The modified condition $\lambda_2^2 = \lambda_4$, leads to N (Alternatively N may be obtained directly as $N = b2^{t(k)} + 2vn_a + n_0$, where n_0 is given in equation (4.2)). Equations (4.4) and (4.5) leads to 'n_a' given in equation (4.1). From the slope rotatability condition (2.6) by using the modified condition $\lambda_2^2 = \lambda_4$ and equations (4.3) and (4.5), we have 'n_0' given in equation (4.2).

Example 4.1: We illustrate the construction of three level modified SOSRD for 5-factors with the help of a BIBD. The design points, $\left[a-(v=5, b=10, r=4, k=2, \lambda=1)\right] 2^2 \cup n_a (a,0,0,...,0)2^1 \cup n_0$ give a three level modified SOSRD in N=100 design points. Here equation (4.1) leads to $n_a = 2$ and equations (4.3), (4.4) and (4.5) are

$$\sum x_{iu}^2 = 16a^2 + 4a^2 = N\lambda_2$$

$$\sum x_{iu}^4 = 16a^4 + 4a^4 = 5N\lambda_4$$

$$\sum x_{iu}^2 x_{ju}^2 = 4a^4 = N\lambda_4$$

Equation (4.2) gives $n_0 = 40$.

Example 4.2: Consider the design points,

$[\mathrm{a}-(\mathrm{v}=7,\mathrm{b}=7,\mathrm{r}=3,\mathrm{k}=3,\lambda=1)]\ 2^3 \cup n_a(a,0,0,...,0)2^1 \cup \mathrm{n}_0$ will give a three level 7-factor modified SOSRD in N=200 design points with $n_a = 4$ and $n_0 = 32$.

Case (ii): Modified SOSRD can be constructed as follows with three level factors using BIBD when $r > 5\lambda$. In this case, consider $b2^{t(k)}$ design points associated to a BIBD. Repeat these design points 'n_a' times. We choose the additional unknown combinations $(\mathrm{a},\mathrm{a},...,\mathrm{a})$ and multiply with $2^{t(v)}$ associate combinations (or a suitable fraction of 2^v associate combination) to obtain $2^{t(v)}$ additional design points. Let n_0 be the number of central points in modified SOSRD.

Theorem (4.2): The design points,

$n_a[\mathrm{a}-(\mathrm{v},\mathrm{b},\mathrm{r},\mathrm{k},\lambda)]\ 2^{\mathrm{t(k)}} \cup (a,a,a,...,a)2^{t(v)} \cup \mathrm{n}_0$ give a three level v-dimensional modified SOSRD in $N = \dfrac{\left[\dfrac{r2^{t(v)+2}}{(r-5\lambda)} + 2^{t(v)}\right]^2}{\left[\dfrac{\lambda 2^{t(v)+2}}{(r-5\lambda)} + 2^{t(v)}\right]}$ design points if,

$$n_a = \frac{2^{t(v)-t(k)+2}}{(r-5\lambda)}, \tag{4.6}$$

$$n_0 = \frac{\left[\dfrac{r2^{t(v)+2}}{(r-5\lambda)} + 2^{t(v)}\right]^2}{\left[\dfrac{\lambda 2^{t(v)+2}}{(r-5\lambda)} + 2^{t(v)}\right]} - \left[b2^{t(k)}n_a + 2^{t(v)}\right] \tag{4.7}$$

and n_0 turns out to be an integer.

Proof: From conditions of modified SOSRD, we have

$$\sum \mathrm{x}_{\mathrm{iu}}^2 = \mathrm{r}2^{\mathrm{t(k)}} n_a a^2 + 2^{\mathrm{t(v)}}\,\mathrm{a}^2 = \mathrm{N}\lambda_2 \tag{4.8}$$

$$\sum \mathrm{x}_{\mathrm{iu}}^4 = \mathrm{r}2^{\mathrm{t(k)}} n_a a^4 + 2^{\mathrm{t(v)}}\mathrm{a}^4 = 5\mathrm{N}\lambda_4 \tag{4.9}$$

$$\sum \mathrm{x}_{\mathrm{iu}}^2 \mathrm{x}_{\mathrm{ju}}^2 = \lambda 2^{\mathrm{t(k)}} n_a a^4 + 2^{\mathrm{t(v)}}\mathrm{a}^4 = \mathrm{N}\lambda_4 \tag{4.10}$$

From equations (4.9) and (4.10) leads to 'n_a' given in equation (4.6). From the slope rotatability condition (2.6) by using the modified condition $\lambda_2^2 = \lambda_4$ and equations

(4.8) and (4.10), we have 'n_0' given in equation (4.7). However, we may mention that large number of design points are required in the above method for the construction of three level modified SOSRD. Further, we may add that three level modified SOSRD can also be constructed alternatively by replicating the BIBD generated points n_a (different from n_a above) times, cube points n_b times and addition n_0 central points where n_a, n_b and n_0 are to be chosen suitably on the above lines.

However, the case with r = 5λ obviously give designs with 0, ± 1 levels, and do not need 'a' and 'n_a' (please see Table 3.1).

5. AUGMENTED SORD AS SOSRD USING BIBD

Victorbabu and Narasimham (1991) noted that the value of level 'a' for the axial points required for slope rotatability is appreciably larger than the value required for rotatability in Das and Narasimham (1962) using BIBD. Now we obtain second order slope rotatable design (with Victorbabu and Narasimham, 1991, slope rotatability) by augmenting Das and Narasimham (1962) second order rotatable design (SORD) with additional axial points and central points. These designs are useful in parts to estimate responses and slopes with spherical variance functions. These augmented SOSRD are obtained by suitably selecting some additional number of replications for the axial points (n_a) in a SORD. More specifically, in this work SORD constructed using BIBD are augmented with additional axial points and central points to form SOSRD.

If D_1 (a) denotes a Das and Narasimham (1962) SORD, D_2 (0) are some addition central points and D_3 (a) are some additional axial points, we augment the SORD D_1 (a) to the SOSRD D_1 (a) U D_2 (0) U D_3 (a) to obtain an augmented design such that D_1 (a) or D_1 (a) U D_2 (0) can be used as Das and Narasimham (1962) SORD for estimating responses [here we may mention that D_1 (a) is enough for Das and Narasimham (1962) SORD but to get pure error we may take additional central points D_2 (0)] and the augmented design D_1 (a) U D_2 (0) U D_3 (a) can be used as Victorbabu and Narasimham (1991) SOSRD for estimating the slopes. We note that we choose the level $\pm a$ in the axial points to be same in both the designs SORD D_1 (a) U D_2 (0) and SOSRD D_1 (a) U D_2 (0) U D_3 (a). The exploration of responses surface can be carried sequentially in parts with these augmented designs for estimation of responses and slopes. The method of construction of augmented SORD using BIBD as SOSRD using BIBD is established in the following theorem 5.1.

Theorem 5.1: Let $(\mathrm{v}, \mathrm{b}, \mathrm{r}, \mathrm{k}, \lambda)$ be a BIBD, then the design points,

$[1-(\mathrm{v}, \mathrm{b}, \mathrm{r}, \mathrm{k}, \lambda)]\, 2^{\mathrm{t(k)}} \cup \mathrm{n}_{\mathrm{a}}(\mathrm{a},0,0,\ldots,0)2^{1} \cup \mathrm{n}_0$ give a v-dimensional modified SOSRD in $N=\dfrac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}$ design points if,

$$\mathrm{a}^4 = \frac{(5\lambda - r)2^{t(k)-1}}{n_a}, \tag{5.1}$$

$$\mathrm{n}_0 = \frac{(\mathrm{r}2^{\mathrm{t(k)}} + 2\mathrm{n}_{\mathrm{a}}\mathrm{a}^2)^2}{\lambda 2^{\mathrm{t(k)}}} - \left[\mathrm{b}2^{\mathrm{t(k)}} + 2\mathrm{n}_{\mathrm{a}}\mathrm{v}\right]. \tag{5.2}$$

Proof: From conditions of modified SOSRD, we have

$$\sum \mathrm{x}_{\mathrm{iu}}^2 = \mathrm{r}2^{\mathrm{t(k)}} + 2\mathrm{n}_{\mathrm{a}}\mathrm{a}^2 = \mathrm{N}\lambda_2 \tag{5.3}$$

$$\sum \mathrm{x}_{\mathrm{iu}}^4 = \mathrm{r}2^{\mathrm{t(k)}} + 2\mathrm{n}_{\mathrm{a}}\mathrm{a}^4 = 5\mathrm{N}\lambda_4 \tag{5.4}$$

$$\sum \mathrm{x}_{\mathrm{iu}}^2 \mathrm{x}_{\mathrm{ju}}^2 = \lambda 2^{\mathrm{t(k)}} = \mathrm{N}\lambda_4 \tag{5.5}$$

The modified condition $\lambda_2^2 = \lambda_4$, leads to $N=\dfrac{(r2^{t(k)}+2n_a a^2)^2}{\lambda 2^{t(k)}}$ (Alternatively N may be obtained directly as $\mathrm{N} = \mathrm{b}2^{\mathrm{t(k)}} + 2\mathrm{vn}_{\mathrm{a}} + \mathrm{n}_0$, where n_0 is given in equation (5.2)). Equations (5.4) and (5.5) leads to a^4 given in equation (5.1).

Example 5.1: Consider the augmented SORD with the help of a BIBD, $(v=8, b=14, r=7, k=4, \lambda=3)$. Here, equations (5.3), (5.4) and (5.5) are

$$\sum \mathrm{x}_{\mathrm{iu}}^2 = 112 + 2\mathrm{n}_{\mathrm{a}}\mathrm{a}^2 = \mathrm{N}\lambda_2 \tag{5.6}$$

$$\sum \mathrm{x}_{\mathrm{iu}}^4 = 112 + 2\mathrm{n}_{\mathrm{a}}\mathrm{a}^4 = 5\mathrm{N}\lambda_4 \tag{5.7}$$

$$\sum \mathrm{x}_{\mathrm{iu}}^2 \mathrm{x}_{\mathrm{ju}}^2 = 48 = \mathrm{N}\lambda_4 \tag{5.8}$$

Equations (5.7) and (5.8) leads to $n_a a^4 = 64$, which implies $a^2 = 4.00$ for $n_a = 4$. From equations (5.6) and (5.8) using the modified condition ($\lambda_2^2 = \lambda_4$), with $a^2 = 4.00$ and $n_a = 4$, we get N=432. Equation (5.2) leads to n_0=144. Here with $c=3$, $n_a=1$, and $n_o=1$, we get Das and Narasimham (1962) rotatability level

$a^4 = (3\lambda - r)2^{t(k)-1} = 16$, i.e., $a^2 = 4.00$ and N=241. Thus with $n_a = 4$, $a^2 = 4.00$, we get an augmented SORD as SOSRD in N=432 design points using BIBD for 8-factors.

Table 3.1 A list of modified SOSRD using BIBD for $4 \le v \le 16$

(v, b, r, k, λ)	t(k)	n_a	a^2	n_0	N	$V\left(\frac{\delta \hat{y}}{\delta x_i}\right)\sigma^{-2}$
(4,6,3,2,1)	2	1	2.00	32	64	(0.0625+0.25d^2)
(5,10,6,3,3)	3	1	6.00	60	150	(0.016667+0.041667d^2)
(6,15,5,2,1)*	2	--	--	40	100	(0.05+0.25d^2)
(7,7,3,3,1)	3	2	2.00	44	128	(0.03125+0.125d^2)
(8,14,7,4,3)	4	4	4.00	144	432	(0.006944+0.020833d^2)
(9,12,4,3,1)	3	1	2.00	48	162	(0.027778+0.125d^2)
(10,18,9,5,4)	4	1	4.00	53	361	(0.006579+0.015625d^2)
(10,15,6,4,2)	4	2	4.00	112	392	(0.008929+0.03125d^2)
(10,45,9,2,1)	7	2	0.125	142	578	(0.014706+0.125d^2)
(11,55,15,3,3)*	3	--	--	160	600	(0.008333+0.041667d^2)
(12,33,11,4,3)	4	2	4.00	192	768	(0.005208+0.020833d^2)
(13,13,4,4,1)	4	2	2.00	140	400	(0.0125+0.0625d^2)
(15,15,7,7,3)	6	1	16.00	210	1200	(0.002083+0.005208d^2)
(16,20,5,4,1)*	4	--	--	80	400	(0.0125+0.0625d^2)
(16,16,6,6,2)	5	1	8.00	132	676	(0.004808+0.015625d^2)

* ‘a’ and ‘n_a’ are not needed for these cases as in these BIBD’s, r = 5λ (vide case (ii) of Theorem 3.1)

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