# Infinite dimensional Morse theory for geodesics on Finsler manifolds

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## 1. Questions and previous methods

Professor Hams-Bert Rademacher had given two beautiful lectures on geodesics. So I only give a few of related notions.

A $C^\infty$ **Finsler metric** on a $C^\infty$ manifold $M^n$ is a $C^0$ function $F : TM \to \mathbb{R}$ with properties:

**(i)** $C^\infty$ in $TM \setminus \{0\}$,

**(ii)** $F(v) > 0\ \forall v \in TM \setminus \{0\}$,

**(iii)** $F(tv) = tF(v)\ \forall t > 0$ and $v \in TM$,

**(iv)** $F^2$ is fiberwise strongly convex, i.e., for any $(x, y) \in TM \setminus 0_{TM}$ the symmetric bilinear form $g^F(x, y) : T_xM \times T_xM \to \mathbb{R}$ given by

$$(u, v) \mapsto \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(x, y + su + tv)\right]\Big|_{s=t=0}$$

is positive definite. Call $g^F$ the **fundamental tensor** of $(M, F)$.

**Geodesics** on $(M,F)$ can be characterized as critical points of the **energy functional** on a Hilbert manifold:

$$W^{1,2}([0,1],M) \ni \gamma \mapsto \mathcal{E}(\gamma) = \int_0^1 F^2(\gamma(t),\dot{\gamma}(t))dt.$$

Since our theory is of local nature we **restrict to** a compact Finsler manifold $(M,F)$ below.

Let $I=[0,1]$ and the Hilbert manifold $W^{1,2}(I,M)$ be equipped with the **Riemannian structure** induced by a Riemann metric $g$ on $M$:

$$\begin{aligned}\langle X,Y\rangle_1 &= \int_0^1 g(x(t))[X(t),Y(t)]dt \\ &+ \int_0^1 g(x(t))[\nabla_t^g X(t),\nabla_t^g Y(t)]dt.\end{aligned}$$

For a $C^\infty$ closed submanifold $Q$ of $M\times M$ we have a Riemannian-Hilbert submanifold

$$\Lambda_Q(M) := \{x \in W^{1,2}(I, M) \,|\, (x(0), x(1)) \in Q\}.$$

**Theorem 1** **(I)** *On* $\Lambda_Q(M)$ *the functional* $\mathcal{E}$ *is* $C^{2-0}$*, and satisfies the* **(PS) condition***.*
**(II)** *A curve* $\gamma \in \Lambda_Q(M)$ *is a (non constant) critical point of* $E$ **iff** *it is a constant (nonzero) speed geodesic on* $(M, F)$ *with (BC):*

$$g^F(\gamma(0), \dot\gamma(0))[V, \dot\gamma(0)] = g^F(\gamma(1), \dot\gamma(1))[W, \dot\gamma(1)]$$

*for any* $(V, W) \in T_{(\gamma(0),\gamma(1))}Q$*.*

[Caponio-Javaloyes-Masiello, Math.Ann. 2010], [Mercuri, Math.Z,156(1977),231-245] for $Q = \triangle_M$, [Kozma-Kristaly-Varga, Contributions to Algebraic Geometry, 45(2004),47-59] for $Q = M_1 \times M_2$ with submanifolds $M_i$ of $M$, $i = 1, 2$.

**Note**: Abbondandolo and Schwarz in [Advanced Nonlinear Studies, **9**(2009),597-623] showed: **the above energy functional $\mathcal{E}$ is twice differentiable at a (nonconstant) critical point $\gamma$** *iff* **$F^2$ is Riemannian along $\gamma$.**

In order to study **existence and multiplicity** of critical points of $\mathcal{E}$ with Morse theory one need:

- to compute **critical groups**

$$C_*(\mathcal{E}, \gamma_0; \mathbb{K}) = H_*(\Lambda_Q(\gamma_0) \cup \{\gamma_0\}, \Lambda_Q(\gamma_0); \mathbb{K})$$

at a critical point $\gamma_0$ (where $\Lambda_Q(\gamma_0) = \{\mathcal{E} < \mathcal{E}(\gamma_0)\}$), **or**

$$C_*(\mathcal{E}, S^1 \cdot \gamma_0; \mathbb{K}) = H_*(\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0); \mathbb{K})$$

at a critical orbit $S^1 \cdot \gamma_0$ if $Q = \triangle_M$,

• to prove the induced homomorphism

$$(\varphi_m)_* : H_*(\Lambda(\gamma_0) \cup \{S^1 \cdot \gamma_0\}, \Lambda(\gamma_0); \mathbb{K})$$
$$\to H_*(\Lambda(\gamma_0^m) \cup \{S^1 \cdot \gamma_0^m\}, \Lambda(\gamma_0^m); \mathbb{K})$$

are isomorphisms under suitable conditions, where the $m$-th iterate

$$\varphi_m : \Lambda M \to \Lambda M, \gamma \to \gamma^m \tag{1}$$

by $\gamma^m(t) = \gamma(mt)\ \forall t \in \mathbb{R}$.

On Riemannian manifolds the energy functional $\mathcal{E}$ are smooth and these can be completed with the **Gromoll-Meyer splitting lemma** for $C^2$-functionals on Hilbert spaces and its corollary— **shifting theorem.**

So one cannot directly apply Gromoll-Meyer theory to the Finsler energy functional $\mathcal{E}$. In past almost all literatures one completed the related arguments by using Gromoll-Meyer theory on the **Morse's finite-dimensional approximation** of $\Lambda_Q(M)$. For $a \in (0, \infty)$ and a large $k \in \mathbb{N}$ let

$$\Lambda_Q^a(M) = \{\gamma \in \Lambda_Q(M) : \mathcal{E}(\gamma) \le a\},$$

$$\Lambda_Q^a(k, M) = \{\gamma \in \Lambda_Q^a(M) : \gamma|[\frac{i}{k}, \frac{i+1}{k}] \text{ is } F\text{-geodesic}\}.$$

Then $\Lambda_Q^a(k, M)^\circ := \Lambda_Q^a(k, M) \cap \mathrm{Int}(\Lambda_Q^a(M))$ is a smooth manifold of finite dimension, and

- the restriction of $\mathcal{E}$ to $\Lambda_Q^a(k, M)^\circ$ is smooth, and has the same critical set as $\mathcal{E}$ in $\mathrm{Int}(\Lambda_Q^a(M))$,
- for each $c < a$ the set

$$\Lambda_Q^c(k, M) = \Lambda_Q^a(k, M) \cap \Lambda_Q^c(M)$$

is compact, and a **deformation retract** of $\Lambda_Q^c(M)$ (this is also true for $c = a$).

Hence by applying Gromoll-Meyer theory to the restriction of $\mathcal{E}$ to $\Lambda^a_Q(k,M)^\circ$ one can complete the desired arguments. These can be found in

- H.H.Matthias [Bonner Math.Schr.128(1980)],
- H.B.Rademacher[[Bonner Math.Schr.229(1992)]
- Z.Shen [Lecture on Finsler Geometry, 2001]
- V.Bangert & Y.Long[Math.Ann.346(2010)335].

As on Riemannian manifolds it is expected to develop infinite-dimensional Morse theory methods for geodesics problem on Finsler manifolds for conveniences in some cases.

We state the expected first result for $Q=\{p\}\times\{q\}$ with $p\ne q$. Let $\gamma_0$ be a nonconstant cpt of

$$\Lambda_Q(M)\ni\gamma\mapsto\mathcal{E}(\gamma)=\int_0^1 F^2(\gamma(t),\dot\gamma(t))dt.$$

Then $\gamma_0 \in C^\infty_Q(I, M)$ and $\exists\ c > 0$ such that

$$F(\gamma_0(t), \dot\gamma_0(t)) = \sqrt{c} \quad \forall t \in [0, 1]. \qquad (2)$$

Note the restriction of $\mathcal{E}$ to Banach manifold

$$C^1_Q(I, M) = \{x \in C^1(I, M) \,|\, (x(0), x(1)) \in Q\}$$

is $C^2$ near each regular curve, but it does not satisfy the **(PS) condition** on this space.

Let exp be the exponential map of a Riemann metric $g$ on $M$, and take

$$0 < 2\rho < \inf\{\mathrm{inj}(\gamma_0(t), g) | t \in [0, 1]\}.$$

Let $e_1, \cdots, e_n$ be a parallel orthonormal frame along $\gamma_0$. Define $\varphi : I \times B^n(0, \rho) \to M$ by

$$\varphi(t, x_1, \cdots, x_n) = \exp_{\gamma_0(t)}(x_1 e_1(t) + \cdots + x_n e_n(t)).$$

It induces a chart

$$\varphi_* : H^1_0(I, B^n(0, \rho)) \to H^1_Q(I, M)$$

by $\varphi_*(x)(t) = \varphi(t, x(t))$. Then $\varphi_*(0) = \gamma_0$ and

$$\widetilde{\mathcal{E}}(x) := \mathcal{E} \circ \varphi_*(x) = \int_0^1 \widetilde{F}^2(t, x(t), \dot{x}(t))dt.$$

Here $\widetilde{F} : I \times B^n(0, \rho) \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\widetilde{F}(t, x, v) := F(\varphi(t, x), d\varphi(t, x)[(1, v)]).$$

It is $C^2$ in $(I \times B^n(0, \rho) \times \mathbb{R}^n) \setminus Z$, where $Z =$

$\{(t, x, v) \in I \times B^n(0, \rho) \times \mathbb{R}^n \,|\, d\varphi(t, x)[(1, v)] = 0\}$.

Let us define

$$\begin{aligned} H &:= H^1_0(I, \mathbb{R}^n), \\ X &:= \{x \in C^1(I, \mathbb{R}^n) \,|\, x(0) = x(1) = 0\}, \\ \mathcal{U} &:= H^1_0(I, B^n(0, \rho)) = \{x \in H : x(I) \subset B^n(0, \rho)\}, \\ \mathcal{U}_X &:= X \cap \mathcal{U} = \{x \in X \,|\, x(I) \subset B^n(0, \rho)\}. \end{aligned}$$

Note that $\varphi_*$ restricts to a chart

$$\varphi_*^X : \mathcal{U}_X \to C^1_Q(I, M).$$

Let $B_\delta^X(0) := \{x \in X : \|x\|_X < \delta\}$. Then

$$B_\delta^X(0) \subset \mathcal{U}_X \quad \text{for small } \delta > 0.$$

Since $F^2(\gamma_0(t), \dot\gamma_0(t)) \equiv c > 0$ by (2), we shrink $\delta > 0$ so that $\forall x = \varphi_*^X(\tilde x)$ with $\tilde x \in B_\delta^X(0)$,

$$F^2(x(t), \dot x(t)) > \frac{2}{3}c \quad \forall t \in [0,1]. \qquad (3)$$

Then

- $\widetilde{\mathcal{E}}^X := \mathcal{E} \circ \varphi_*^X = \widetilde{\mathcal{E}}|_{\mathcal{U}_X}$ is $C^2$ on $B_\delta^X(0)$ and
- $\exists$ a Fredholm operator $B_0 \in \mathcal{L}_s(H)$ s.t.

$$d^2\widetilde{\mathcal{E}}^X(0)(u,v) = (B_0 u, v)_H \ \forall u, v \in X. \qquad (4)$$

($B_0$ is the second G-differential at 0 of $\widetilde{\mathcal{E}}^X$. $m^-(\gamma_0) = \max\{\dim S | \text{subspace}\, S \subset H, B_0|_S < 0\}$ is called **Morse index** $\mathcal{E}$ at $\gamma_0$.)

- $N := \mathrm{Ker}(B_0) \subset X$ and so orthogonal decomposition $H = N \oplus N^\perp$ induces a topological direct sum decomposition of closed subspaces

$X = N \oplus (N^{\perp} \cap X)$ with projection operator $(I - P_N)|_X : X \to N^{\perp} \cap X$.

• The gradient $\nabla\widetilde{\mathcal{E}}$ on $H$ maps $B^X_\delta(0)$ into $X$, and thus give $C^1$-maps

$$A : B^X_\delta(0) \to X,\ x \mapsto \nabla\widetilde{\mathcal{E}}(x), \tag{5}$$
$$E : (N \cap B^X_\delta(0)) \oplus (N^{\perp} B^X_\delta(0)) \to N^{\perp} \cap X$$

given by $E(u + v) = (I - P_N)|_X \circ A(u + v)$. Applying **IFT** to $E$ we get $r \in (0, \delta)$, $C^1$-map

$$h : B^X_r(0) \cap N \to N^{\perp} \cap B^X_\delta(0) \tag{6}$$

such that

$$(I - P_N)|_X \circ A(u + h(u)) = E(u + h(u)) = 0 \tag{7}$$

for all $u \in B^X_r(0) \cap N$. Define

$$\begin{aligned}\widetilde{\mathcal{E}}^{\circ} : B^X_r(0) \cap N \to \mathbb{R}, u &\mapsto \widetilde{\mathcal{E}}(u + h(u)) \\ &= \mathcal{E}(\varphi_*(u + h(u)).\end{aligned}$$

It is $C^2$ and has an isolated critical point 0.

(**Expected**) shifting theorem: $\forall q \in \mathbb{N} \cup \{0\}$,

$$
\begin{aligned}
C_q(\mathcal{E}, \gamma_0; \mathbb{K}) &\cong C_q(\widetilde{\mathcal{E}}, 0; \mathbb{K}) && (8)\\
&\cong C_{q-m^-(\gamma_0)}(\widetilde{\mathcal{E}}^\circ, 0; \mathbb{K})? && (9)
\end{aligned}
$$

Since $\varphi_*$ is a coordinate chart, (8) is obvious.

E. Caponio, M. A. Javaloyes, A. Masiello [ Analyse Nonlinéaire 27(2010)857] used some ideas of K.-C. Chang in [Sci.Sinica Ser.A26(1983) 1241] and book(1993) to prove a splitting lemma of $\widetilde{\mathcal{E}}_X$ near $0 \in X = \{x \in C^1(I, \mathbb{R}^n) | x(0) = x(1) = 0\}$:$\exists$ **homeomorphism** $\phi : B^X_\eta(0) \to \phi(B^X_\eta(0))$ **with** $\phi(0) = 0$ **(**$\eta < r$**) such that**

$$\widetilde{\mathcal{E}}^X(\phi(x)) = \frac{1}{2}(B_0 v, v)_H + \widetilde{\mathcal{E}}(u + h(u))$$

**for any** $x \in B^X_r(0)$, $u = P_N x$ **and** $v = x - u$.

(**Actually**, this splitting lemma can also be proved by a generalization of Morse lemma by M. Jiang [Nonl.Anal,36(1999)943], see author's paper [arXiv:0909.0609 v1, 3 Sep 2009].)

As usual this implies for any $q \in \mathbb{N} \cup \{0\}$,

$$C_q(\widetilde{\mathcal{E}}^X, 0; \mathbb{K}) \cong C_{q-m^-(\gamma_0)}(\widetilde{\mathcal{E}}^\circ, 0; \mathbb{K}). \qquad (10)$$

However, they can only prove

$$C_*(\widetilde{\mathcal{E}}, 0; \mathbb{K}) \cong C_*(\widetilde{\mathcal{E}}^X, 0; \mathbb{K}) \qquad (11)$$

and so the shifting theorem in (9) if

$\gamma_0$ (so 0) is a nondegenerate cpt.

See [E. Caponio, M. A. Javaloyes, A. Masiello, Addendum to "Morse theory... of a Finsler metric" [Ann.I.H.Poincaré-AN 27 (3)(2010)857-876].Ann.I.H.Poincaré-AN 30(5)(2013)961-968].

Their proof used ideas and techniques in

- Abbondandolo and Schwarz [Advanced Nonlinear Studies, **9**(2009),597-623],
- K.-C. Chang in [Sci.Sinica Ser.A26(1983) 1241-1255],
- Palais [Th. 16,17, Topology 5(1966)1-16].

And nondegeneracy of cpt $\gamma_0$ was essential to their Proposition 3: $\exists\ \mu_0 > 0$ s.t.

$$d\widetilde{\mathcal{E}}(u)[B_0 u] \geq \mu_0 \|\nabla\widetilde{\mathcal{E}}(u)\|^2\ \forall u \text{ near } 0 \in H,$$

where $B_0$ represents the second G-differential at 0 of $\widetilde{\mathcal{E}}^X$ as in (4).

## Our methods

Follow author's preprint [arXiv:1212.2078v5]. We begin with our new splitting lemma in [Corrigendum: The Conley conjecture..., J.Funct. Anal.261(2011)542-589] (a more general version is in [The splitting lemmas for nonsmooth functionals on Hilbert spaces, Discr.Cont.Dyna. Syst-A. 33(2013)2939, arXiv:1102.2062v1]).

Hilbert space $(H, (\cdot,\cdot)_H)$, $\|\cdot\| = \sqrt{(\cdot,\cdot)_H}$, and Banach space $(X, \|\cdot\|_X)$ satisfy
**(S)** $X \subset H$ is dense in $H$ and $\|x\| \le \|x\|_X$ $\forall x$.
$\mathcal{U}$— open neighborhood of $0 \in H$,
$\mathcal{U}_X := \mathcal{U} \cap X$ open neighborhood of $0 \in X$.
$\mathcal{L} \in C^1(\mathcal{U}, \mathbb{R})$ has 0 as an isolated cpt.
<u>Assumptions</u>: $\exists$ maps $A \in C^1(\mathcal{U}_X, X)$ and $B \in$

$C(\mathcal{U}_X, \mathcal{L}_s(H))$ such that

$$\mathcal{L}'(x)(u) = (A(x), u)_H \quad \forall x \in \mathcal{U}_X \ \&\ u \in X,$$
$$(A'(x)(u), v)_H = (B(x)u, v)_H \ \forall x \in \mathcal{U}_X, \ \ u, v \in X.$$

(These imply: (a) $\mathcal{L}|_{\mathcal{U}_X} \in C^2(\mathcal{U}_X, \mathbb{R})$,
(b) $d^2\mathcal{L}|_{\mathcal{U}_X}(x)(u,v) = (B(x)u, v)_H \ \forall x \in \mathcal{U}_X \ \&\ u, v \in X$, (c) $B(x)(X) \subset X \ \forall x \in \mathcal{U}_X$).

Furthermore we also assume $B$ to satisfy:

**(B1)** $\{u \in H \,|\, B(0)(u) \in X\} \subset X$, and
$B(0)u = \lambda u$ for $u \in H \ \&\ \lambda < 0 \Rightarrow u \in X$.

**(B2)** $B : \mathcal{U}_X \to \mathcal{L}_s(H)$ has a decomposition

$$B(x) = P(x) + Q(x) \quad \forall x \in \mathcal{U}_X,$$

$P(x) \in \mathcal{L}_s(H)$ is positive definite, $Q(x) \in \mathcal{L}_s(H)$ is compact, and also satisfy:

**(i)** $\forall\ (x_k) \subset \mathcal{U}$ with $\|x_k\| \to 0$ it holds that

$$\|P(x_k)u - P(0)u\| \to 0 \ \forall u \in H;$$

**(ii)** $\forall\,(x_k) \subset \mathcal{U} \cap X$ with $\|x_k\| \to 0$ we have

$$\|Q(x_k) - Q(0)\|_{\mathcal{L}_s(H)} \to 0 \text{ as } k \to \infty;$$

**(iii)** $\exists$ constants $\eta_0 > 0$ and $C_0 > 0$ such that

$$(P(x)u, u) \geq C_0\|u\|^2 \quad \forall u \in H,\ \forall x \in B^H_{\eta_0}(0) \cap X.$$

$N := \text{Ker}(B(0))$, $H^-$ =maximal negative definite subspace of $B(0)$. ((B1)+(B2)$\Rightarrow$

$$\dim(N + H^-) < \infty \ \&\ N + H^- \subset X.$$

Call $m^0 = \dim N$ **nullity**, $m^- = \dim H^-$ **Morse index**, the cpt 0 **nondegenerate** if $m^0 = 0$.

$P_N : H \to N$ is the orthogonal projection, $X = N \oplus (N^\perp \cap X)$ is a topological direct sum decomposition. For $\delta > 0$ let $B^H_\delta(0) = \{x \in H\ :\ \|x\| < \delta\}$. (**Note**: norms $\|\cdot\|$

and $\|\cdot\|_X$ are equivalent on $N$ since $\dim N < \infty$.) The following is our splitting lemma.

**Theorem 2** *Under (S) and (B1)-(B2),* $\exists \epsilon > 0$, $C^1$ *map* $h : B^N_\epsilon(0) = B^H_\epsilon(0) \cap N \to N^\perp \cap X$ *satisfying* $h(0) = 0$ *and*

$$(I - P_N)A(z + h(z)) = 0 \quad \forall z \in B^N_\epsilon(0), \quad (12)$$

- *an open neighborhood* $W$ *of* $0$ *in* $H$,
- *an origin-preserving homeomorphism*

$$\Phi : B^N_\epsilon(0) \times B^{N^\perp}_\epsilon(0) \to W$$

*such that for all* $(z, u) \in B^N_\epsilon(0) \times B^{N^\perp}_\epsilon(0)$,

$$\begin{aligned} \mathcal{L} \circ \Phi(z, u) &= \|u - P_N u - P_{H^-} u\|^2 - \|P_{H^-} u\|^2 \\ &+ \mathcal{L}(z + h(z)). \end{aligned}$$

**(i)** $\Phi(z, 0) = z + h(z) \; \forall z \in B_N(0, \epsilon)$.

**(ii)** $\forall z \in B^N_\epsilon(0)$, $h'(z)$ *is equal to*

$-[P_{N^\perp} A'(z + h(z))|_{N^\perp \cap X}]^{-1} \circ P_{N^\perp} A'(z + h(z))|_N$.

**(iii)** $B^N_\epsilon(0) \ni z \mapsto \mathcal{L}^\circ(z) := \mathcal{L}(z + h(z))$ *is* $C^2$, *has* 0 *as an isolated cpt* , $d^2\mathcal{L}^\circ(0) = 0$, *and*

$$d\mathcal{L}^\circ(z_0)(z) = (A(z_0 + h(z_0)), z)_H$$

*for all* $(z_0, z) \in B^N_\epsilon(0) \times N$.

**(iv)** *Let* $c = \mathcal{L}(0)$, $\mathcal{L}^X = \mathcal{L}|_{\mathcal{U}^X}$, $\mathcal{W} \subset \mathcal{U}$ *be a neighborhood of* 0 *(so* $\mathcal{W}_X := \mathcal{W} \cap X$ *a neighborhood of* 0 *in* $X$*),* $\mathbb{K}$ *be an* Abel group. *Then the inclusion*

$$\left(\mathcal{L}^X_c \cap \mathcal{W}_X, \mathcal{L}^X_c \cap \mathcal{W}_X \setminus \{0\}\right) \hookrightarrow (\mathcal{L}_c \cap \mathcal{W}, \mathcal{L}_c \cap \mathcal{W} \setminus \{0\}) \tag{13}$$

*induces* **surjective homomorphisms**

$$H_*\left(\mathcal{L}^X_c \cap \mathcal{W}_X, \mathcal{L}^X_c \cap \mathcal{W}_X \setminus \{0\}; \mathbb{K}\right) \to H_*\left(\mathcal{L}_c \cap \mathcal{W}, \mathcal{L}_c \cap \mathcal{W} \setminus \{0\}; \mathbb{K}\right). \tag{14}$$

**Corollary 3 (Shifting)** *For any Abel group* $\mathbb{K}$

$$C_q(\mathcal{L}, 0; \mathbb{K}) \cong C_{q-m^-}(\mathcal{L}^\circ, 0; \mathbb{K}) \quad \forall q = 0, 1, \cdots .$$

**Corollary 4** *For any field* $\mathbb{K}$ *the surjective homomorphisms in (14) are all* **isomorphisms !**

• Th. 2⇒Cor. 3 is standard, see Mawhin-Willem's book (89) and Chang's book (93).

• Under assumptions of Th.2 we use a generalized Morse lemma by M.Jiang [Nonl.Anal,36(1999) 943] to prove a splitting lemma for $\mathcal{L}^X$ at 0:
$\exists$ **ball** $B^X_\delta(0) \subset \mathcal{U}_X \cap B^H_\epsilon(0)$**, an origin-preserving local homeomorphism** $\varphi$ **from** $B^X_\delta(0)$ **to a neighbor. of** $0$ **in** $\mathcal{U}_X$ **such that**

$$\mathcal{L}^X \circ \varphi(x) = \frac{1}{2}(B(0)x^\perp, x^\perp)_H + \mathcal{L}^\circ(z) \quad (15)$$

**for** $x \in B^X_\delta(0)$**,where** $z = P_N(x)$**,** $x^\perp = x - z$**.**

So for any Abel group $\mathbb{K}$ and $q \in \mathbb{N} \cup \{0\}$,

$$C_q(\mathcal{L}^X, 0; \mathbb{K}) \cong C_{q-m^-}(\mathcal{L}^\circ, 0; \mathbb{K}). \tag{16}$$

When $\mathbb{K}$ is a field, this, Cor.3 and (14) lead to Cor.4, which is important for our methods.

A Lagrangian $L : [0,1] \times TM \to \mathbb{R}$ is called **convex quadratic growth** (CQG) if it satisfies:

**(L1)** $\exists$ constant $\ell_0 > 0$ such that

$$\partial_{vv} L(t,x,v) \geq \ell_0 I,$$

**(L2)** $\exists$ constant $\ell_1 > 0$ such that

$$\begin{aligned}
&\|\partial_{vv} L(t,x,v)\| \leq \ell_1 \quad \text{and} \\
&\|\partial_{xv} L(t,x,v)\| \leq \ell_1(1 + |v|_x), \\
&\|\partial_{xx} L(t,x,v)\| \leq \ell_1(1 + |v|_x^2)
\end{aligned}$$

with respect to some Riemann metric $g$ (with $|v|_x^2 = g_x(v,v)$).

In [J.Funct. Anal.261(2011)542-589] we had proved that for a CQG Lagrangian $L : [0,1] \times TM \to \mathbb{R}$ the corresponding energy functional on $W^{1,2}(S^1, M)$ satisfies the conditions of Theorem 2 in a suitable chart.

Assume $(M, F)$ is a compact Finsler manifold for simplicity. Since $F^2$ is not $C^2$ on $TM$, **our idea is to deform $F^2$ to a CQG Lagrangian** $L^*$. Fix a Riemann metric $g$ and write $|v|_x^2 = g_x(v,v)$. $\exists$ constant $C_1 \geq 1$,

$$|v|_x^2 \leq F^2(x,v) \leq C_1 |v|_x^2 \quad \forall (x,v) \in TM. \quad (17)$$

Given $c > 0$, choose $0 < \varepsilon < \delta < \frac{2c}{3C_1}$. We have suitable constants $\kappa > 0$, $\varrho_0 < 0$, $\mu > 0$ and $b > 0$, and $C^\infty$ functions

$$\psi_{\varepsilon,\delta} : [0,\infty) \to \mathbb{R}, \quad \phi_{\mu,b} : [0,\infty) \to \mathbb{R}$$

as in the following figure:

Define $L^* : TM \to \mathbb{R}$ by

$$L^*(x,v) = \frac{\psi_{\varepsilon,\delta}(F^2(x,v)) + \phi_{\mu,b}(|v|_x^2) + \mu\delta - \varrho_0}{\kappa}.$$

Clearly, $L^*$ is of $C^k$ if $F$ is only of $C^k$ ($k \geq 2$).

**Proposition 5** **(i)** *$L^*$ is CQG,*

**(ii)** $L^*(x,v) = F^2(x,v) \quad$ *if* $F^2(x,v) \geq \frac{2c}{3C_1}$,

**(iii)** $L^* \geq 0$, *and* $L^*(x,v) = 0 \Longleftrightarrow v = 0$,

**(iv)** $L^*(x,v) \leq F^2(x,v)\ \forall (x,v) \in TM$,

**(v)** *if $F$ is reversible, so is $L^*$,*

For $\tau \in [0,1]$ we define $L^\tau : TM \to \mathbb{R}$ by

$$L^\tau(x,v) = (1-\tau)F^2(x,v) + \tau L^*(x,v). \quad (18)$$

We present our results for two kinds of boundary conditions:

- $Q = M_0 \times M_1$, $M_0$ and $M_1$ are two disjoint boundaryless submanifolds of $M$,
- $Q = \triangle_M$.

## Case 1

Let $\gamma_0 \in \Lambda_Q(M)$ be an isolated nonconstant cpt of $\mathcal{L}$ on $\Lambda_Q(M)$. Then $\exists c > 0$, s.t. $F(\gamma_0(t), \dot{\gamma}_0(t)) \equiv \sqrt{c} > 0$. We can choose the Riemannian metric $g$ such that:

> **$M_0$ (resp. $M_1$) is totally geodesic near $\gamma_0(0)$ (resp. $\gamma_0(1)$).**

$\Lambda_Q(M)$ is equipped with *Hilbert-Riemannian structure* induced by $g$. Let exp denote the exponential map of $g$, and for $\rho > 0$ let

$$\mathbf{B}_{2\rho}(T_{\gamma_0}\Lambda_Q(M)) = \{\xi \in T_{\gamma_0}\Lambda_Q(M) \mid \|\xi\|_1 < 2\rho\}.$$

$\exists \rho > 0$ such that

$$\mathsf{EXP}_{\gamma_0} : \mathbf{B}_{2\rho}(T_{\gamma_0}\Lambda_Q(M)) \to \Lambda_Q(M) \qquad (19)$$

given by $\mathsf{EXP}_{\gamma_0}(\xi)(t) = \exp_{\gamma_0(t)}(\xi(t))$, is a coordinate chart around $\gamma_0$ on $\Lambda_Q(M)$. Define

$$\mathcal{L}^\tau(\gamma) = \int_0^1 L^\tau(\gamma(t), \dot\gamma(t))dt\ \forall \gamma \in \Lambda_Q(M) \quad (20)$$

for $L^\tau$ in (18).

Then $\mathcal{L}^\tau \circ \mathsf{EXP}_{\gamma_0}$ is $C^{2-0}$ and has an isolated cpt $0 \in T_{\gamma_0}\Lambda_Q(M) = W^{1,2}_Q(\gamma_0^* TM)$. Let $I = [0, 1]$. Consider the Banach manifold

$$\mathcal{X} = C^1_Q(I, M) = \{\gamma \in C^1(I, M) \mid (\gamma(0), \gamma(1)) \in Q\}.$$

$T_{\gamma_0}\mathcal{X} = \{\xi \in C^1(\gamma_0^* TM) \mid (\xi(0), \xi(1)) \in TQ\}$

with usual $C^1$-norm. Let $\mathcal{E}^X = \mathcal{E}|_{\mathcal{X}}$ and

$$\mathbf{B}_{2\rho}(T_{\gamma_0}\mathcal{X}) = \{\xi \in T_{\gamma_0}\mathcal{X} : \|\xi\|_{C^1} < 2\rho\}.$$

Then $\mathbf{B}_{2\rho}(T_{\gamma_0}\mathcal{X}) \subset \mathbf{B}_{2\rho}(T_{\gamma_0}\Lambda_Q(M)) \cap T_{\gamma_0}\mathcal{X}$. Let

$$\mathcal{A}^\tau = \nabla(\mathcal{L}^\tau \circ \mathsf{EXP}_{\gamma_0})|\mathbf{B}_{2\rho}(T_{\gamma_0}\mathcal{X}).$$

We can shrink $\rho > 0$ so that

$$\min_t F^2(\gamma(t), \dot\gamma(t)) \geq \frac{2c}{3C_1}\ \forall \gamma \in \mathsf{EXP}_{\gamma_0}\Big(\mathbf{B}_{2\rho}(T_{\gamma_0}\mathcal{X})\Big).$$

By Prop.5(ii), for any $\tau \in [0, 1]$

$$\mathcal{A}^\tau = \mathcal{A} = \nabla(\mathcal{E} \circ \mathsf{EXP}_{\gamma_0})|\mathbf{B}_{2\rho}(T_{\gamma_0}\mathcal{X})$$

is a $C^1$-map to $T_{\gamma_0}\mathcal{X}$ and

$$\langle d\mathcal{A}^\tau(0)[\xi], \eta\rangle_1 = d^2\mathcal{E}^X(\gamma_0)[\xi, \eta] \quad \forall \xi, \eta \in T_{\gamma_0}\mathcal{X}.$$

The symmetric bilinear form $d^2\mathcal{E}^X(\gamma_0)$ can be extended into such a form on $T_{\gamma_0}\Lambda_Q(M)$, also denoted by $d^2\mathcal{E}^X(\gamma_0)$. The associated self-adjoint operator is Fredholm, has finite dimensional **negative definite and null spaces**

$$\mathbf{H}^-(d^2\mathcal{E}^X(\gamma_0)) \quad \text{and} \quad \mathbf{H}^0(d^2\mathcal{E}^X(\gamma_0)),$$

which are actually contained in $T_{\gamma_0}\mathcal{X}$. Call

$$m^-(\gamma_0) := \dim \mathbf{H}^-(d^2\mathcal{E}^X(\gamma_0)) \quad \text{and}$$
$$m^0(\gamma_0) := \dim \mathbf{H}^0(d^2\mathcal{E}^X(\gamma_0))$$

**Morse index** and **nullity** of $\gamma_0$, respectively. $\exists$ the orthogonal decomposition

$$\begin{aligned} T_{\gamma_0}\Lambda_Q(M) &= \mathbf{H}^-(d^2\mathcal{E}^X(\gamma_0)) \oplus \mathbf{H}^0(d^2\mathcal{E}^X(\gamma_0)) \\ &\oplus \mathbf{H}^+(d^2\mathcal{E}^X(\gamma_0)), \end{aligned} \tag{21}$$

which induces a (topological) direct sum decomposition of Banach spaces

$$\begin{aligned} T_{\gamma_0}\mathcal{X} &= \mathbf{H}^-(d^2\mathcal{E}^X(\gamma_0)) \dot{+} \mathbf{H}^0(d^2\mathcal{E}^X(\gamma_0)) \\ &\dot{+} \Big(\mathbf{H}^+(d^2\mathcal{E}^X(\gamma_0)) \cap T_{\gamma_0}\mathcal{X}\Big). \end{aligned}$$

Using IFT, $\exists\ \delta \in (0, 2\rho]$ and a unique $C^1$-map

$h$ from ball $\mathbf{B}_\delta\Big(\mathbf{H}^0(d^2\mathcal{E}^X(\gamma_0))\Big) \subset \mathbf{B}_{2\rho}(T_{\gamma_0}\mathcal{X})$ to

$$\mathbf{H}^-(d^2\mathcal{E}^X(\gamma_0))\dot{+}\Big(\mathbf{H}^+(d^2\mathcal{E}^X(\gamma_0)) \cap T_{\gamma_0}\mathcal{X}\Big)$$

such that $h(0) = 0$, $dh(0) = 0$ and

$(I-P^0)\mathcal{A}(\xi+h(\xi)) = 0\ \forall\xi \in \mathbf{B}_\delta\Big(\mathbf{H}^0(d^2\mathcal{E}^X(\gamma_0))\Big)$,

where $P^\star : T_{\gamma_0}\Lambda_Q(M) \to \mathbf{H}^\star(d^2\mathcal{E}^X(\gamma_0))$, $\star = -, 0, +$, are orthogonal projections given by (21). Define $\mathcal{E}^\circ : \mathbf{B}_\delta\Big(\mathbf{H}^0(d^2\mathcal{E}^X(\gamma_0))\Big) \to \mathbb{R}$ by

$$\mathcal{E}^\circ(\xi) = \mathcal{E} \circ \mathsf{EXP}_{\gamma_0}\Big(\xi + h(\xi)\Big). \qquad (22)$$

It is $C^2$, has an isolated cpt 0, and $d^2\mathcal{E}^\circ(0) = 0$.

**Theorem 6** *Let* $\mathbb{K}$ *be an* Abel group*.*

**(i)** $C_*(\mathcal{L}^\tau, \gamma_0; \mathbb{K}) \cong C_*(\mathcal{E}, \gamma_0; \mathbb{K})\ \forall\tau \in [0, 1]$.

**(ii)** $\exists$ *a splitting lemma for* $\mathcal{L}^* \circ \mathsf{EXP}_{\gamma_0} := \mathcal{L}^1 \circ \mathsf{EXP}_{\gamma_0}$ *at* $0 \in T_{\gamma_0}\Lambda_Q(M)$*, i.e. (by shrinking* $\delta > 0$*)* $\exists$ *an origin-preserving homeomorphism*

$\psi$ *from* $\mathbf{B}_\delta(T_{\gamma_0}\Lambda_Q(M))$ *to an open neighborhood of* 0 *in* $T_{\gamma_0}\Lambda_Q(M)$ *such that*

$$\mathcal{L}^*\circ \mathsf{EXP}_{\gamma_0}\circ\psi(\xi) = \|P^+\xi\|_1^2 - \|P^-\xi\|_1^2 + \mathcal{E}^\circ(P^0\xi)$$

$\forall\xi\in\mathbf{B}_\delta(T_{\gamma_0}\Lambda_Q(M))$, *and hence* $\forall q\in\mathbb{N}\cup\{0\}$

$$C_q(\mathcal{L}^*, 0;\mathbb{K}) \cong C_{q-m^-(\gamma_0)}(\mathcal{E}^\circ, 0;\mathbb{K}). \qquad (23)$$

**(iii)** $\exists$ *a splitting lemma for* $\mathcal{E}^X\circ\mathsf{EXP}_{\gamma_0}$ *at* $0\in T_{\gamma_0}\mathcal{X}$, *i.e.* $\exists$ $\epsilon\in(0,\delta)$ *and an origin-preserving homeomorphism* $\varphi$ *from* $\mathbf{B}_\epsilon(T_{\gamma_0}\mathcal{X})$ *to an open neighborhood of* 0 *in* $T_{\gamma_0}\mathcal{X}$ *such that for any* $\xi\in\mathbf{B}_\epsilon(T_{\gamma_0}\mathcal{X})$,

$$\begin{aligned}\mathcal{E}^X\circ\mathsf{EXP}_{\gamma_0}\circ\varphi(\xi) &= \frac{1}{2}d^2\mathcal{E}^X(\gamma_0)[P^+\xi, P^+\xi]\\ &- \|P^-\xi\|_1^2 + \mathcal{E}^\circ(P^0\xi),\end{aligned}$$

*and hence* $\forall q\in\mathbb{N}\cup\{0\}$

$$C_q(\mathcal{E}^X, 0;\mathbb{K}) \cong C_{q-m^-(\gamma_0)}(\mathcal{E}^\circ, 0;\mathbb{K}). \qquad (24)$$

**(iv)** *Let $\mathcal{L}^{*X}=\mathcal{L}^*|_{\mathcal{X}}$, $\mathcal{W}\subset\Lambda_Q(M)$ be a neighborhood of $\gamma_0$ (so $\mathcal{W}_X:=\mathcal{W}\cap\mathcal{X}$ a neighborhood of $\gamma_0$ in $\mathcal{X}$). Then the inclusion*

$$\left(\mathcal{L}_c^{*X}\cap\mathcal{W}_X,\mathcal{L}_c^{*X}\cap\mathcal{W}_X\setminus\{\gamma_0\}\right)\hookrightarrow \\ (\mathcal{L}_c^*\cap\mathcal{W},\mathcal{L}_c^*\cap\mathcal{W}\setminus\{\gamma_0\}) \qquad (25)$$

*induces* **surjective homomorphisms**

$$H_*\left(\mathcal{L}_c^{*X}\cap\mathcal{W}_X,\mathcal{L}_c^{*X}\cap\mathcal{W}_X\setminus\{\gamma_0\};\mathbb{K}\right)\to \\ H_*\left(\mathcal{L}_c^*\cap\mathcal{W},\mathcal{L}_c^*\cap\mathcal{W}\setminus\{\gamma_0\};\mathbb{K}\right), \qquad (26)$$

*which are also* **isomorphisms** *if $\mathbb{K}$ is a* **field***.*

When $\mathcal{W}\subset\Lambda_Q(M)$ is a closed neighborhood of $\gamma_0$, $\mathcal{W}_X:=\mathcal{W}\cap\mathcal{X}$ is also a closed neighborhood of $\gamma_0$ in $\mathcal{X}$). Choose an open neighborhood $\mathcal{O}$ of $\gamma_0$ in $\mathcal{X}$ such that $\mathcal{W}_X\setminus\mathcal{O}$ is contained in the interior of $\mathcal{W}_X\setminus\{\gamma_0\}$ and that $\mathcal{L}^*=\mathcal{E}$ in $\mathcal{O}$. Then by **excision theorem for singular**

**homology groups** the inclusion

$$\left(\mathcal{E}_c^X \cap \mathcal{O}, \mathcal{E}_c^X \cap \mathcal{O} \setminus \{\gamma_0\}\right) \hookrightarrow \left(\mathcal{L}_c^{*X} \cap \mathcal{W}_X, \mathcal{L}_c^{*X} \cap \mathcal{W}_X \setminus \{\gamma_0\}\right) \quad (27)$$

induces **isomorphisms**

$$H_*\left(\mathcal{E}_c^X \cap \mathcal{O}, \mathcal{E}_c^X \cap \mathcal{O} \setminus \{\gamma_0\}; \mathbb{K}\right) \to H_*\left(\mathcal{L}_c^{*X} \cap \mathcal{W}_X, \mathcal{L}_c^{*X} \cap \mathcal{W}_X \setminus \{\gamma_0\}; \mathbb{K}\right) (28)$$

for any abel group $\mathbb{K}$.

These and Theorem 6 are sufficient for Morse theory arguments needed.

The above versions are convenient in applications. Their proofs can be completed in another chart. Since $M_0$ (resp. $M_1$) is totally geodesic near $\gamma_0(0)$ (resp. $\gamma_0(1)$) with respect to the metric $g$ on $M$. Since $\gamma_0$ is of class $C^\infty$

we may take a parallel orthogonal $C^\infty$ frame field along $\gamma_0$ with respect to the metric $g$, $I \ni t \to (e_1(t), \cdots, e_n(t))$. For a small open ball $B^n(0, 2\rho) \subset \mathbb{R}^n$ we get a $C^\infty$ map

$$\phi: I \times B^n(0, 2\rho) \to M,\ (t, v) \mapsto \exp_{\gamma_0(t)}\left(\sum_{i=1}^n v_i e_i(t)\right).$$

Since $\exists$ linear subspaces $V_i \subset \mathbb{R}^n$, $i = 0, 1$, such that $v \in V_i \Leftrightarrow \sum_{k=1}^n v_k e_k(i) \in T_{\gamma_0(i)} M_i$, $i = 0, 1$, by shrinking $\rho > 0$ (if necessary) we get

$$v \in V_i \cap B^n(0, 2\rho) \Leftrightarrow \phi(i, v) \in M_i, i = 0, 1.$$

Set $V := V_0 \times V_1$ and

$$H_V := \{\zeta \in W^{1,2}(I, \mathbb{R}^n) \,|\, (\zeta(0), \zeta(1)) \in V\},$$
$$X_V := \{\zeta \in C^1(I, \mathbb{R}^n) \,|\, (\zeta(0), \zeta(1)) \in V\}.$$

Use $(\cdot, \cdot)_{W^{1,2}}$ and $\|\cdot\|_{W^{1,2}}$ to denote the inner product and norm in $H_V$. Let $\mathbf{B}_{2\rho}(H_V) := \{\zeta \in$

$H_V \mid \|\zeta\|_{W^{1,2}} < 2\rho\}$. Then the map

$$\Phi : \mathbf{B}_{2\rho}(H_V) \to \Lambda_Q(M) \qquad (29)$$

defined by $\Phi(\zeta)(t) = \phi(t, \zeta(t))$, gives a coordinate chart around $\gamma_0$ on $\Lambda_Q(M)$. Define $\tilde{L}^\tau : I \times B^n_{2\rho}(0) \times \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{L}^\tau(t, x, v) = L^\tau\Big(\phi(t, x), d\phi(t, x)[(1, v)]\Big),$$

and $\widetilde{\mathcal{L}}^\tau : \mathbf{B}_{2\rho}(H_V) \to \mathbb{R}$ by

$$\widetilde{\mathcal{L}}^\tau(\xi) = \int_0^1 \tilde{L}^\tau(t, \xi(t), \dot{\xi}(t))dt \quad \forall \xi \in \mathbf{B}_{2\rho}(H_V).$$

We can prove that this family of functionals satisfies **stability theorem of critical groups** (see Th.8.8 in Mawhin-Willem's book (89) or Th.5.6 in Chang's book (93), or Cingolani and Degiovanni [Adv.Nonl.Stud.**9**(2009)679]). So for any $q = 0, 1, \cdots$, we get

$$\begin{aligned} C_q(\mathcal{E}, \gamma_0; \mathbb{K}) &= C_q(\widetilde{\mathcal{L}}^0, 0; \mathbb{K}) \\ &= C_q(\widetilde{\mathcal{L}}^1, 0; \mathbb{K}) = C_q(\mathcal{L}^*, \gamma_0; \mathbb{K}). \end{aligned}$$

Moreover we can prove that Theorem 2 can be applied to $\widetilde{\mathcal{L}}^* := \widetilde{\mathcal{L}}^1$. Hence Theorem 6 follows.

## Case 2

For $Q = \triangle_M$,

$$\begin{aligned} &\Lambda_Q(M) = \Lambda M := W^{1,2}(S^1, M) \\ &= \{\gamma \in W^{1,2}_{loc}(\mathbb{R}, M) \,|\, \gamma(t+1) = \gamma(t)\ \forall t \in \mathbb{R}\}. \end{aligned}$$

Here $S^1 := \mathbb{R}/\mathbb{Z} = \{[s] \,|\, [s] = s + \mathbb{Z},\, s \in \mathbb{R}\}$. $\exists$ equivariant and isometric operations of $S^1$-action on $W^{1,2}(S^1, M)$ and $TW^{1,2}(S^1, M)$:

$$\begin{aligned} &[s] \cdot \gamma(t) = \gamma(s+t), \quad \forall [s] \in S^1,\ \gamma \in \Lambda M, \\ &[s] \cdot \xi(t) = \xi(s+t), \quad \forall [s] \in S^1,\ \xi \in T_\gamma \Lambda M, \end{aligned}$$

which are **continuous**, but **not differentiable**. $\mathcal{E}$ is $S^1$-invariant. Let $\gamma_0 \in \Lambda M$ be a (nonconstant) cpt of $\mathcal{E}$ with critical value $c > 0$. The orbit $\mathcal{O} := S^1 \cdot \gamma_0$ is a **smooth critical submanifold** of $\mathcal{E}$ in $\Lambda M$. We assume: $\mathcal{O}$ **is an isolated critical orbit**.

Let $\mathcal{X} = C^1(S^1, M)$, $\pi : N\mathcal{O} \to \mathcal{O}$ be the normal bundle of $\mathcal{O}$ in $\Lambda M$. Then $XN\mathcal{O} := T_{\mathcal{O}}\mathcal{X} \cap N\mathcal{O}$ is a Banach vector bundle over $\mathcal{O}$. Define

$$\left.\begin{array}{l} N\mathcal{O}(\varepsilon) = \{(x,v) \in N\mathcal{O} \mid \|v\|_1 < \varepsilon\}, \\ XN\mathcal{O}(\varepsilon) = \{(x,v) \in XN\mathcal{O} \mid \|v\|_{C^1} < \varepsilon\}. \end{array}\right\}$$

Clearly, $XN\mathcal{O}(\varepsilon) \subset N\mathcal{O}(\varepsilon)$. (Replacing $M$ by a compact neighborhood of $\mathcal{O}$) we may assume $\varepsilon > 0$ so small that the map

$$\mathrm{EXP} : T\Lambda M(\varepsilon) = \{(x,v) \in T\Lambda M \mid \|v\|_1 < \varepsilon\} \to \Lambda M$$

defined by $\mathrm{EXP}(x,v)(t) = \exp_{x(t)} v(t)\ \forall t \in \mathbb{R}$, restricts to a $S^1$-equivariant diffeomorphism from the **normal disk bundle** $N\mathcal{O}(\varepsilon)$ onto a $S^1$-invariant open neighborhood of $\mathcal{O}$ in $\Lambda M$,

$$F : N\mathcal{O}(\varepsilon) \to \mathcal{N}(\mathcal{O}, \varepsilon). \tag{30}$$

Then

$$\mathcal{F} := \mathcal{E} \circ F \quad \text{and} \quad \mathcal{F}^* := \mathcal{L}^* \circ F \tag{31}$$

are $C^{2-0}$, $S^1$-invariant and satisfy (PS). Let

$\mathcal{F}^X = \mathcal{F}|N\mathcal{O}(\varepsilon) \cap XN\mathcal{O}$, $\mathcal{F}^{*X} = \mathcal{F}^*|N\mathcal{O}(\varepsilon) \cap XN\mathcal{O}$

and let $\mathcal{F}_x, \mathcal{F}_x^*, \mathcal{F}_x^X, \mathcal{F}_x^{*X}$ be restrictions of $\mathcal{F}$,$\mathcal{F}^*$ and $\mathcal{F}^X$, $\mathcal{F}^{*X}$ to the fibres at $x \in \mathcal{O}$. Let

$$A_x := \nabla \mathcal{F}_x | N\mathcal{O}(\varepsilon)_x \cap XN\mathcal{O}_x.$$

When $\delta > 0$ is small $A_x$ is a $C^1$ map from $XN\mathcal{O}(\delta)_x$ to $XN\mathcal{O}_x$ (so $\mathcal{F}_x^X$ is $C^2$ on $XN\mathcal{O}(\delta)_x$).

$A_{s\cdot x}(s\cdot v) = s\cdot A_x(v)\ \forall s \in S^1,\ v \in N\mathcal{O}(\varepsilon)_x \cap XN\mathcal{O}_x.$

Denote by $B_x$ the symmetric bilinear form $d^2\mathcal{F}_x^X(0)$ and by its extension on $N\mathcal{O}_x$. The associated self-adjoint operator is Fredholm, has finite dimensional **negative definite and null spaces**

$$\mathbf{H}^-(B_x) \quad \text{and} \quad \mathbf{H}^0(B_x).$$

Moreover, $\mathbf{H}^-(B_x) + \mathbf{H}^0(B_x) \subset XN\mathcal{O}_x$, and $\exists$ orthogonal decomposition

$$N\mathcal{O}_x = \mathbf{H}^-(B_x) \oplus \mathbf{H}^0(B_x) \oplus \mathbf{H}^+(B_x). \quad (32)$$

Since $B_{s\cdot x}(s\cdot\xi, s\cdot\eta) = B_x(\xi,\eta) \forall s \in S^1, x \in \mathcal{O}$, (32) leads to a natural Hilbert vector bundle orthogonal decomposition

$$N\mathcal{O} = \mathbf{H}^-(B) \oplus \mathbf{H}^0(B) \oplus \mathbf{H}^+(B) \qquad (33)$$

with $\mathbf{H}^\star(B)_x = \mathbf{H}^\star(B_x)$ for $x \in \mathcal{O}$ and $\star = +, 0, -$, which induces a Banach vector bundle (topological) direct sum decomposition

$$XN\mathcal{O} = \mathbf{H}^-(B) \dot{+} \mathbf{H}^0(B) \dot{+} (\mathbf{H}^+(B) \cap XN\mathcal{O}).$$

$m^-(\mathcal{O}) := \text{rank}\mathbf{H}^-(B)$ and $m^0(\mathcal{O}) := \text{rank}\mathbf{H}^0(B)$ are called **Morse index** and **nullity** of $\mathcal{O}$. When $m^0(\mathcal{O}) = 0$ the orbit $\mathcal{O}$ is called **nondegenerate**. Moreover $0 \le m^0(\mathcal{O}) \le 2n-1$.

Let $\mathbf{P}^\star : N\mathcal{O} \to \mathbf{H}^\star(B)$ be the orthogonal bundle projections, $\star = +, 0, -$, and let

$$\mathbf{H}^0(B)(\epsilon) = \mathbf{H}^0(B) \cap N\mathcal{O}(\epsilon) \text{ for } \epsilon > 0.$$

Then $\mathbf{H}^0(B)(\epsilon) \subset XN\mathcal{O}$ and we may shrink $\epsilon > 0$ so that $\mathbf{H}^0(B)(\epsilon) \subset XN\mathcal{O}(\delta)$ since $\text{rank}\mathbf{H}^0(B) < \infty$ and $\mathcal{O}$ is compact. By IFT, shrinking $\epsilon > 0$ assures: $\forall x \in \mathcal{O}$ $\exists$ unique $S^1_x$-equivariant $C^1$

map

$$\mathfrak{h}_x : \mathbf{H}^0(B)(\epsilon)_x \to \mathbf{H}^-(B)_x \dot{+} (\mathbf{H}^+(B)_x \cap XN\mathcal{O}_x) \tag{34}$$

such that $\mathfrak{h}_x(0_x) = 0_x$, $d\mathfrak{h}_x(0_x) = 0_x$ and

$$(\mathbf{P}_x^+ + \mathbf{P}_x^-) \circ A_x\Big(v + \mathfrak{h}_x(v)\Big) = 0 \; \forall v \in \mathbf{H}^0(B)(\epsilon)_x.$$

Moreover, the functional $\mathcal{E}^\circ_\triangle$ given by

$$\mathbf{H}^0(B)(\epsilon) \ni (x, v) \to \mathcal{E} \circ \mathsf{EXP}_x\Big(v + \mathfrak{h}_x(v)\Big) \tag{35}$$

is $C^1$, has the **isolated critical orbit** $\mathcal{O}$ and restricts to a $C^2$ functional $\mathcal{E}^\circ_{\triangle x}$ in fiber $\mathbf{H}^0(B)(\epsilon)_x$.

**Theorem 7** **(i)** $C_*(\mathcal{L}^\tau, \mathcal{O}; \mathbb{K}) = C_*(\mathcal{E}, \mathcal{O}; \mathbb{K}) \; \forall \tau$.
**(ii)** $\exists$ *a splitting lemma for* $\mathcal{F}^* := \mathcal{L}^* \circ F$ *near* $\mathcal{O} \subset N\mathcal{O}(\varepsilon)$*:* shrinking $\epsilon > 0$, $\exists$ a $S^1$-invariant open neighborhood $U$ of $\mathcal{O} \subset N\mathcal{O}$, a $S^1$-equivariant fiber-preserving, $C^1$ map $\mathfrak{h}$ given by (34), and a

$S^1$-equivariant fiber-preserving homeomorphism $\Upsilon : N\mathcal{O}(\epsilon) \to U$ such that for all $(x, u) \in N\mathcal{O}(\epsilon)$,

$$\begin{aligned}\mathcal{F}^* \circ \Upsilon(x, u) &= \mathcal{L}^* \circ \mathsf{EXP} \circ \Upsilon(x, u)\\ &= \|\mathbf{P}_x^+ u\|_1^2 - \|\mathbf{P}_x^- u\|_1^2 + \mathcal{E}^{\circ}_{\triangle x}(\mathbf{P}_x^0 u).\end{aligned}$$

**(iii)** $\exists$ *a splitting lemma for* $\mathcal{F}^X := \mathcal{E}^X \circ F$ *near* $\mathcal{O} \subset XN\mathcal{O}(\varepsilon)$*:* shrinking $\epsilon > 0$ $\exists$ $S^1$-invariant open neighborhood $V$ of $\mathcal{O} \subset XN\mathcal{O}$, $S^1$-equivariant fiber-preserving $C^1$ map $\mathfrak{h}$ given by (34), $S^1$-equivariant fiber-preserving homeomorphism $\Psi : XN\mathcal{O}(\epsilon) \to V$ such that for all $(x, v) \in XN\mathcal{O}(\epsilon)$,

$$\begin{aligned}\mathcal{E}^X \circ \mathsf{EXP} \circ \Psi(x, v) &= \frac{1}{2} d^2\mathcal{L}|_{\mathcal{X}}(x)[\mathbf{P}_x^+ v, \mathbf{P}_x^+ v]\\ &- \|\mathbf{P}_x^- v\|_1^2 + \mathcal{L}^{\circ}_{\triangle x}(\mathbf{P}_x^0 v).\end{aligned}$$

**(iv)** *For any open neighborhood* $\mathcal{W}$ *of* $\mathcal{O}$ *in* $\Lambda M$ *and a field* $\mathbb{K}$*, write* $\mathcal{W}_X = \mathcal{W} \cap \mathcal{X}$ *as an open subset of* $\mathcal{X}$*, then the inclusion*

$$((\mathcal{L}^*|_{\mathcal{X}})_c \cap \mathcal{W}_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap \mathcal{W}_X \setminus \mathcal{O}) \hookrightarrow (\mathcal{L}_c^* \cap \mathcal{W}, \mathcal{L}_c^* \cap \mathcal{W} \setminus \mathcal{O})$$

*induces isomorphisms*

$$H_*\left((\mathcal{L}^*|_{\mathcal{X}})_c \cap \mathcal{W}_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap \mathcal{W}_X \setminus \mathcal{O}; \mathbb{K}\right) \to H_*\left(\mathcal{L}^*_c \cap \mathcal{W}, \mathcal{L}^*_c \cap \mathcal{W} \setminus \mathcal{O}; \mathbb{K}\right).$$

*The corresponding conclusion is still true if* $((\mathcal{L}^*|_{\mathcal{X}})_c \cap \mathcal{W}_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap \mathcal{W}_X \setminus \mathcal{O})$ *and* $(\mathcal{L}^*_c \cap \mathcal{W}, \mathcal{L}^*_c \cap \mathcal{W} \setminus \mathcal{O})$ *are replaced by* $\left((\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c \cap \mathcal{W}_X \cup \mathcal{O}, (\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c \cap \mathcal{W}_X\right)$ *and* $\left(\mathring{\mathcal{L}}^*_c \cap \mathcal{W} \cup \mathcal{O}, \mathring{\mathcal{L}}^*_c \cap \mathcal{W}\right)$*, respectively, where* $\mathring{\mathcal{L}}^*_c = \{\mathcal{L}^* < c\}$ *and* $(\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c = \{\mathcal{L}^*|_{\mathcal{X}} < c\}$.

Let $\mathbf{H}^{0-}(B) = \mathbf{H}^0(B) + \mathbf{H}^-(B)$ and

$$\mathbf{H}^{0-}(B)(\epsilon) = (\mathbf{H}^0(B) + \mathbf{H}^-(B)) \cap N\mathcal{O}(\epsilon).$$

Then $\mathbf{H}^{0-}(B) \subset XN\mathcal{O}$. Define

$$\mathfrak{L} : \mathbf{H}^{0-}(B)(\epsilon) \to \mathbb{R} \tag{36}$$

by $\mathfrak{L}(x, v) = -\|\mathbf{P}^-_x v\|^2_1 + \mathcal{L}^\circ_{\triangle x}(\mathbf{P}^0_x v)$. The usual deformation arguments and two splitting lemmas in Th.7(ii)-(iii) lead to

$$\begin{aligned} C_*(\mathcal{L}^*, \mathcal{O}; \mathbb{K}) &\cong C_*(\mathfrak{L}, \mathcal{O}; \mathbb{K}) \\ &\cong C_*(\mathcal{E}^X, \mathcal{O}; \mathbb{K}), \end{aligned} \tag{37}$$

which can also be derived from Th.7(iv). This and Th.7(i) yield

$$\begin{aligned} C_*(\mathcal{E}, \mathcal{O}; \mathbb{K}) &\cong C_*(\mathcal{E}^X, \mathcal{O}; \mathbb{K}) \\ &\cong C_*(\mathfrak{L}, \mathcal{O}; \mathbb{K}). \end{aligned} \tag{38}$$

Let $S^1_x \subset S^1$ denote the stabilizer of $x \in \mathcal{O}$. It is a finite cyclic group and $\mathcal{L}^\circ_{\triangle x}$ is $S^1_x$-invariant. Let $C_*(\mathcal{L}^\circ_{\triangle x}, 0; \mathbb{K})^{S^1_x}$ denote the subgroup of all elements in $C_*(\mathcal{L}^\circ_{\triangle x}, 0; \mathbb{K})$, which are fixed by the induced action of $S^1_x$ on the homology. Since $\mathbf{H}^{0-}(B) \subset XN\mathcal{O}$ is a smooth manifold of finite dimension using (38) and repeating the arguments by Bangert and Long [Prop.3.7, *Math. Ann.*, **346**(2010)335] we can obtain their following generalization of the **Gromoll-Meyer shifting theorem** for Finsler manifolds.

**Theorem 8** *Let $\mathbb{K}$ be a field of characteristic 0 or prime to order $|S^1_{\gamma_0}|$ of $S^1_{\gamma_0}$. Then for any $x\in\mathcal{O}=S^1\cdot\gamma_0$ and $q=0,1,\cdots$,*

$$C_q(\mathcal{E},\mathcal{O};\mathbb{K})=$$

$$\Big(H_{m^-(\mathcal{O})}(\mathbf{H}^-(B)_x,\mathbf{H}^-(B)_x\backslash\{0_x\};\mathbb{K})\otimes C_{q-m^-(\mathcal{O})}(\mathcal{E}^\circ_{\triangle x},0;\mathbb{K})\Big)^{S^1_x}$$

$$\oplus\Big(H_{m^-(\mathcal{O})}(\mathbf{H}^-(B)_x,\mathbf{H}^-(B)_x\backslash\{0_x\};\mathbb{K})\otimes C_{q-m^-(\mathcal{O})-1}(\mathcal{L}^\circ_{\triangle x},0;\mathbb{K})\Big)^{S^1_x}.$$

*provided $m^-(\mathcal{O})m^0(\mathcal{O})>0$. Moreover,*

$$C_q(\mathcal{E},\mathcal{O};\mathbb{K})=\big(C_{q-1}(\mathcal{E}^\circ_{\triangle x},0;\mathbb{K})\big)^{S^1_x}\oplus\big(C_q(\mathcal{E}^\circ_{\triangle x},0;\mathbb{K})\big)^{S^1_x}$$

*if $m^-(\mathcal{O})=0$ and $m^0(\mathcal{O})>0$, and*

$$\begin{aligned}C_q(\mathcal{E},\mathcal{O};\mathbb{K}) &= H_q(\mathbf{H}^-(B),\mathbf{H}^-(B)\setminus\mathcal{O};\mathbb{K})\\ &= \Big(H_{q-1}(\mathbf{H}^-(B)_x,\mathbf{H}^-(B)_x\setminus\{0_x\};\mathbb{K})\Big)^{S^1_x}\\ &\oplus \Big(H_q(\mathbf{H}^-(B)_x,\mathbf{H}^-(B)_x\setminus\{0_x\};\mathbb{K})\Big)^{S^1_x}\end{aligned}$$

*if* $m^-(\mathcal{O})>0$ *and* $m^0(\mathcal{O})=0$. *Finally,*

$$C_q(\mathcal{E}, S^1\cdot\gamma_0;\mathbb{K}) = H_q(S^1;\mathbb{K})$$

*for any Abel group* $\mathbb{K}$ *if* $m^-(\mathcal{O})=m^0(\mathcal{O})=0$.

For a field $\mathbb{K}$ Th. 7(iv) claimed the inclusion

$$\left((\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c\cap\mathcal{W}_X\cup\mathcal{O}, (\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c\cap\mathcal{W}_X\right)\hookrightarrow\left(\mathring{\mathcal{L}}^*_c\cap\mathcal{W}\cup\mathcal{O}, \mathring{\mathcal{L}}^*_c\cap\mathcal{W}\right)$$

induces isomorphisms

$$\begin{aligned}&H_*\left((\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c\cap\mathcal{W}_X\cup\mathcal{O}, (\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c\cap\mathcal{W}_X;\mathbb{K}\right)\\ &\quad\to H_*\left(\mathring{\mathcal{L}}^*_c\cap\mathcal{W}\cup\mathcal{O}, \mathring{\mathcal{L}}^*_c\cap\mathcal{W};\mathbb{K}\right),\end{aligned}$$

and hence (by excision) the inclusion

$$\left((\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c\cup\mathcal{O}, (\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c\right)\hookrightarrow\left(\mathring{\mathcal{L}}^*_c\cup\mathcal{O}, \mathring{\mathcal{L}}^*_c\right)$$

induces isomorphisms

$$H_*\left((\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c\cup\mathcal{O}, (\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c;\mathbb{K}\right)\to H_*\left(\mathring{\mathcal{L}}^*_c\cup\mathcal{O}, \mathring{\mathcal{L}}^*_c;\mathbb{K}\right).$$

Take a neighborhood $\mathcal{V}$ of $\mathcal{O}$ in $\mathcal{X}$ such that $\mathcal{L}^* = \mathcal{E}$ in $\mathcal{V}$. Using the excision again we derive that the inclusion

$$\Big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V} \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V}\Big) \hookrightarrow \Big(\mathring{\mathcal{L}}_c^* \cup \mathcal{O}, \mathring{\mathcal{L}}_c^*\Big)$$

induces isomorphisms

$$H_*\left((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V} \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V}; \mathbb{K}\right) \to H_*\left(\mathring{\mathcal{L}}_c^* \cup \mathcal{O}, \mathring{\mathcal{L}}_c^*; \mathbb{K}\right).$$

It is $\mathcal{L}^* \leq \mathcal{E}$ (since $L^* \leq F^2$ by Prop.5(iv)) that we have the commutative diagram

$$\begin{array}{ccc}
\big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V} \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V}\big) & \xrightarrow{\text{Inclusion}} & \big(\mathring{\mathcal{E}}_c \cup \mathcal{O}, \mathring{\mathcal{E}}_c\big) \\
\text{Identity}\big\downarrow & & \big\downarrow\text{Inclusion} \\
\big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V} \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V}\big) & \xrightarrow{\text{Inclusion}} & \big(\mathring{\mathcal{L}}_c^* \cup \mathcal{O}, \mathring{\mathcal{L}}_c^*\big)
\end{array}$$

and hence commutative diagram

$$\begin{array}{ccc}
H_*\big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V} \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V}; \mathbb{K}\big) & \xrightarrow{\text{Homeomorphism}} & H_*\big(\mathring{\mathcal{E}}_c \cup \mathcal{O}, \mathring{\mathcal{E}}_c; \mathbb{K}\big) \\
\text{Identity}\big\downarrow & & \big\downarrow\text{Homeomorphism} \\
H_*\big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V} \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V}; \mathbb{K}\big) & \xrightarrow{\text{Isomorphism}} & H_*\big(\mathring{\mathcal{L}}_c^* \cup \mathcal{O}, \mathring{\mathcal{L}}_c^*; \mathbb{K}\big)
\end{array}$$

It follows that the homeomorphisms

$$H_*\Big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V} \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cap \mathcal{V}; \mathbb{K}\Big) \to H_*\Big(\mathring{\mathcal{E}}_c \cup \mathcal{O}, \mathring{\mathcal{E}}_c; \mathbb{K}\Big)$$

are injective, and hence isomorphisms since the associated groups are all vector spaces of same finite dimension. Using the excision again we deduce

**Claim 9** *The inclusion*

$$\Big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c\Big) \hookrightarrow \Big(\mathring{\mathcal{E}}_c \cup \mathcal{O}, \mathring{\mathcal{E}}_c\Big)$$

*induces isomorphisms*

$$H_*\Big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c; \mathbb{K}\Big) \to H_*\Big(\mathring{\mathcal{E}}_c \cup \mathcal{O}, \mathring{\mathcal{E}}_c; \mathbb{K}\Big).$$

For an integer $m > 0$ define the $m$-th iterate

$$\varphi_m : \Lambda M \to \Lambda M, \gamma \to \gamma^m \qquad (39)$$

by $\gamma^m(t) = \gamma(mt)\ \forall t \in \mathbb{R}$. For some integer $m > 1$, suppose $\varphi_m(\mathcal{O}) = S^1 \cdot \gamma_0^m$ is an isolated critical orbit of $\mathcal{E}$ in $\Lambda M$. Claim 9 implies that the inclusion

$$\Big((\mathring{\mathcal{E}}|_{\mathcal{X}})_{m^2c}\cup\varphi_m(\mathcal{O}),(\mathring{\mathcal{E}}|_{\mathcal{X}})_{m^2c}\Big)\hookrightarrow\Big(\mathring{\mathcal{E}}_{m^2c}\cup\varphi_m(\mathcal{O}),\mathring{\mathcal{E}}_{m^2c}\Big)$$

induces isomorphisms

$$H_*\big((\mathring{\mathcal{E}}|_{\mathcal{X}})_{m^2c}\cup\varphi_m(\mathcal{O}),(\mathring{\mathcal{E}}|_{\mathcal{X}})_{m^2c};\mathbb{K}\big)\to H_*\big(\mathring{\mathcal{E}}_{m^2c}\cup\varphi_m(\mathcal{O}),\mathring{\mathcal{E}}_{m^2c};\mathbb{K}\big). \tag{40}$$

Furthermore, let

$$m^-(\mathcal{O})=m^-(\varphi_m(\mathcal{O})),\qquad m^0(\mathcal{O})=m^0(\varphi_m(\mathcal{O})). \tag{41}$$

As in Riemannian geometry using the splitting lemma in Th. 7(iii) we may prove that

$$\varphi_m:\big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c\cup\mathcal{O},(\mathring{\mathcal{E}}|_{\mathcal{X}})_c\big)\to\big((\mathring{\mathcal{E}}|_{\mathcal{X}})_{m^2c}\cup\varphi_m(\mathcal{O}),(\mathring{\mathcal{E}}|_{\mathcal{X}})_{m^2c}\big)$$

induces isomorphisms

$$\begin{aligned}(\varphi_m)_*:\quad & H_*\Big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c\cup\mathcal{O},(\mathring{\mathcal{E}}|_{\mathcal{X}})_c;\mathbb{K}\Big)\\ & \to H_*\Big((\mathring{\mathcal{E}}|_{\mathcal{X}})_{m^2c}\cup\varphi_m(\mathcal{O}),(\mathring{\mathcal{E}}|_{\mathcal{X}})_{m^2c};\mathbb{K}\Big).\end{aligned}$$

Hence commutative diagram

$$\begin{array}{ccc}
\big((\mathring{\mathcal{E}}|_{\mathcal{X}})_c \cup \mathcal{O}, (\mathring{\mathcal{E}}|_{\mathcal{X}})_c\big) & \xrightarrow{\varphi_m} & \big(\mathring{\mathcal{E}}_{m^2c} \cup \varphi_m(\mathcal{O}), \mathring{\mathcal{E}}_{m^2c}\big) \\
\text{Inclusion}\big\downarrow & & \text{Inclusion}\big\downarrow \\
\big(\mathcal{E}_c \cup \mathcal{O}, \mathcal{E}_c\big) & \xrightarrow{\varphi_m} & \big(\mathcal{E}_{m^2c} \cup \varphi_m(\mathcal{O}), \mathcal{E}_{m^2c}\big)
\end{array}$$

lead to the following generalization of a result by Gromoll-Meyer on Finsler manifolds.

**Theorem 10** *For some integer $m > 1$, suppose that $\mathcal{O} = S^1 \cdot \gamma_0$ and $\varphi_m(\mathcal{O}) = S^1 \cdot \gamma_0^m$ are two isolated critical orbits of $\mathcal{E}$ in $\Lambda M$ and that (41) is satisfied. Then for any field $\mathbb{K}$,*

$$\varphi_m : \Big(\mathring{\mathcal{E}}_c \cup \mathcal{O}, \mathring{\mathcal{E}}_c\Big) \to \Big(\mathring{\mathcal{E}}_{m^2c} \cup \varphi_m(\mathcal{O}), \mathring{\mathcal{E}}_{m^2c}\Big)$$

*induces isomorphisms*

$$(\varphi_m)_* : H_*\Big(\mathring{\mathcal{E}}_c \cup \mathcal{O}, \mathring{\mathcal{E}}_c; \mathbb{K}\Big) \to H_*\Big(\mathring{\mathcal{E}}_{m^2c} \cup \varphi_m(\mathcal{O}), \mathring{\mathcal{E}}_{m^2c}; \mathbb{K}\Big)$$

*(or in the usual notation*

$$(\varphi_m)_*: \quad H_*\Big(\Lambda(\gamma_0)\cup S^1\cdot\gamma_0, \Lambda(\gamma_0); \mathbb{K}\Big) \\ \to H_*\Big(\Lambda(\gamma_0^m)\cup S^1\cdot\gamma_0^m, \Lambda(\gamma_0^m); \mathbb{K}\Big)\ \Big)$$

As applications we may prove the following generalization of a famous result on Riemannian manifolds by Bangert and Klingenberg [Topology, 23(1983)379].

**Theorem 11** *A connected closed Finsler manifold $(M, F)$ of dimension $n > 1$ has infinitely many geometrically distinct closed geodesics provided that there exists a nonconstant closed geodesics $\bar{\gamma}$ such that $m^-(\bar{\gamma}^k) \equiv 0$ and*

$$H_{\bar{p}}(\Lambda(\bar{\gamma})\cup S^1\cdot\bar{\gamma}, \Lambda(\bar{\gamma}); \mathbb{Q}) \neq 0$$

*with some integer $\bar{p} \geq 2$.*

A slightly different version of it was proved by H.B.Rademacher [Th.7.5, Bonner Math.Schr.229 (1992)] with finite-dimensional approximations.

Thank you!