Escape, collisions and regularization in the variational approach to the N-body problem

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Workshop on Symplectic Dynamics
and Hamiltonian Systems
Nankai Univeristy, May 19-23 2014

Outline

The N-body problem

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Motion equation:

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where $x_k(t)$ denotes the position of the k-th body at time t, and $m_k > 0$ its mass.

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Generally, global complex dynamics is connected with the existence of periodic solutions featuring a complex behaviour (Poincaré conjecture).

H. Poincaré: Les Méthodes Nouvelles de la Mécanique Céleste (1892)

"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputée inabordable...".

According to Poincaré, periodic orbits catch the complexity of the global dynamics:

"...voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans le n. 13[1] *et une solution particulière quelconque de ces équations, one peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut."*

[1] Formula n. 13 is Hamilton equation.

Periodic solutions: a global approach

We may try to take advantage of

- Symmetries: the problem is invariant with respect to
  - the orthogonal group $O(d)$,
  - if $m_i = m_j$, the permutations of x_i and x_j,
  - time shift and reversal.
- Topology: of the loop space over the N-body configuration space.

Due to the difficulty of the general problem, we may also think to some (yet highly nontrivial) simplifications:

- the circular planar restricted N-body problem;
- the N-center problem.

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Settings

- N point particles with masses m_1, m_2, $\dots m_N$ and positions x_1, x_2, $\dots$, $x_N \in \mathbb{R}^d$, with $d \geq 2$, which form the configuration space $\mathcal{X} = \mathbb{R}^{Nd}$.
- Interaction potential: $U(x) = \displaystyle\sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$; when $\alpha = 1$ we have the gravitational Newton potentials.
- On collisions ($x_i = x_j$ for some $i \neq j$) potential $U = +\infty$. Admissible configurations: $\tilde{\mathcal{X}} = \mathcal{X} \setminus \{\mathit{collisions}\}$.
- Collisionless T-periodic orbits: solutions of the Newton equations (such that $\forall t : x(t+T) = x(t) \in \tilde{\mathcal{X}}$).

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}.$$

Action and Maupertuis' functionals

- Lagrangian: $L(x,\dot{x}) = \overbrace{\sum_i \frac{1}{2} m_i |\dot{x}_i|^2}^{K} + \overbrace{\sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}}^{U}.$
- Action functional: for $x \in H^1_T$ ($T > 0$ fixed period)
  $$\mathcal{A}_T(x) = \int_0^T L(x(t),\dot{x}(t))dt.$$
- Maupertuis functional: $x \in H^1_1$ ($h \in \mathbb{R}$ fixed energy)
  $$\mathcal{J}_h(x) = \left(\int_0^1 K dt\right)\left(\int_0^1 (h+U)dt\right).$$

Variational approach

We seek critical points of the action functional, (or the Maupertuis one) on

$$\mathcal{A}\colon \Lambda \to \mathbb{R} \cup \{+\infty\}, \quad x \mapsto \int_0^T \sum_i \frac{1}{2} m_i |\dot{x}_i|^2 + \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$$

constrained on suitable admissible linear subspaces $\Lambda_0 \subset \Lambda$.

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Special case: two bodes and $\alpha = 1$.

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minimize $\mathcal{A}$ over Λ. No minimizer.

- Second attempt: minimize $\mathcal{A}$ over $\{x \in \Lambda \ : \ \deg(x_1 - x_2; 0) \neq 0\}$.

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Success (?)

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  - Other type of critical points (mountain pass).
- The action functional $\mathcal{A}$ does not satisfy the Palais-Smale compactness condition on Λ: sequences of almost–critical points may diverge.
- The potential U is singular on collisions, and thus miminizers or other critical points can *a priori* have collisions.

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Action minimizing G-equivariant solutions

- A. Chenciner and R. Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses. Ann. of Math. (2) 152 (2000)
- Ferrario, D. L. and Terracini, S., On the existence of collisionless equivariant minimizers for the classical n-body problem. Invent. Math. 155 (2004)
- Ferrario, Davide L. Transitive decomposition of symmetry groups for the n-body problem. Adv. Math. 213 (2007)
- Chen, Kuo-Chang, Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses. Ann. of Math. (2) 167 (2008),
- G. Fusco, G. F. Gronchi and P. Negrini, Platonic polyhedra, topological constraints and periodic solutions of the classical N-body problem Invent. Math., (2011)

Symmetry groups and equivariant orbits

- G finite group.
- $\tau\colon G \to O(2)$ orthogonal representation of dimension 2 (on cyclic time $\mathbb{T} = \mathbb{R} \mod T \cong S^1$).
- $\rho\colon G \to O(d)$ orthogonal representation on the euclidean space $\mathbb{R}^d$.
- $\sigma\colon G \to \Sigma_n$ homomorphism on the symmetric group on n elements ($\implies G$ on the index set $\mathbf{n} = \{1, 2, \ldots, n\}$)
    - G acts on time (translation and reversal) $\mathbb{T}$ via τ;
    - G acts on the configuration space $\mathcal{X}$ via ρ and σ:

$$\forall i = 1 \ldots n : (gx)_i = \rho(g) x_{\sigma(g)^{-1}(i)}.$$

From Ferrario and T., 2004.

Definition

Consider the linear subspace of $\Lambda_0 = \Lambda^G \subset \Lambda$ of periodic curves in Λ which are equivariant with respect to the G-action:
$\forall g \in G : x(gt) = (gx)(t)$.

Consequences:

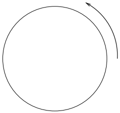
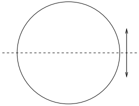
- (1) Λ^G is a natural constraint if $m_{\sigma(g(i))} = m_i$, for all i and $g \in G$, i.e. critical points of the action constrained to Λ^G are free critical points.
- (2) Gain of coercivity:

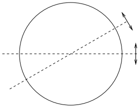
Proposition

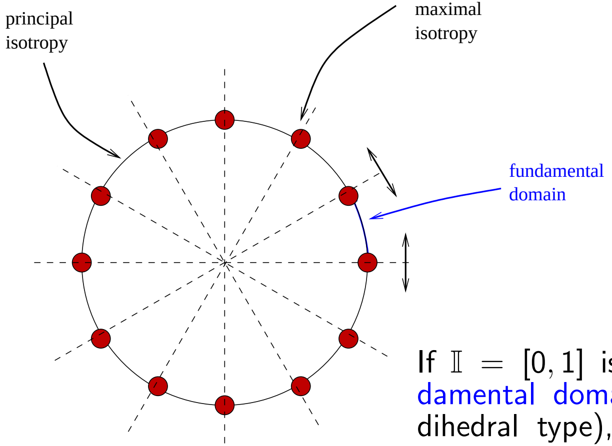
if $\mathcal{X}^G$ is trivial we have $x \in \Lambda_0, |x| \to \infty \implies \mathcal{A}(x) \to \infty$.

Cyclic and dihedral actions

Consider the normal subgroup $\ker\tau \lhd G$ and the quotient $\bar{G} = G/\ker\tau$. Since $\bar{G}$ acts effectively on $\mathbb{T}$, it is either a *cyclic* group or a *dihedral* group.

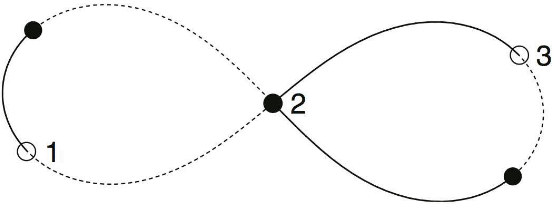
- If the group $\bar{G}$ acts trivially on the orientation of $\mathbb{T}$, then $\bar{G}$ is cyclic  and we say that the action of G on Λ is of cyclic type.
- If the group $\bar{G}$ consists of a single reflection on $\mathbb{T}$, then we say that  action of G on Λ is of brake type.
- Otherwise, we say that the action of G on Λ is of dihedral type.





If $\mathbb{I} = [0,1]$ is the fundamental domain (for a dihedral type), then G-equivariant trajectories correspond to paths $x\colon \mathbb{I} \to \mathcal{X}^{\ker\tau}$ with $x(0) \in \mathcal{X}^{H_0}$ and $x(1) \in \mathcal{X}^{H_1}$, where H_0 and H_1 are the maximal isotropy subgroups of the boundary of $\mathbb{I}$.

Symmetries of the Chenciner and Montgomery eight



The group is generated by two following space-time reflections, which fix the following space of loops.

$$x_1(-t) = -x_3(t)\ ,\ x_2(-t) = -x_2(t)\ ,\ x_3(-t) = -x_1(t)\ .$$

$$x_1(1-t) = -x_2(t)\ ,\ x_2(1-t) = -x_1(t)\ ,\ x_3(1-t) = -x_3(t)$$

G-equivariancy and boundary conditions

G-equivariance can be split into:

- proper boundary conditions on the fundamental domain;
- a time-independent constraint on the space of configurations.

A plethora of periodic trajectories

A systematic use of equivariant variational methods involves:

- The classification of all the admissible symmetry groups.
- The analysis of possible collisions for equivariant minimizers and the determination of those groups whose minimizers are free of collisions.
- A further study of qualitative properties of equivariant minimizers to understand whether different classes of symmetric loops may share the same minimizers.

Further possible developments

1. Develop an equivariant Morse Theory specific for the N–body problem, taking into account of all possible collisions.
2. Fully understand the impact of collisions on the variational characterization (Morse index) of periodic trajectories.

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Absence of collision for locally minimal paths

Solutions to the Newtonian n–body problem which are minimals for the action are, very likely, free of any collision. This fact was observed by the construction of suitable local variation arguments for the 2 and 3–body cases by Serra and Terracini (1992 and 1994). The 4–body case was treated afterward by Dell'Antonio (non really rigorously) and then by A. Venturelli in his PhD thesis. In general, the proof goes by the sake of the contradiction and involves the construction of a suitable variation that lowers the action in presence of a collision. A powerful breakthrough in this direction is due of the neat idea, due to C. Marchal, of averaging over a family of variations parameterized on a sphere. This method has been developed and exposed by Chenciner, and then extended to α–homogeneous potentials and various constrained minimization problems by Ferrario and Terracini. This argument can be used in many of the known cases to prove that minimizing trajectories are collisionless.

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Notation

Assume time $t = 0$ is an isolated collision (possibly more than one colliding cluster).

- $\mathbf{k} \subset \mathbf{n}$: colliding cluster.
- The momentum of inertia with respect to the center of mass:

$$I_{\mathbf{k}} = \sum_{i \in \mathbf{k}} m_i (x_i - x_0)^2$$

  ($x_0 = \sum_{i \in \mathbf{k}} m_i x_i / m_0$, $m_0 = \sum_{i \in \mathbf{k}} m_i$). All the bodies in $\mathbf{k}$ collide in x_0 if and only if $I_{\mathbf{k}} = 0$

- The partial kinetic energy $K_{\mathbf{k}}$ and the partial potential function

$$K_{\mathbf{k}} = \sum_{i \in \mathbf{k}} \frac{m_i}{2} |\dot{x}_i|^2 , \qquad U_{\mathbf{k}} = \sum_{i,j \in \mathbf{k}, i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}.$$

- The partial energy and the partial Lagrangian:

$$E_{\mathbf{k}} = K_{\mathbf{k}} - U_{\mathbf{k}} \; ; \qquad L_{\mathbf{k}} = K_{\mathbf{k}} + U_{\mathbf{k}}.$$

Generalized Sundman–Sperling estimates

- For every colliding cluster $\mathbf{k} \subset \mathbf{n}$ the partial energy $E_{\mathbf{k}}$ is bounded.
- There is $\kappa > 0$ such that the following asymptotic estimates hold:

$$I_{\mathbf{k}} \sim (\kappa t)^{\frac{4}{2+\alpha}}$$
$$\dot{I}_{\mathbf{k}} \sim \frac{4}{2+\alpha}\kappa(\kappa t)^{\frac{2-\alpha}{2+\alpha}},$$

and

$$K_{\mathbf{k}} \sim U_{\mathbf{k}} \sim \frac{1}{4-2\alpha}\ddot{I}_{\mathbf{k}} \sim \frac{2}{(2+\alpha)^2}\kappa^2(\kappa t)^{\frac{-2\alpha}{2+\alpha}}.$$

- Let s be the normalized configuration of a colliding cluster $s = I_{\mathbf{k}}^{-1/2}(x - x_0)$. Then, by a classical monotonicity formula, the trajectory $s(t)$ of normalized configurations converge, as $t \to +\infty$ to the set central configuration (for the $\mathbf{k}$-body problem).

Blow-ups

For every $\lambda > 0$ let
$$x^\lambda(t) = \lambda^{-2/(2+\alpha)} x(\lambda t)$$
If $\{\lambda_n\}_n$ is a sequence of positive real numbers such that $s(\lambda_n)$ converges to a normalized configuration $\bar{s}$, then
$\forall t \in (0,1) : \lim_{n\to\infty} s(\lambda_n t) = \lim_{n\to\infty} s(\lambda_n) = \bar{s}$. Hence the rescaled sequence will converge uniformly to the blow-up of $x(t)$ relative to the colliding cluster $\mathbf{k} \subset \mathbf{n}$ (in $t = 0$).

The blow-up $\bar{x}$ is parabolic: where a parabolic collision trajectory for the cluster $\mathbf{k}$ is the path

$$\bar{x}_i(t) = |t|^{2/(2+\alpha)} \xi_i, \quad i \in \mathbf{k},\ t \in \mathbb{R}$$

where $\xi = (\xi_i)_{i\in\mathbf{k}}$ is a central configuration with k bodies.

Proposition; The sequences x^{λ_n} and $\dfrac{dx^{\lambda_n}}{dt}$ converge to the blow-up $\bar{x}$ and its derivative $\dot{\bar{x}}$ respectively, uniformly in $[0, T]$.

Proposition: Let φ be a variation of the particles in $\mathbf{k}$ which is C^1 in a neighborhood of $T > 0$, defined and centered for every $t \in [0, T]$. Then there is a sequence ψ_n of H^1-functions with support in $[0, T]$, converging to 0 with the following property:

$$\lim_{n\to\infty} \int_0^T \left[L_{\mathbf{k}}(x^{\lambda_n} + \varphi + \psi_n) - \mathcal{L}(x^{\lambda_n}) \right] dt =$$
$$\int_0^T \left[L_{\mathbf{k}}(\bar{x} + \varphi) - \mathcal{L}(\bar{q}) \right] dt.$$

The standard variation

Let G_0 be the isotropy group at the collision time, then the blow–up procedure implies the existence of q, a G_0-equivariant minimizing parabola homiletic collision trajectory.

The standard variation associated to δ and T is defined as

$$v^\delta(t) = \begin{cases} \delta & \text{if } 0 \le |t| \le T - |\delta| \\ (T-t)\frac{\delta}{|\delta|} & \text{if } T - |\delta| \le |t| \le T \\ 0 & \text{if } |t| \ge T. \end{cases}$$

Our next goal is to find a G_0-equivariant standard variation v^δ such that the trajectory $q + v^\delta$ does not have a collision at $t = 0$ and

$$\Delta\mathcal{A} := \int_{-\infty}^{+\infty} [\mathcal{L}_{\mathbf{k}}(q + v^\delta) - \mathcal{L}_{\mathbf{k}}(q)]dt < 0.$$

Introduce the potential displacement function

$$S(\xi,\delta)=\int_0^{+\infty}\left(\frac{1}{\left|\xi t^{2/(2+\alpha)}-\delta\right|^{\alpha}}-\frac{1}{\left|\xi t^{2/(2+\alpha)}\right|^{\alpha}}\right)dt$$

where $\xi,\delta\in\mathbb{R}^2$.

Theorem: *Let $q=\{q\}_i=\{t^{2/(2+\alpha)}\xi_i\}$, $i=1,\dots,k$ be a parabolic collision trajectory and v^δ a G_0-equivariant standard variation. Then, as $\delta\to 0$*

$$\Delta\mathcal{A}=2|\delta|^{1-\alpha/2}\sum_{\substack{i<j\\ i,j\in\mathbf{k}}} m_i m_j S(\xi_i-\xi_j,\frac{\delta_i-\delta_j}{|\delta|})+O(|\delta|).$$

The function S

We observe that

$$S(\lambda\xi,\mu\delta) = |\lambda|^{-1-\alpha/2}|\mu|^{1-\alpha/2}S(\xi,\delta)$$

and hence the sign of S only depends on the angle between ξ and δ. Let

$$\Phi(\vartheta) = \int_0^{+\infty} \frac{1}{\left(t^{\frac{4}{\alpha+2}} - 2\cos\vartheta\, t^{\frac{2}{\alpha+2}} + 1\right)^{\alpha/2}} - \frac{1}{t^{\frac{2\alpha}{\alpha+2}}}dt, \quad \alpha \in (0,2)$$

$\Phi(\theta)$ represents the potential differential needed for displacing the colliding particle from zero to $e^{i\theta}$. We can expand interns of hypergeometric functions:

$$\begin{aligned}\Phi(\vartheta) \quad &= \quad \frac{\alpha(\alpha+2)}{2}\left\{\frac{1}{\alpha-2}\beta\left(\frac{\alpha+2}{4},\frac{\alpha+2}{4}\right) + \frac{1}{\alpha}\sum_{k=1}^{+\infty}\binom{-\alpha/2}{k}\right.\\ &\qquad \left.(-1)^k 2^{k-1}(\cos\vartheta)^k\beta\left(\frac{\alpha}{4}-\frac{1}{2}+\frac{k}{2},\frac{\alpha}{4}+\frac{1}{2}+\frac{k}{2}\right)\right\}.\end{aligned}$$

Some properties of Φ

The value of $\Phi(\theta)$ ranges from $+\infty$ to some negative value, depending on α. However, thanks to some harmonic analysis one can prove that suitable averages are always negative: the first inequality is particularity useful for dealing with reflected triple collisions from the Lagrange central configuration:

$$\Phi(\frac{2\pi}{3}+\gamma)+\Phi(\frac{2\pi}{3}-\gamma)<0, \quad \forall\gamma\in[0,\pi/2].$$

A key remark was made by Christian Marchal: being the Newton potential a harmonic map averaging it on a sphere results in a truncation in the interior. In fact, is not so much a matter of harmonicity.

A crucial estimate

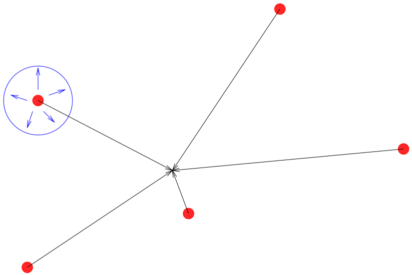
A crucial estimate was proved in [FT] about the averages of Φ on circles:

For every $\alpha > 0$, $\xi \in \mathbb{R}^3 \setminus \{0\}$ and for every circle $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ with center in 0,

$$\tilde{S}(\xi, \mathbb{S}^{d-1}) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} S(\xi, \delta) d\delta$$
$$= |\xi|^{-1-\alpha/2} |\delta|^{1-\alpha/2} \frac{1}{2\pi} \int_0^{2\pi} \Phi(\theta) d\theta < 0.$$

Consider $\xi = x_i - x_j$ and δ ranging in a circle. Then we obtain the principle, a generalization of the result announced in :
Chenciner, A., Action minimizing solutions of the Newtonian n–body problem: from homology to symmetry, August 2002, *ICM, Peking*

Marchal's Principle



It is more convenient (from the point of view of the integral of the potential on the time line) to replace one of the point particles with a homogeneous circle of same mass and fixed radius which is moving keeping its center in the position of the original particle

If the action of G on $\mathbb{T}$ and $\mathcal{X}$ fulfills some conditions (computable) then (local) minimizers of the action functional $\mathcal{A}^G$ in $\Lambda^G \subset \Lambda$ do not have collisions.

The rotating circle property

The rotating circle property

For a group H acting orthogonally on $\mathbb{R}^d$, a circle $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ (with center in 0) is termed rotating under H if $\mathbb{S}^{d-1}$ is invariant under H (that is, for every $g \in H$ $g\mathbb{S}^{d-1} = \mathbb{S}^{d-1}$) and for every $g \in H$ the restriction $g|\mathbb{S}^{d-1} \colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ is a rotation (the identity is meant as a rotation of angle 0).
Let $i \in \mathbf{n}$ be an index and $H \subset G$ a subgroup. A circle $\mathbb{S}^{d-1} \subset \mathbb{R}^d = V$ (with center in 0) is called *rotating for i under H* if $\mathbb{S}^{d-1}$ is rotating under H and

$$\mathbb{S}^{d-1} \subset V^{H_i} \subset V = \mathbb{R}^d,$$

where $H_i \subset H$ denotes the isotropy subgroup of the index i in H relative to the action of H on the index set $\mathbf{n}$ induced by restriction (that is, the isotropy $H_i = \{g \in H \mid gi = i\}$).

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A group G acts with the *rotating circle property* if for every $\mathbb{T}$-isotropy subgroup $G_t \subset G$ and for at least $n-1$ indexes $i \in \mathbf{n}$ there exists in $\mathbb{R}^d$ a rotating circle $\mathbb{S}^{d-1}$ under G_t for i.

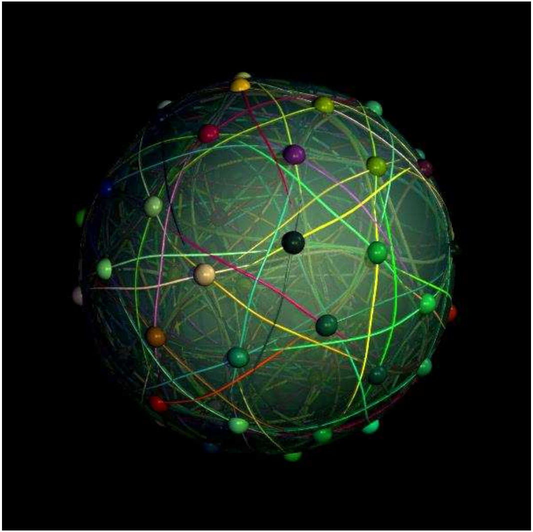
- If the action has the rotating circle property, then for every $g \in G$ the linear map $1 - g$ sends the rotating circle into another circle (thus we can use the averaging trick).
- In most of the known examples the property is fulfilled.
- There are several infinite families with the rotating circle property.

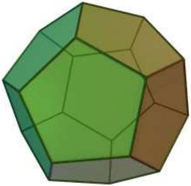
Theorems with the RCP

- **Theorem**: Consider a finite group K acting on Λ with the rotating circle property. Then a minimizer of the K-equivariant fixed–ends (Bolza) problem is free of collisions.
- **Corollary**: For every $\alpha > 0$, minimizers of the fixed-ends (Bolza) problem are free of interior collisions.
- **Corollary**: If the action of G on Λ is of cyclic type and $\ker\tau$ has the rotating circle property (or it is trivial) then any local minimizer of $\mathcal{A}^G$ in Λ^G is collisionless.

Theorem: Consider a finite group G acting on Λ so that every maximal $\mathbb{T}$-isotropy subgroup of G either has the rotating circle property or acts trivially on the index set $\mathbf{n}$. Then any local minimizer of $\mathcal{A}^G$ yields a collision-free periodic solution of the Newton equations for the n-body problem in $\mathbb{R}^d$.

Klein groups



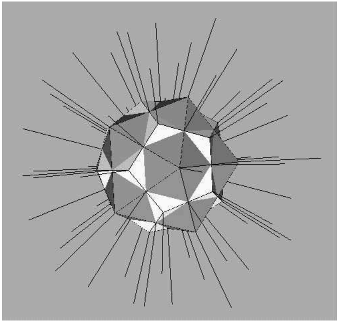


The reduced potential

Let us consider the quotient configuration space in the case of a Klein group. The reduced potential takes the form:

$$U(x) = \sum_{r\in\mathcal{R}} \frac{m_r}{\mathrm{d}(x,r)^\alpha}$$

where $\mathcal{R}$ is the set of rotation axes and $m_r > 0$ depends depends on the order of the rotation.



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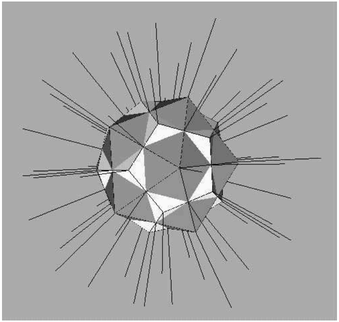

The reduced potential is anisotropic and homogenous. It has many lines of singularity

Collisions

The general Marchal-Chenciner-Ferrario-T. theorem, has many useful applications, but fails a number of relevant cases.

- For potentials of the form

$$U(x) = \sum_{r \in \mathcal{R}} \frac{m_r}{\mathrm{d}(x, r)^\alpha}$$

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- Moreover Averaging destroys topology.

For topologically constrained minimizers Marchal's argument does not work, and other devices have to be designed to avoid the occurrence of collisions.

Problems:

- Include topological constraints.
- Anisotropic and logarithmic potentials (for which the averaged variation does not hold).
- Study the contributions to the Morse index given by the possible collisions (Barutello, Secchi, 2006).
- Connection between collision and parabolic trajectories.
- Symbolic dynamics.

Blowing up

On the other hand, topology plays a fundamental role in the very same existence of non trivial motions. We wish to study the possible occurrence of collisions in the minimization process in the presence of topological contraints. Let us assume that, when minimizing:

$$\min\{\mathcal{A}(x) \ : \ x \in \Lambda \ : \ \deg(x;0) \neq 0 \ \& \ |x_1(t) - x_2(t)| \geq \varepsilon\}$$

the minimizer hits the obstacle for every $\epsilon > 0$. By rescaling and letting $\epsilon \to 0$, in the limit we find two minimal parabolic arcs connected by a circular arc:

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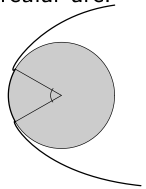

the excess angle indicates the tendency to collisions of minimal arcs.

Absence of collisions for minimizers of Bolza problems

In the planar case, we use polar coordinates (r, θ).

Definition

We say that $x = (r, \vartheta) \in \mathcal{AC}(t_1, t_2)$ is a fixed-time Bolza minimizer associated to the ends $x_1 = r_1 e^{i\varphi_1}$, $x_2 = r_2 e^{i\varphi_2}$, in the sector $(\vartheta^-, \vartheta^+)$, if

- $\vartheta^- \leq \vartheta(t) \leq \vartheta^+ \ \forall t \in [t_1, t_2]$;
- $r(t_i) = r_i$ and $\vartheta(t_i) = \varphi_i$, $i = 1, 2$;
- for every $z = (\rho, \zeta) \in \mathcal{AC}(t_1, t_2)$ taking values in the sector $(\vartheta^-, \vartheta^+)$, there holds

$$\rho(t_i) = r_i, \ \zeta(t_i) = \varphi_i, \quad \Longrightarrow \quad \mathcal{A}([t_1, t_2]; x) \leq \mathcal{A}([t_1, t_2]; z).$$

If $\min_{t \in [t_1, t_2]} r(t) > 0$ we say that the Bolza minimizer is collisionless.

Theorem

Consider a perturbed Kepler potential $V = \dfrac{1}{r^{\alpha}} + W$, *with* $\alpha > \alpha'$ *and*

$$\lim_{r \to 0} r^{\alpha'} \left(W(x) + r |\nabla W(x)| \right) = 0 \, .$$

Given any pair of points x_1 *and* x_2 *in the sector* $(\vartheta^-, \vartheta^+)$, *if* $\vartheta^+ - \vartheta^- < 2\pi/(2-\alpha)$ *then all fixed-time Bolza minimizers associated to* x_1, x_2 *within the sector* $(\vartheta^-, \vartheta^+)$ *are free of collisionless.*

The restriction that the minimizing path stays in the sector $(\vartheta^-, \vartheta^+)$ can be removed, when $W \equiv 0$, as it is implied by the conservation of the angular momentum. The theorem easily extends to $\mathbb{R}^3 \setminus \{x_1 = x_2 = 0\}$ for potentials with cylindrical symmetry (Hip-hop, T.- Venturelli)

Some remarks

- If conversely $\vartheta^+ - \vartheta^- \geq 2\pi/(2-\alpha)$, then there are always some Bolza problems which admit only collision minimizers. It is enough to chose $x_1 = \vartheta^-$ and $x_2 = \vartheta^+$ and T as the natural time of the free time minimizer.

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- If conversely $\vartheta^+ - \vartheta^- \geq 2\pi/(2-\alpha)$, then there are always some Bolza problems which admit only collision minimizers. It is enough to chose $x_1 = \vartheta^-$ and $x_2 = \vartheta^+$ and T as the natural time of the free time minimizer.
- If $\vartheta^+ - \vartheta^- = 2\pi/(2-\alpha)$ and $W \equiv 0$, then the following alternative holds:
  - either the minimizer is collisionless,
  - or $(x_1, x_2) = (\vartheta^-, \vartheta^+)$ and the minimizer is a collision–ejection homothetic trajectory.
- The latter statement is a generalization of the Marchal's statement about the existence of direct and inverse action–minimizing keplerian arcs.

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Parabolic trajectories for homogeneous potentials

Let us give an exponent $\alpha \in (0,2)$, and a

positive potential V, homogeneous of degree $-\alpha$,

possibly singular on some cone Σ.

Definition

A (global) *completely parabolic trajectory* of

$$\ddot{x}(t) = \nabla V(x(t)), \qquad x \in \mathbb{R}^d \setminus \Delta,$$

is a collisionless solution having null energy and vanishing velocities at infinity:

$$\frac{1}{2}|\dot{x}(t)|^2 = V(x(t)), \text{ for every } t \in \mathbb{R} \quad \text{and} \quad \lim_{|t|\to+\infty} |\dot{x}(t)| = 0 \; .$$

In the Kepler problem ($V(x) = 1/|x|$) all global zero-energy trajectories are indeed parabola.

Central configurations

Similar to collisions, parabolic trajectories enjoy nice asymptotic properties, regarding $\dot{x}$, $|x|$ and $x/|x|$. First of all, $\dot{x}(t) \to 0$ and $|x(t)| \to \infty$ as $t \to \pm\infty$ with a definite rate; recall that

Definition

A *central configuration* for V is a unitary vector which is a critical point of the restriction of V to the sphere $\mathbb{S}^{d-1}$.

The normalized configuration $x(t)/|x(t)|$ has infinitesimal distance from the set of central configurations of V, as $t \to \pm\infty$. In particular, whenever this set is discrete, we have that

$$\frac{x(t)}{|x(t)|} \to \xi^{\pm}, \qquad \text{as } t \to \pm\infty,$$

where $\xi^{\pm}$ are central configurations.

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A Lagrangian version of Mc Gehee coordinates

Start with a Morse minimizer path $x = rs$, using polar coordinates, and write the formal Lagrangian

$$\int \frac{1}{2}(\dot{r}^2 + r^2|\dot{s}|^2) + \frac{U(s)}{r^\alpha}$$

Now, we change time and space variables:

$$\mathrm{d}t = r^{(2+\alpha)/2}\mathrm{d}\tau\ , \qquad (\cdot)' = \frac{\mathrm{d}}{\mathrm{d}\tau}\ , \qquad \rho = r^{(2-\alpha)/4}$$

and find

$$\int \frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \rho^2\left(\frac{1}{2}|s'|^2 + U(s)\right)\ .$$

A new Lagrangian system

The metric has been deformed in the radial direction, The Euler-Lagrange equations for the new functional are

$$\begin{cases} -\left(\frac{4}{2-\alpha}\right)^2 \rho'' + \left(\frac{1}{2}|s'|^2 + U(s)\right)\rho = 0 \\ -\left(\rho^2 s'\right)' + \rho^2 \nabla_s U(s) = \rho^2 |s'|^2 s \end{cases}$$

and the null energy condition transforms into a new null energy condition

$$\frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \rho^2\left(\frac{1}{2}|s'|^2 - U(s)\right) = 0 \,.$$

The monotonicity formula

Let $v = \frac{4}{2-\alpha}\frac{\rho'}{\rho}$, so that

$$|s'|^2 - 2U(s) = -v^2 = -\left(\frac{4}{2-\alpha}\right)^2\left(\frac{\rho'}{\rho}\right)^2 .$$

Then we have, as we my assume $\rho' > 0$,

$$(v^2)' = 4\frac{\rho'}{\rho}|s'|^2 = (2-\alpha)v|s'|^2 \implies v' = \frac{2-\alpha}{2}|s'|^2.$$

So that we can eliminate ρ from the system:

$$\begin{cases} v' = \dfrac{2-\alpha}{2}|s'|^2 \\ -\dfrac{2-\alpha}{2}vs' - s'' + \nabla_s U(s) = |s'|^2 s . \end{cases}$$

Back to Morse minimizers

Given ξ^- and ξ^+ ingoing and outgoing asymptotic directions, we consider the following class of minimizers.

Definition

We say that $x \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a *(free) minimizer of $\mathcal{A}$ of parabolic type, in the sense of Morse* with asymptotic configurations $\xi^\pm$, if

- $\min_{t\in\mathbb{R}} |x(t)| > 0$;
- $|x(t)| \to +\infty$, $x(t)/|x(t)| \to \xi^\pm$ as $t \to \pm\infty$;
- for every $a < b$, $a' < b'$, and $z \in H^1(a', b')$, there holds

$$z(a') = x(a),\ z(b') = x(b) \qquad \Longrightarrow \qquad \mathcal{A}([a,b];x) \le \mathcal{A}([a',b'];z).$$

In many cases, one may be also interested in Morse minimizers in a local sense, for instance imposing some topological constraints.

Structural instability of parabolic trajectories connecting minimal central configurations

A parabolic Morse minimizer is a minimal geodesic for the Jacobi metric.

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As we shall see, a potential V needs not to admit a parabolic Morse minimizer connecting two minimal central configurations.

To deal with this intrinsic structural instability we need to introduce an auxiliary parameter (the homogeneity) and look for parabolic orbits as pairs trajectory-parameter. To clarify the role of the additional parameter, it is worthwhile to let the potential vary in a class.

Definition

We denote by $\mathcal{V} = \mathcal{V}(\xi^-, \xi^+)$ the class of all α homogenous potentials sharing $\xi^{\pm}$ as minimal central configurations.

The property of a potential to admit parabolic minimizers is related to its behavior with respect to the following fixed-endpoints problem. For any $V \in \mathcal{V}$, let us define

$$c(U,\alpha) := \inf\left\{\mathcal{A}\left([a,b];x\right) : a<b,\ x(a)=\xi^-,\, x(b)=\xi^+\right\};$$

such ithe minimum is achieved by a possibly colliding solution.

Proposition

Let $U \in \mathcal{V}$; then one of the following alternatives is satisfied:

(1) $c(U,\alpha) = 4\sqrt{2U_{\min}}/(2-\alpha)$ *is achieved by the juxtaposition of two self-similar (homothetic) motions, the first connecting ξ^- to the origin and the second the origin to ξ^+;*

(2) $c(U,\alpha) < 4\sqrt{2U_{\min}}/(2-\alpha)$, *and it is achieved by trajectories which are uniformly bounded away from the origin.*

Focal behavior

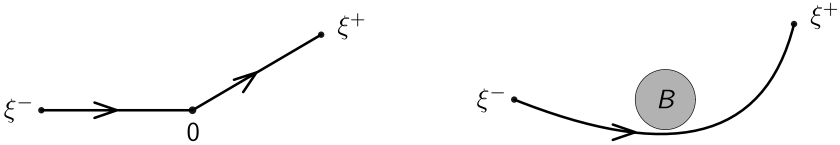
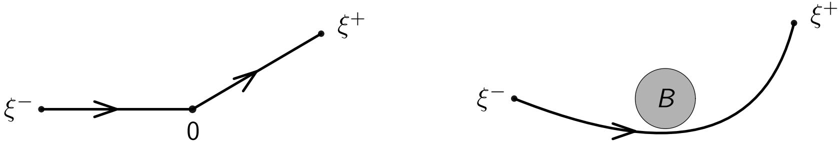


Figure: at left, $c(U, \alpha)$ is achieved by a double-homothetic motion (case (1) of the Proposition); at right $c(U, \alpha)$ is achieved by a non-collision trajectory (case (2) of the Proposition). When the second situation occurs, there exists a ball B, centered at the origin, such that any trajectory that achieves $c(U, \alpha)$ does not intersect B.

We distinguish potentials with "inner" minimizers (i.e. minimizers which pass through the origin) from potential with "outer" ones:

Focal behavior



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We distinguish potentials with "inner" minimizers (i.e. minimizers which pass through the origin) from potential with "outer" ones:

$$\mathrm{In} := \left\{(U, \alpha) \in \mathcal{V} : c(U, \alpha) = 4\sqrt{2U_{\min}}/(2-\alpha)\right\},$$

$$\mathrm{Out} := \left\{(U, \alpha) \in \mathcal{V} : c(U, \alpha) < 4\sqrt{2U_{\min}}/(2-\alpha)\right\}.$$

It is easy to see that these two sets are disjoint and their union is the whole $\mathcal{V}$; moreover, we can show that In is closed while Out is open. We are interested in their common boundary, that is

$$\Pi := \partial \mathrm{In} \cap \partial \mathrm{Out}.$$

The separating property of the common boundary is highlighted by the following result.

Lemma (Barutello, Verzini, T.)

There exists an open nonempty set $\Sigma \subset \mathcal{U}$, and a continuous function $\bar{\alpha} : \Sigma \to (0, 2)$ such that

$$\Pi = \{(V, \bar{\alpha}(U)) : \; U \in \Sigma\}.$$

Of course, we can exhibit explicit criteria in order to establish whether a potential $U \in \mathcal{U}$ belongs to the domain of the function $\bar{\alpha}$.

The Structure Theorem

Our main result states that the above graph coincides with the set of potentials admitting parabolic Morse minimizers.

Theorem (Barutello, Verzini, T.)

$V \in \mathcal{V}$ admits a parabolic Morse minimizer if and only if $V \in \Pi$.

Of course, due to the invariance by homothetity of the problem, such Morse minimizing parabolic trajectories always come in one-parameter families and give rise to a 2-dimensional Lagrangian submanifold having boundary corresponding to the two homothetic solutions.

Smooth solutions of Hamilton–Jacobi equation in two dimensions

By a conformal map, we can always reduce to the case $\vartheta^+ = \vartheta^- + 2\pi$. Let us chose the threshold exponent $\alpha = \bar{\alpha}(U, \vartheta^*, \vartheta^* + 2\pi)$. The above mentioned Lagrangian submanifold is laminated by the family of parabolic trajectories. Such a structure determines a smooth solution of the stationary Hamilton-Jacobi equation associated to the newton equation, of class $\mathcal{C}^1$ on the double covering of $\mathbb{R}^2 \setminus \{0\}$.

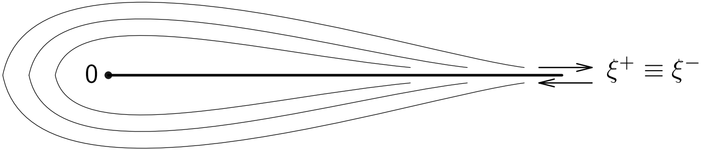

Figure: one parameter family of Morse minimizing parabolic trajectories having the same asymptotic configuration at $+\infty$ e $-\infty$ ond a nontrivial winding number.

Remarks

In spite of their fragility, parabolic trajectories carry precious information:

- they mark a transition between different focal properties of the origin with respect to the minimal geodesics;
- are connected with the possible absence/occurrence of collisions for solutions to the Bolza problems (fixed ends);
- they can be used as carriers to travel at very low cost from one to another region of the phase space;

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- are connected with the possible absence/occurrence of collisions for solutions to the Bolza problems (fixed ends);
- they can be used as carriers to travel at very low cost from one to another region of the phase space;
- As remarked by de Luz and Maderna, the property to be collisionless for all Bolza minimizers implies the absence of parabolic trajectories which are Morse minimal for the usual n–body problem with $\alpha = 1$ (without topological constraints). This can be easily seen by a rescaling and limiting argument.
- In contrast, minimal parabolic arcs (i.e., defined only on the half line) exist for every starting configuration, as proven recently by Maderna and Venturelli.

Parabolic solutions and complex dynamics

Moreover, parabolic trajectories are building blocks to construct collisionless complex trajectories featuring strong oscillations:

Theorem (Soave-T., DCDS 2012)

The planar N-center problem displays (slightly) negative energy non collision symbolic dynamics if $N \geq 3$ and $\alpha \in [1, 2)$, interacting with the boundary of the Hill's region.

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Devaney's Work: the Anisotropic Kepler Problem in $\mathbb{R}^2$

In 1978 R.L. Devaney (*Invent. Math., 45*) considered the planar anisotropic Kepler potential

$$V(r\cos\vartheta, r\sin\vartheta) = \frac{U(\vartheta)}{r^\alpha}, \qquad \vartheta \in \mathbb{R},\ r > 0,$$

where U is a 2π-periodic function such that $U(\vartheta) \geq U_{\min} > 0$, $\forall \vartheta \in \mathbb{R}$.

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where U is a 2π-periodic function such that $U(\vartheta) \geq U_{\min} > 0$, $\forall \vartheta \in \mathbb{R}$.

Following Devaney, a remarkable variant of MacGehee coordinates makes the parabolic motion equations equivalent to a planar first order system. Let

$$z = \sqrt{2U(\vartheta)}$$

and, assuming $x = re^{i\theta}$, $\dot{x} = r^{-\alpha/2} z e^{i\varphi}$, introduce the new parameter τ as

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = z r^{1+\alpha/2}$$

Now rewrite the dynamical system as (here "$'$" denotes the derivative with respect to τ)

$$\begin{cases} r' = rz^2\cos(\varphi-\vartheta) = 2rU(\vartheta)\cos(\varphi-\vartheta) \\ z' = zU'(\vartheta)\sin(\varphi-\vartheta) \\ \vartheta' = z^2\sin(\varphi-\vartheta) = 2U(\vartheta)\sin(\varphi-\vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi-\vartheta) + \alpha U(\vartheta)\sin(\varphi-\vartheta), \end{cases}$$

The above system contains the independent planar system

$$\begin{cases} \vartheta' = 2U(\vartheta)\sin(\varphi-\vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi-\vartheta) + \alpha U(\vartheta)\sin(\varphi-\vartheta). \end{cases}$$

- Stationary points: (ϑ^*, φ^*), where $U'(\vartheta^*) = 0$ and $\varphi^* = \vartheta^* + h\pi$, for some $h \in \mathbb{Z}$. Minima of U correspond to saddles, maxima to sinks/sources. Other trajectories: heteroclinics between the above.

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- The function

$$v(\tau) = \sqrt{U(\vartheta(\tau))}\cos\left(\varphi(\tau) - \vartheta(\tau)\right),$$

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- The function

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  is non-decreasing on the solutions.
- The corresponding solutions of the systems are:
  global and unbounded if $\cos(\varphi-\vartheta) \to \pm 1$ as $\tau \to \pm\infty$;
  colliding in finite t if $\cos(\varphi-\vartheta) \to \mp 1$ as $\tau \to \pm\infty$
  (in the future/past).

Corollary

Let $\vartheta^- < \vartheta^+$ belong to $\Theta_{\vartheta_1\vartheta_2}$ and let $x = (r, \vartheta)$ be an associated parabolic Morse minimizer for (U, α). Then the corresponding (ϑ, φ) is a heteroclinic connection between the saddles

$$(\vartheta^-, \vartheta^- + \pi) \text{ and } (\vartheta^+, \vartheta^+).$$

Moreover ϑ is strictly increasing between ϑ^- and ϑ^+.

Corollary

Let $\vartheta^- < \vartheta^+$ belong to $\Theta_{\vartheta_1\vartheta_2}$ and let $x = (r, \vartheta)$ be an associated parabolic Morse minimizer for (U, α). Then the corresponding (ϑ, φ) is a heteroclinic connection between the saddles

$$(\vartheta^-, \vartheta^- + \pi) \text{ and } (\vartheta^+, \vartheta^+).$$

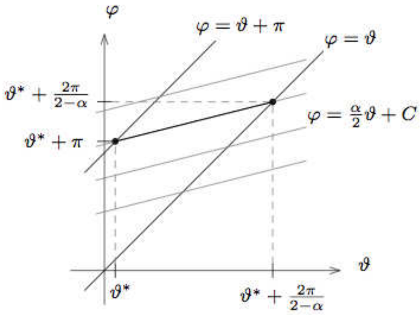
Moreover ϑ is strictly increasing between ϑ^- and ϑ^+.

In the isotropic case ($U \equiv 1$) we have

$$\begin{cases} \vartheta' = 2\sin(\varphi - \vartheta) \\ \varphi' = \alpha \sin(\varphi - \vartheta). \end{cases}$$

Heteroclinic connections must satisfy

$$\vartheta^+ - \vartheta^- = \frac{2\pi}{2-\alpha}$$



Structural instability of Morse minimizing parabolic trajectories

- Parabolic trajectories for (DS) correspond to saddle-saddle heteroclinic connections for a planar dynamical system.

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- Parabolic trajectories for (DS) correspond to saddle-saddle heteroclinic connections for a planar dynamical system.
- But generically the unstable manifold at a saddle falls into a sink, while the stable one emanates from a source, implying that parabolic trajectories do not exist.

Parabolic Trajectories as Phase Transition

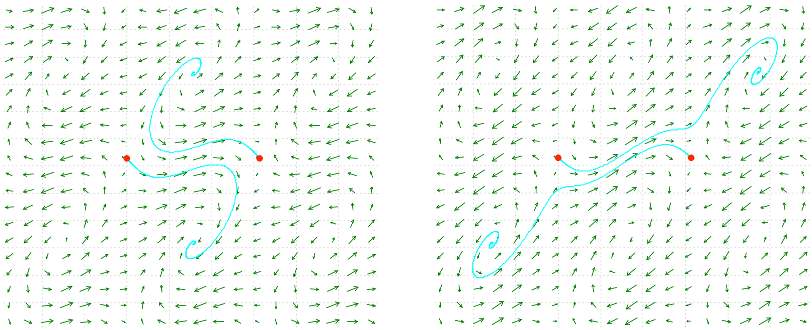

Figure: the two pictures represent the phase portrait of the planar dynamical system with $U(\vartheta) = 2 - \cos(2\vartheta)$, when $\alpha = 0.5$ (at left) or $\alpha = 1$ (at right). We focus our attention on the saddles $(0, \pi)$ and (π, π): from the mutual positions of the unstable manifold departing from $(0, \pi)$ and the stable one ending in (π, π) we deduce that the two vector fields are not topologically equivalent. By structural stability we infer the existence, for some $\bar{\alpha} \in (0.5, 1)$, of a saddle connection between $(0, \pi)$ and (π, π).

Introducing a Transition Parameter

We deal with the anisotropic Kepler problem in any dimension: more precisely V is such that:

- $V \in \mathcal{C}^2(\mathbb{R}^d \setminus \{0\})$, in particular $\mathcal{X} = \{0\}$;
- $V(x) = V(s)/r^\alpha$, $\alpha \in (0, 2)$;
- $V > 0$;
- V admits (at least) two non-degenerate and globally minimal c.c. $\xi^\pm$.

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The previous discussion suggests to choose the homogeneity exponent $-\alpha$ as parameter. To clarify the role of the parameter, we let the potential vary in a class.
For $\xi^+ \neq \xi^-$ in $\mathbb{S}^{d-1}$ and $V_{\min} > 0$, and let us define the metric spaces

$$\begin{aligned}
\mathcal{U} &= \left\{ V \in \mathcal{C}^2(\mathbb{S}^{d-1}) : \begin{array}{l} s \in \mathbb{S}^{d-1} \text{ implies } V(s) \geq V(\xi^\pm) = V_{\min}; \\ \exists \delta > 0, \mu > 0 \text{ such that } |s - \xi^\pm| < \delta \\ \text{implies } V(s) - V(\xi^\pm) \geq \mu |s - \xi^\pm|^2 \end{array} \right\}, \\
\mathcal{V} &= \left\{ (V, \alpha) \in \mathcal{C}^2(\mathbb{S}^{d-1}) \times (0,2) : \ V \in \mathcal{U} \right\},
\end{aligned}$$

the latter being equipped with the product distance.

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Inner and Outer Potentials

The property of a potential to admit parabolic minimizers is related to its behavior w.r.t. the fixed-endpoints problem

$$c(V) := \inf \left\{ \mathcal{A}\left([a,b];x\right) : a < b,\, x \in H^1(a,b),\, x(a) = \xi^-,\, x(b) = \xi^+ \right\}.$$

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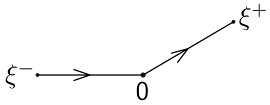

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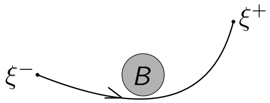

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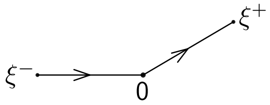

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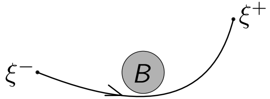

The sets In and Out enjoy the following properties:

- $\mathrm{In} \cap \mathrm{Out} = \emptyset$, $\mathrm{In} \cup \mathrm{Out} = \mathcal{V}$;
- In is closed;
- Out is open.

Structure Theorem

The role of the homogeneity parameter can be now clarified by the following property. Let $\Pi := \partial\mathrm{In} \cap \partial\mathrm{Out}$.

Lemma (Separation Property)

There exists an open nonempty set $\Sigma \subset \mathcal{U}$, and a continuous function $\bar{\alpha} : \Sigma \to (0, 2)$ such that

$$\Pi = \{(V, \bar{\alpha}(V)) : V \in \Sigma\} .$$

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We can now characterize the set of potentials admitting parabolic Morse minimizers as the graph of the above function.

Main Theorem.

$V \in \mathcal{V}$ admits a parabolic Morse minimizer $\iff$ $V \in \Pi$.

Back to $\mathbb{R}^2$: Topological Constraints

Let $d = 2$, $U(\vartheta) := V(\cos\vartheta, \sin\vartheta)$. In this last part, for the sake of simplicity, let U be a positive, $\mathcal{C}^2$ Morse function such that every local minimum is indeed a global one.

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Motivated by this, we introduce the set

$$\Theta := \{\vartheta \in \mathbb{R} : \vartheta \text{ is a (non-degenerate global) minimum for } u\}.$$

Parabolic Threshold and Collisionless Minimizers

Theorem

Let $\vartheta^-, \vartheta^+ \in \Theta$, $\vartheta^- \neq \vartheta^+$. Then there exists at most one $\bar{\alpha} = \bar{\alpha}(\vartheta^-, \vartheta^+) \in (0, 2)$ such that $V = (U, \alpha)$ admits a corresponding parabolic Morse minimizer associated with $(\vartheta^-, \vartheta^+, U)$ if and only if $\alpha = \bar{\alpha}$.

If such a $\bar{\alpha}$ does not exist, we define $\bar{\alpha}(\vartheta^-, \vartheta^+) := 0$.

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Theorem

For any $k \in \mathbb{Z} \setminus \{0\}$ and $T > 0$, if $\alpha > \bar\alpha(\vartheta^*, \vartheta^* + 2k\pi)$, for every minimum $\vartheta^* \in \Theta$, then there exists an action minimizing collisionless T–periodic trajectory winding k times around zero.

Proof: From Bolza to Morse Minimizers

- To construct Morse minimizers of parabolic type, we will first consider analogous problems on bounded intervals (Bolza problems), and then pass to the limit.

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Proof: From Bolza to Morse Minimizers

- To construct Morse minimizers of parabolic type, we will first consider analogous problems on bounded intervals (Bolza problems), and then pass to the limit.
- This procedure may fail for two main reasons: sequences of approximating trajectories may either converge to the singularity, or escape to infinity.
- This naturally leads to introduce some constraints and to study the constrained minimization problem

$$\boxed{m = m(\varepsilon, x_1, x_2) := \inf_{x\in\Gamma} \mathcal{A}(x)} \quad \text{where } \Gamma := \bigcup_{T>0} \Gamma_T, \quad \text{and}$$

$$\Gamma_T := \Big\{ x \in H^1(-T,T) : \; x(-T) = x_1, x(T) = x_2, \\ \min_{t\in[-T,T]} |x(t)| = \varepsilon \Big\},$$

where $\varepsilon > 0$ and $x_1, x_2 \in \mathbb{R}^d \setminus B_{2\varepsilon}(0)$ are fixed.

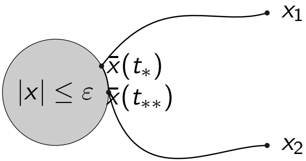
Basic Properties of Bolza Constrained Minimizers

- If $\bar{x} = \bar{r}\bar{s} \in \Gamma_{\bar{T}}$ is a constrained minimizer and $\bar{r}(t) > \varepsilon$ for $t \in (a, b)$, then

$$\ddot{\bar{x}}(t) = \nabla V(\bar{x}(t)) \quad \text{and} \quad \frac{1}{2}|\dot{\bar{x}}(t)| = V(\bar{x}(t)), \quad \text{for every } t \in (a, b).$$

- If $\bar{x}$ achieves m and it does interact with the constraint, then there exist $t_* \leq t_{**}$ such that

$$\begin{aligned} &\bar{r}(t) = \varepsilon \ \Leftrightarrow \ t \in [t_*, t_{**}];\\ &t \in (-T, t_*) \ \Rightarrow \ \dot{\bar{r}}(t) < 0;\\ &t \in (t_{**}, T) \ \Rightarrow \ \dot{\bar{r}}(t) > 0;\end{aligned}$$



if $t_* < t_{**}$ and $t \in (t_*, t_{**}) \Rightarrow$

$$\begin{cases} \ddot{\bar{x}}(t) = \nabla_T V(\bar{x}(t)) - \dfrac{1}{\varepsilon^2}|\dot{\bar{x}}(t)|^2 \bar{x}(t),\\ \dfrac{1}{2}|\dot{\bar{x}}(t)| = V(\bar{x}(t)).\end{cases}$$

Interaction with the Constraint

$\Rightarrow$ If $\bar{x}$ achieves m then it may be not regular only in t_* and t_{**}.

Proposition

If $\bar{x}$ achieves m, then one of the following situations occurs:

(a) $t_* < t_{**}$ and $\bar{x} \in \mathcal{C}^1(-\bar{T}, \bar{T})$;

(b) $t_* = t_{**}$ and $\bar{x} \in \mathcal{C}^1(-\bar{T}, \bar{T})$;

(c) $t_* = t_{**}$ and $\dot{\bar{x}}(t_*^-) \neq \dot{\bar{x}}(t_*^+)$; in such a case $\bar{x}$ undergoes a radial reflection,

$$\text{that is } \dot{\bar{r}}(t_*^-) = -\dot{\bar{r}}(t_*^+) \neq 0 \quad \text{and} \quad \dot{\bar{s}}(t_*^-) = \dot{\bar{s}}(t_*^+).$$

$\Rightarrow$ We can classify Bolza minimizers with respect to the discontinuity of the quantities x and $\dot{x}$ on the constraint.

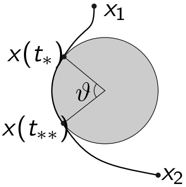
Definition

Given a constrained Bolza minimizer $x = rs$ we define :

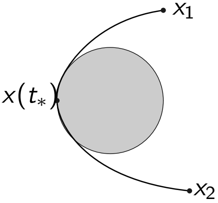
$$\Delta_{\mathrm{pos}}(x) := |s(t_{**}) - s(t_*)| = \frac{|x(t_{**}) - x(t_*)|}{\varepsilon},$$

$$\Delta_{\mathrm{vel}}(x) := \varepsilon^{\alpha/2}\left[\dot{r}(t_{**}^+) - \dot{r}(t_*^-)\right] = \frac{\dot{x}(t_{**}^+)\cdot x(t_{**}) - \dot{x}(t_*^-)\cdot x(t_*)}{\varepsilon^{-\alpha/2}\cdot\varepsilon},$$

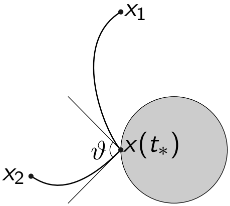
respectively as the normalized position-jump and velocity-jump of x.



(a) $t_* < t_{**}$, $\bar{x} \in \mathcal{C}^1$, $\Delta_{\mathrm{pos}} > 0$,
$\Delta_{\mathrm{vel}} = 0$,
x is position-jumping



(b) $t_* = t_{**}$, $\bar{x} \in \mathcal{C}^1$, $\Delta_{\mathrm{pos}} = 0$,
$\Delta_{\mathrm{vel}} = 0$,
x is parabolic



(c) $t_* = t_{**}$, $\dot{\bar{x}}(t_*^-) \neq \dot{\bar{x}}(t_*^+)$,
$\Delta_{\mathrm{pos}} = 0$, $\Delta_{\mathrm{vel}} > 0$,
x is velocity-jumping

Existence of Constrained Morse Minimizers

Definition

We say that $x \in H^1_{\mathrm{loc}}(\mathbb{R})$ is an *ε-constrained Morse minimizer* if

- $\min_t |x(t)| = \varepsilon$;
- $|x(t)| \to +\infty$ and $\dfrac{x(t)}{|x(t)|} \to \xi^{\pm}$, as $t \to \pm\infty$;
- for every $a < b$ and $T > 0$, and for every $z \in H^1(-T, T)$, with $\min_{t\in[-T,T]} |z(t)| = \min_{t\in[a,b]} |x(t)|$, there holds

$$z(-T) = x(a),\ z(T) = x(b) \quad \Longrightarrow \quad \mathcal{A}([a,b];x) \leq \mathcal{A}([-T,T];z).$$

Proposition

$\mathcal{M} = \{\varepsilon\text{-constrained Morse minimizer}\} \neq \emptyset.$

We argue by approximation, solving the Bolza problem with $x_1 = R\xi^-$ and $x_2 = R\xi^+$ and then letting $R \to +\infty$.

Jump-classification of Constrained Morse Minimizers

Of course, also Morse minimizers can be classified according to their jumps.

Proposition

If $x = rs \in \mathcal{M}$ then there exist $t_* \leq t_{**}$ such that:

- $r(t) = \varepsilon$ if and only if $t \in [t_*, t_{**}]$, $\dot{r}(t) < 0$ (resp. > 0) if and only if $t < t_*$ (resp. $t > t_{**}$);
- $\ddot{x}(t) = \nabla V(x(t))$, for every $t \notin [t_*, t_{**}]$;
- both $\Delta_{\mathrm{pos}}(x)$ and $\Delta_{\mathrm{vel}}(x)$ are non-negative and at least one vanishes;
- $\dfrac{1}{2}|\dot{x}(t)|^2 = V(x(t))$, for every $t \in \mathbb{R}$.

In general, for any fixed ε and potential V, we do not expect uniqueness for the Morse minimizers. Nevertheless, it is possible to show that, with respect to the jump classification, they are all of the same type.

Constrained Morse Minimizers have all the same jumps

Theorem

Let $V \in \mathcal{V}$ be fixed. For any x_1, x_2 in $\mathcal{M}$ it holds:

$$\Delta_{\mathrm{pos}}(x_1) = \Delta_{\mathrm{pos}}(x_2) =: \Delta_{\mathrm{pos}}(V),$$

$$\Delta_{\mathrm{vel}}(x_1) = \Delta_{\mathrm{vel}}(x_2) =: \Delta_{\mathrm{vel}}(V),$$

Lemma

Let $0 < \alpha_1 < \alpha_2 < 1$ and let us assume that there exists $x_i \in \mathcal{M}(U, \alpha_i)$, $i = 1, 2$, such that

$$\Delta_{\mathrm{pos}}(x_1) > 0 \quad \text{and} \quad \Delta_{\mathrm{vel}}(x_2) > 0.$$

Then there exist $\bar{\alpha} \in (\alpha_1, \alpha_2)$ and $\bar{x} \in \mathcal{M}(U, \bar{\alpha})$ such that

$\Delta_{\mathrm{pos}}(\bar{x}) = \Delta_{\mathrm{vel}}(\bar{x}) = 0$ and $\bar{x}$ is a corresponding free Morse minimizer.