HOLOMORPHIC FRAMES FOR WEAKLY CONVERGING HOLOMORPHIC VECTOR BUNDLES

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Perhaps the most useful analytic tool in gauge theory is the Uhlenbeck compactness theorem for sequences of unitary connections on hermitian vector bundles [U]. Given connections $\{D_j\}$ on a bundle E over a compact manifold M, the result returns a subsequence converging weakly in L^p_1, up to unitary gauge transformations, provided the original sequence has uniform L^p bounds on curvature, where $2p > \dim M$. Convergence away from some singular set can also be obtained in the critical case $2p = \dim M$.

In the case where M is a complex manifold and the connections satisfy an integrability condition, the theorem implies weak L^p_1 convergence of the induced holomorphic structures D''_j on E. In applications, it is useful to have control on local holomorphic frames, since then one may use techniques from several complex variables. The purpose of this note is to show that under the circumstances described above one may find local holomorphic trivializations of E which also converge with the optimal regularity, provided $p > \dim M$.

The argument we give is based largely on Webster's proof of the Newlander-Nirenberg theorem [W]. A notable difference is the somewhat more linear character of the problem for vector bundles. For this reason, the proof in [W] may be adapted to the weak L^p_1 convergence that is natural to Uhlenbeck compactness, whereas stronger control of derivatives is generally required for holomorphic structures on manifolds.

For background on connections on hermitian vector bundles, we refer the reader to [K].

Theorem 1. *Let $\{D_j\}$ be a sequence of integrable unitary connections on a complex vector bundle E over a complex manifold M of complex dimension n. Assume that $D_j \to D_\infty$ weakly in $L^p_{1,loc.}(M)$ for some integrable connection D_∞ and some $p > 2n$. Then for each $x \in M$ there is:*

1. *a coordinate neighborhood Ω of x,*
2. *a sequence $\{\mathbf{s}_j\}$ of D''_j-holomorphic frames on Ω,*
3. *a D''_∞-holomorphic frame $\mathbf{s}_\infty$ on Ω,*
4. *and a subsequence $\{j_k\} \subset \{j\}$,*

such that $\mathbf{s}_{j_k} \to \mathbf{s}_\infty$ weakly in $L^p_2(\Omega)$ and strongly in $C^{1,\alpha}(\Omega)$ for $0 < \alpha < 1 - 2n/p$.

In the following, B_r will denote the open ball of radius r about the origin in $\mathbb{C}^n$. For k a nonnegative integer, and α a real number $0 < \alpha < 1$, $\|\cdot\|_{k+\alpha;r}$ will denote the $C^{k,\alpha}$ norm on B_r. We

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recall the Leray-Koppelman operators P and Q for matrix valued $(0,1)$ and $(0,2)$-forms on B_r, respectively. Given σ, $0<\sigma<1$, have the following important properties [W, eq. (1.7) and Lemma 2.2]:

$$\varphi = \bar\partial P(\varphi) + Q(\bar\partial\varphi) \tag{1}$$

$$\|P(\varphi)\|_{1+\alpha;r(1-\sigma)} \leq rK\|\varphi\|_{\alpha;r} \tag{2}$$

$$\|Q(\psi)\|_{1+\alpha;r(1-\sigma)} \leq rK\|\psi\|_{\alpha;r} \tag{3}$$

where $K = c_\alpha\sigma^{-s}$ for c_α constant and $s>0$ an integer. We will prove the following:

Proposition 2. *Fix a positive integer R and a real number α, $0<\alpha<1$. Given $r>0$ there are constants $\theta>0$, $B>0$, and $r'>0$, $0<r'<r$, such that the following holds: if a is any $R\times R$-matrix valued $(0,1)$-form on B_r satisfying:*

- $\bar\partial a + a\wedge a = 0$,
- $\|a\|_{\alpha;r} \leq \theta$,

then there exists an $R\times R$-matrix valued function G on $B_{r'}$ satisfying:

- $\bar\partial G + aG = 0$,
- $\|G\|_{1+\alpha;r'} \leq B$,
- $\inf_{B_{r'}} |\det G| \geq B^{-1}$.

Proof of Theorem 1. Choose a coordinate neighborhood centered at x, identified with B_r for some $r>0$, and over which there exists a D''_∞-holomorphic trivialization of E. With respect to this trivialization we may regard $a_j = D''_j - D''_\infty$ as matrix valued $(0,1)$-forms satisfying $\bar\partial a_j + a_j\wedge a_j = 0$. By the compactness of the embedding $L^p_1 \hookrightarrow C^\alpha$ for $0<\alpha<1-2n/p$ and the weak convergence $a_j \to 0$ in $L^p_{1,loc.}$, we may assume $\|a_j\|_{\alpha;r}\to 0$. Hence, by Proposition 2, for each sufficiently large j we may find matrix valued functions G_j satisfying:

$$\bar\partial G_j + a_jG_j = 0\ , \tag{4}$$

$$\|G_j\|_{1+\alpha;r'} \leq B\ , \tag{5}$$

$$\inf_{B_{r'}} |\det G_j| \geq B^{-1}\ , \tag{6}$$

for some B and $r'>0$ independent of j. In particular, the column vectors of G_j are linearly independent and define D''_j-holomorphic frames on $B_{r'}$. By (5) and the elliptic estimate for $\bar\partial$ applied to (4), it follows that the G_j are bounded in $L^p_{2,loc.}(B_{r'})$ uniformly in j. After passing to a subsequence, we may assume that there is some G such that $G_j\to G$ weakly in $L^p_{2,loc.}(B_{r'})$ and strongly in $C^{1,\alpha}_{loc.}(B_{r'})$. In particular, again using (4), $\bar\partial G = 0$. Finally, by the uniform bound (6), G is invertible on $B_{r'}$ and so its column vectors define a D''_∞-holomorphic frame. This completes the proof. □

It remains to prove Proposition 2. We will need the following:

Lemma 3. *Suppose T_j is a sequence of $R \times R$ complex matrices with $|T_j| \le 1/2$ and $\sum_{j=1}^{\infty} |T_j| = C < \infty$. Set $S_k = (\mathbf{I} + T_1)(\mathbf{I} + T_2) \cdots (\mathbf{I} + T_k)$ where $\mathbf{I}$ is the $R \times R$ identity matrix. Then $|\det S_k| \ge e^{-2RC}$ for all k.*

Proof. For each T_j we have

$$|\det(\mathbf{I} + T_j)| \ge (1 - |T_j|)^R \ ,$$
$$\log |\det(\mathbf{I} + T_j)| \ge R \log(1 - |T_j|) \ ,$$

Since $\log(1-x) \ge -2x$ for $0 \le x \le 1/2$, $\log |\det(\mathbf{I} + T_j)| \ge -2R|T_j|$. Hence,

$$\log |\det S_k| = \sum_{j=1}^{k} \log |\det(\mathbf{I} + T_j)| \ge -2R \sum_{j=1}^{k} |T_j| \ge -2RC \ .$$

□

Proof of Proposition 2. Set $a_0 = a$, $h_0 = 0$. We define sequences a_j, h_j recursively, where h_j are $R \times R$ matrix valued functions defined on $B_{r'}$. The initial bound θ on a will be chosen presently so that $\sup_{B_{r'}} |h_j| \le 1/4$. Hence, $g_j = \mathbf{I} + h_j$ will be uniformly invertible. The recursive definition proceeds as follows:

$$h_{j+1} = -P(a_j) \tag{7}$$

$$g_{j+1} = \mathbf{I} + h_{j+1} \tag{8}$$

$$a_{j+1} = (g_{j+1})^{-1}(\bar{\partial} g_{j+1} + a_j g_{j+1}) \tag{9}$$

Notice that with this definition the integrability condition $\bar{\partial} a_j + a_j \wedge a_j = 0$ is satisfied for all j. Following [W], set $\sigma_j = 4^{-j-1}$ and $r_{j+1} = r_j(1 - \sigma_j)$ with $r_0 = r$. It follows that the r_j are decreasing and that $r' = \lim_{j\to\infty} r_j > 0$. Recalling the constants $K_j = c_\alpha \sigma_j^{-s}$ in (2) and (3), and using (7), we have:

$$\|h_{j+1}\|_{1+\alpha; r_{j+1}} \le r K_j \|a_j\|_{\alpha; r_j} \ . \tag{10}$$

From (8) and (9) we have:

$$a_{j+1} = (g_{j+1})^{-1}(\bar{\partial} h_{j+1} + a_j + a_j h_{j+1})$$

and by (1) and (7):

$$\bar{\partial} h_{j+1} + a_j = Q(\bar{\partial} a_j) = -Q(a_j \wedge a_j) \ .$$

Assuming the uniform invertibility of g_{j+1} mentioned above, it follows from (3) and (10) that there is a constant C independent of j such that:

$$\|a_{j+1}\|_{\alpha; r_{j+1}} \le C K_j \|a_j\|^2_{\alpha; r_j} \ . \tag{11}$$

After absorbing constants into the definition of K_j, (10) and (11) may be written:

$$\|h_{j+1}\|_{1+\alpha; r_{j+1}} \le K_j \|a_j\|_{\alpha; r_j} \tag{12}$$

$$\|a_{j+1}\|_{\alpha; r_{j+1}} \le K_j \|a_j\|^2_{\alpha; r_j} \ . \tag{13}$$

Moreover, there is a constant b (e.g. $b = 4^s$) such that $K_{j+1} \le bK_j$ for all j. Next, we *define*: $\theta_j = K_j \|a_j\|_{\alpha;r_j}$. Then by assumption: $\theta_0 = K_0\|a\|_{\alpha;r} \le K_0\theta$. We assume that θ has been chosen so small that $bK_0\theta \le 1/4$, say. We then deduce inductively, using (13), that:

$$\theta_{j+1} \le b\theta_j^2 \le \theta_j/4 \ . \tag{14}$$

It follows that $\theta_j \to 0$. Furthermore, we can rewrite (12) and (13) as:

$$\|h_{j+1}\|_{1+\alpha;r_{j+1}} \le \theta_j \tag{15}$$

$$\|a_{j+1}\|_{\alpha;r_{j+1}} \le \theta_j \|a_j\|_{\alpha;r_j} \ . \tag{16}$$

It follows from (16) that $\|a_j\|_{\alpha;r'} \to 0$. Notice also that $\|h_j\|_{1+\alpha;r_j} \le 1/4$ for all j. Hence, the g_j are uniformly invertible, as desired. We now define gauge transformations

$$G_k = g_1 g_2 \cdots g_k \ . \tag{17}$$

First, note that $|G_k|$ is uniformly bounded. Indeed, $|G_k| \le \prod_{j=1}^k |g_j| \le \prod_{j=1}^k (1+\theta_j)$, by (15), and the right hand side converges as $k \to \infty$ by (14). The derivatives $|\nabla G_k|$ are similarly bounded:

$$|\nabla G_k| = \Big|\sum_{j=1}^k g_1 \cdots g_{j-1}\nabla g_j g_{j+1}\cdots g_k\Big| \le \sum_{j=1}^k |g_1|\cdots|g_{j-1}||\nabla g_j||g_{j+1}|\cdots|g_k| \le \Big(\sum_{j=1}^k \theta_j\Big)\prod_{j=1}^k (1+\theta_j) \ ,$$

which also converges as $k \to \infty$. In particular, we have a bound on $\|G_k\|_{\alpha;r'}$ that is uniform in k. Next, from (17) we have: $G_{k+1} = G_k g_{k+1} = G_k + G_k h_{k+1}$, so by (15):

$$\|G_{k+1} - G_k\|_{\alpha;r'} \le c\|G_k\|_{\alpha;r'}\|h_{k+1}\|_{\alpha;r'} \le C\theta_k \ ,$$

for a constant C independent of k. It follows again by (14) that G_k converges in $C^\alpha(B_{r'})$ to some G. To improve the convergence, use the definition (9) to write

$$\bar\partial G_k + aG_k - G_k a_k = 0 \ , \tag{18}$$

for all k. Hence,

$$\|\bar\partial G_j - \bar\partial G_k\|_{\alpha;r'} \le C\big(\|G_j - G_k\|_{\alpha;r'} + \|a_j - a_k\|_{\alpha;r'}\big) \ ,$$

and since $\|a_k\|_{\alpha;r'} \to 0$ and $\|G_j - G_k\|_{\alpha;r'} \to 0$ it follows that $\bar\partial G_k$ converges in $C^\alpha(B_{r'})$. By the elliptic estimate for $\bar\partial$, $G_k \to G$ in $C^{1,\alpha}(B_{r'})$, and moreover: $\bar\partial G + aG = 0$, (cf. (18)). Finally, we claim that G is nonsingular. Now $\det G_k \to \det G$, so this follows from (8), (14), (15), (17), and Lemma 3. Since r', the $C^{1,\alpha}$ bound on G, and the bound on the determinant all stem from the initial choice of θ, which in turn depends only on r, the proof of the Proposition is complete. □

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