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On self-consistent estimators and kernel density estimators with doubly censored data

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Abstract

We study the detailed structure (in a large sample) of the self-consistent estimators of the survival functions with doubly censored data. We also introduce the kernel-type density estimators based on the self-consistent estimators, and using our results on the structure of the self-consistent estimators, we establish the strong uniform consistency and the asymptotic normality of the kernel density estimators for doubly censored data. From these, the strong uniform consistency and the asymptotic normality of the failure rate estimators for doubly censored data are derived. © 1997 Elsevier Science B.V.

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1. Introduction

Incomplete lifetime data are often encountered in medical follow-up studies and in biometry and reliability studies. In these studies, the estimation of the density function of the underlying lifetime is an important problem in its own right and also in the estimation of the *failure rate function or hazard function* of the underlying lifetime. In the right censored case, the kernel-type density estimators have been studied based on the product limit estimator of Kaplan and Meier (1958) (KM estimator) by Blum and Susarla (1980), Földes et al. (1981), Mielniczuk (1986) and Marron and Padgett (1987), among others. Recently, some more complicated types of censoring, such as doubly

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censored data, interval censored data, truncated data, etc., have started to catch the attention of statisticians, as these data occur in important clinical trials. For instance, doubly censored data were encountered in a recent study of age-dependent growth rate of primary breast cancer (Peer et al., 1993). Other examples of doubly censored data encountered in practical situations were given by Gehan (1965) and Turnbull (1974), among others. In this paper, we consider the kernel-type density estimators when the data are censored from right or left, called *doubly censored data.*

For doubly censored data, Turnbull (1974) constructed a self-consistent estimator $S_X^{(n)}$ of the survival function $S_X(t) = P\{X > t\}$, where X denotes the lifetime. Chang and Yang (1987), Chang (1990) and Gu and Zhang (1993) have established the strong consistency and the weak convergence of $S_X^{(n)}$. For more discussion on $S_X^{(n)}$, see Tsai and Crowley (1985) and Gill (1989). The kernel-type density estimators considered in this paper are based on $S_X^{(n)}$.

In the right censored case, the asymptotic properties of the kernel density estimators are studied through the structure of the KM estimator (viz., Mielniczuk, 1986). The KM estimator can be expressed explicitly as a step function with the jump sizes given clearly, and does not have jumps at those right censored observations (Efron, 1967). However, in the doubly censored case, the self-consistent estimator is given implicitly through an integral equation (see (2.5) in Section 2), and the structure of the self-consistent estimator $S_X^{(n)}$ is quite complex. We easily have examples to show that the jumps of $S_X^{(n)}$ may occur at non-censored, or right censored, or left censored observations (see the examples in Gu and Zhang, 1993). This makes it difficult to study the asymptotic properties of the kernel density estimators for doubly censored data. Moreover, in other studies of the statistical inference problems based on $S_X^{(n)}$ such as extended L-, M- and R-estimators for doubly censored data (Ren and Zhou, 1993, 1994), there is also a need to investigate the structure of $S_X^{(n)}$.

The main results of this paper consist of two parts: (1) the structure of the self-consistent estimator $S_X^{(n)}$ in a large sample case; (2) the strong uniform consistency and the asymptotic normality of the kernel density estimators for doubly censored data. As a corollary of the second part of the results, the strong uniform consistency and the asymptotic normality of the estimators of the failure rate function for doubly censored data are also obtained. The results are presented in Section 2 with the proofs deferred to Sections 3 and 4.

2. Main results

Let X be a nonnegative random variable (r.v.) denoting the lifetime under investigation, and let X_i, $i = 1, 2, \ldots, n$, be n independent observations on X with d.f. F. In this research, one observes not $\{X_i\}$ but a doubly censored sample:

$$W_i = \max\{\min\{X_i, Y_i\}, Z_i\}$$

with the index

$$\delta_i = \begin{cases} 1 & \text{if } Z_i \leqslant X_i \leqslant Y_i, \\ 2 & \text{if } X_i > Y_i, \\ 3 & \text{if } X_i < Z_i, \end{cases}$$

where (Y_i, Z_i), $i = 1, 2, \ldots, n$, are independent from X_i and are independent observations on (Y, Z) for nonnegative random variables Y and Z with $P\{Y \geqslant Z\} = 1$. The r.v.s Y_i and Z_i are called *right* and *left censoring variables*, respectively. This means that X_i is observable whenever X_i lies in the interval $[Z_i, Y_i]$, and otherwise we know whether $X_i < Z_i$ or $X_i > Y_i$ and observe the value of Z_i or Y_i accordingly. The problems considered here are the structure of the estimators of S_X, S_Y, S_Z in a large sample and the estimation of the density functions of X, Y, Z based on (W_i, δ_i), where $S_Y(t) = P\{Y > t\}$ and $S_Z(t) = P\{Z > t\}$.

We present the estimators of S_X, S_Y and S_Z as follows. Let (W_i, δ_i) be distributed as (W, δ), and let

$$Q_j(t) = P\{W > t, \delta = j\}, \quad j = 1, 2, 3, \tag{2.1}$$

$$Q_j^{(n)}(t) = \frac{1}{n} \sum_{i=1}^{n} I\{W_i > t, \delta_i = j\}, \quad j = 1, 2, 3, \tag{2.2}$$

then the estimators $S_X^{(n)}, S_Y^{(n)}, S_Z^{(n)}$ *of* S_X, S_Y, S_Z (Chang and Yang, 1987) are given by the solutions of the following equations:

$$\begin{aligned} Q_1^{(n)}(t) &= -\int_t^{\infty} (S_Y^{(n)} - S_Z^{(n)})\,\mathrm{d}S_X^{(n)}, \\ Q_2^{(n)}(t) &= -\int_t^{\infty} S_X^{(n)}\,\mathrm{d}S_Y^{(n)}, \\ Q_3^{(n)}(t) &= -\int_t^{\infty} (1 - S_X^{(n)})\,\mathrm{d}S_Z^{(n)}. \end{aligned} \tag{2.3}$$

Imposing the conditions on $S_Y^{(n)}$ and $S_Z^{(n)}$,

$$S_Y^{(n)}(0) = 1 \quad \text{and} \quad S_Z^{(n)}(\infty) = 0, \tag{2.4}$$

from the system (2.3), Chang and Yang (1987) obtain

$$S_X^{(n)}(t) = Q^{(n)}(t) - \int_{u \leqslant t} \frac{S_X^{(n)}(t)}{S_X^{(n)}(u)}\,\mathrm{d}Q_2^{(n)}(u) + \int_{t<u} \frac{1 - S_X^{(n)}(t)}{1 - S_X^{(n)}(u)}\,\mathrm{d}Q_3^{(n)}(u), \tag{2.5}$$

where $t \geqslant 0$ with

$$\int_{u \leqslant t} = 0 \quad \text{if } S_X^{(n)}(t) = 0 \quad \text{and} \quad \int_{t<u} = 0 \quad \text{if } S_X^{(n)}(t) = 1,$$

and obtain

$$S_Y^{(n)}(t) = 1 + \int_{u \leqslant t} \frac{\mathrm{d}Q_2^{(n)}(u)}{S_X^{(n)}(u)}, \quad t < B_n, \tag{2.6}$$

$$S_Z^{(n)}(t) = -\int_{t < u} \frac{\mathrm{d}Q_3^{(n)}(u)}{1 - S_X^{(n)}(u)}, \quad t \geqslant A_n, \tag{2.7}$$

$$Q^{(n)}(t) = S_Z^{(n)}(t) + S_X^{(n)}(t)[S_Y^{(n)}(t) - S_Z^{(n)}(t)], \quad t \geqslant 0, \tag{2.8}$$

where

$$A_n = \min\{W_i;\ S_X^{(n)}(W_i) < 1\}, \tag{2.9}$$

$$B_n = \max\{W_i;\ S_X^{(n)}(W_i -) > 0\}, \tag{2.10}$$

$$Q^{(n)}(t) = \sum_{j=1}^{3} Q_j^{(n)}(t), \quad t \geqslant 0. \tag{2.11}$$

One may note that a *self-consistent estimator* of S_X is given by a solution of (2.5) and that the self-consistent estimating equation (2.5) is originally given by Tsai and Crowley (1985). Moreover, one may note that for any self-consistent estimator $S_X^{(n)}$ of S_X, the range in which we can obtain information for estimating S_X is $[A_n, B_n]$, thus the restriction on t by A_n and B_n in (2.6) and (2.7) is natural.

From Gu and Zhang (1993), we know that the solution of (2.5) is not unique and that the self-consistent estimators are asymptotically equivalent. Hence, for our investigation in this paper, we consider a particular type of self-consistent estimators as follows. Let

$$W_{(1)} = \min\{W_i;\ 1 \leqslant i \leqslant n\} \quad \text{and} \quad W_{(n)} = \max\{W_i;\ 1 \leqslant i \leqslant n\}.$$

We note that for any solution $S_X^{(n)}$ of (2.5), suppose we define $\bar{S}_X^{(n)}(t) = 1$, if $t < W_{(1)}$; $S_X^{(n)}(t)$, if $W_{(1)} \leqslant t < W_{(n)}$; 0, if $t \geqslant W_{(n)}$, then $\bar{S}_X^{(n)}$ is still a solution of (2.5). For the rest of the paper, we will always consider those solutions $S_X^{(n)}$ of (2.5) which satisfy

$$S_X^{(n)}(t) = \begin{cases} 1 & \text{if } t < W_{(1)}, \\ 0 & \text{if } t \geqslant W_{(n)}. \end{cases} \tag{2.12}$$

Since an arbitrary solution $S_X^{(n)}$ of (2.5) is not necessarily a proper survival function, the condition (2.12) is to restrict our attention to those proper estimators. This is an already adopted convention for right censored data (Efron, 1967; Miller, 1976).

One may note that for a step function $S_X^{(n)}$ satisfying (2.5) and (2.12), A_n and B_n given by (2.9) and (2.10) are the smallest and the largest jump points of $S_X^{(n)}$, respectively, with $W_{(1)} \leqslant A_n < B_n \leqslant W_{(n)}$, and that (2.5) and (2.8) are actually the same equations for $S_Y^{(n)}$ and $S_Z^{(n)}$ satisfying (2.6) and (2.7), respectively.

Throughout, we assme that there are no ties among $W_1, \ldots, W_n$ and impose the following conditions on S_X, S_Y, S_Z.

Assumption A.

(A1) The random variable X_i and the vector (Y_i, Z_i) are independent for each i and the vectors (X_i, Y_i, Z_i), $i = 1, \ldots, n$, are independently and identically distributed;

(A2) $P\{Z \leqslant Y\} = 1$;

(A3) $S_Y(t) - S_Z(t) > 0$ on $[0, \infty)$;

(A4) S_X, S_Y and S_Z are continuous functions of t for $t \geqslant 0$, and $0 < S_X(t) < 1$ for $t > 0$;

(A5) $S_X(0) = S_Y(0) = 1$, $S_X(\infty) = S_Y(\infty) = S_Z(\infty) = 0$.

Remark 1. Because of (A3), our conditions here are slightly stronger than those required by Chang and Yang (1987) for the strong consistency of $S_X^{(n)}, S_Y^{(n)}, S_Z^{(n)}$. Our (A3) implies $P\{Z = 0\} > 0$ and $S_Z(0) < 1$, which ensures that for any $0 < T < \infty$, $[S_Y(t) - S_Z(t)]$ has a positive lower bound for $t \in [0, T]$. In Section 4, we will see that it is naturally required for the strong uniform consistency of the kernel density estimators with doubly censored data that $[S_Y(t) - S_Z(t)]$ has a positive lower bound on $[0, T]$. One may see the comments by Chang (1990) about the condition $S_Z(0) < 1$.

From (2.6) and (2.7), we easily have

$$S_Y^{(n)}(t) = 1 - \frac{1}{n} \sum_{i=1}^{n} \frac{I\{\delta_i = 2, W_i \leqslant t\}}{S_X^{(n)}(W_i)}, \quad t < B_n \tag{2.13}$$

$$S_Z^{(n)}(t) = \frac{1}{n} \sum_{i=1}^{n} \frac{I\{\delta_i = 3, W_i > t\}}{1 - S_X^{(n)}(W_i)}, \quad t \geqslant A_n. \tag{2.14}$$

In the appendix, we show that for a solution $S_X^{(n)}$ of (2.5) satisfying (2.12), we have

$$\delta_i = 2 \text{ if } W_i < A_n \quad \text{and} \quad \delta_i = 3 \text{ if } W_i > B_n. \tag{2.15}$$

Note that if we consider the following extensions of (2.13) and (2.14) beyond the interval $[A_n, B_n)$:

$$S_Y^{(n)}(t) = \begin{cases} 1 - \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \dfrac{I\{\delta_i = 2, W_i \leqslant t\}}{S_X^{(n)}(W_i)} & \text{if } t < B_n, \\ S_Y^{(n)}(B_n -) & \text{if } t \geqslant B_n, \end{cases} \tag{2.16}$$

$$S_Z^{(n)}(t) = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \dfrac{I\{\delta_i = 3, W_i > t\}}{1 - S_X^{(n)}(W_i)} & \text{if } t \geqslant A_n, \\ S_Z^{(n)}(A_n) & \text{if } t < A_n, \end{cases} \tag{2.17}$$

then $S_Y^{(n)}$ and $S_Z^{(n)}$ satisfy (2.5)–(2.8), and from (2.15), they satisfy

$$Q_2^{(n)}(t) = -\int_t^{\infty} S_X^{(n)} \mathrm{d}S_Y^{(n)} + I\{\delta_{B_n} = 2, B_n > t\}/n,$$

$$Q_3^{(n)}(t) = -\int_t^{\infty} (1 - S_X^{(n)}) \mathrm{d}S_Z^{(n)} + I\{\delta_{A_n} = 3, A_n > t\}/n,$$

where δ_{A_n} and δ_{B_n} are the indexes of A_n and B_n, respectively. Hence, from integration by parts and (2.8), we have that for a solution $S_X^{(n)}$ of (2.5) satisfying (2.12), our $S_Y^{(n)}$ and $S_Z^{(n)}$ given by (2.16) and (2.17) satisfy the following equations:

$$Q_1^{(n)}(t) = -\int_t^{\infty} (S_Y^{(n)} - S_Z^{(n)})\,\mathrm{d}S_X^{(n)} - (I\{\delta_{A_n} = 3, A_n > t\} + I\{\delta_{B_n} = 2, B_n > t\})/n$$
$$+\int_t^{\infty} [S_X^{(n)}(u-) - S_X^{(n)}(u)]\,\mathrm{d}S_Y^{(n)}(u) - \int_t^{\infty} [S_X^{(n)}(u-) - S_X^{(n)}(u)]\,\mathrm{d}S_Z^{(n)}(u), \tag{2.18}$$

$$Q_2^{(n)}(t) = -\int_t^{\infty} S_X^{(n)}\,\mathrm{d}S_Y^{(n)} + I\{\delta_{B_n} = 2, B_n > t\}/n,$$

$$Q_3^{(n)}(t) = -\int_t^{\infty} (1 - S_X^{(n)})\,\mathrm{d}S_Z^{(n)} + I\{\delta_{A_n} = 3, A_n > t\}/n,$$

where $t \in [0, \infty)$. Under Assumption A, we have that from Gu and Zhang (1993), Chang and Yang (1987), our $S_Y^{(n)}$ and $S_Z^{(n)}$ given by (2.16) and (2.17) for a solution $S_X^{(n)}$ of (2.5) obeying (2.12) satisfy

$$\lim_{n\to\infty} S_X^{(n)}(t) = S_X(t), \tag{2.19}$$

$$\lim_{n\to\infty} S_Y^{(n)}(t) = S_Y(t), \tag{2.20}$$

$$\lim_{n\to\infty} S_Z^{(n)}(t) = S_Z(t), \tag{2.21}$$

uniformly for $t \in [0, \infty)$ with probability 1. For the rest of this paper, we will always consider $S_Y^{(n)}$ and $S_Z^{(n)}$ given by (2.16) and (2.17).

In the following theorem, we give the detailed structure of $S_X^{(n)}$ in a large sample under Assumption A.

Theorem 1. *Let $S_X^{(n)}$ be a solution of* (2.5) *satisfying* (2.12) *and assume that there are no ties among W_i's. Then,*

(i) *the jump size of $S_X^{(n)}$ at A_n is given by*

$$\Delta_{A_n}^X = \frac{1}{n[S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)]}, \tag{2.22}$$

and we have

$$A_n = \min\{W_i; \delta_i = 1 \text{ or } 3\}; \tag{2.23}$$

(ii) *assuming $[S_Y^{(n)}(B_n-) - S_Z^{(n)}(B_n-)] > 0$, the jump size of $S_X^{(n)}$ at B_n is given by*

$$\Delta_{B_n}^X = \frac{1}{n[S_Y^{(n)}(B_n) - S_Z^{(n)}(B_n)]}, \tag{2.24}$$

and we have

$$B_n = \max\{W_i; \delta_i = 1 \text{ or } 2\}; \tag{2.25}$$

(iii) *under Assumption* A, *we have that for any* $A_n < T < \infty$ *and sufficiently large n,*

$$S_X^{(n)}(t) = 1 - \left\{ \frac{I\{A_n \leq t\}}{n[S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)]} + \frac{1}{n} \sum_{A_n < W_i < C_n} \frac{I\{\delta_i = 1, W_i \leq t\}}{[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]} \right.$$

$$\left. + \frac{I\{B_n \leq \min(t, T)\}}{n[S_Y^{(n)}(B_n) - S_Z^{(n)}(B_n)]} + \frac{I\{B_n > T, \delta_{C_n} = 1, C_n \leq t\}}{n[S_Y^{(n)}(C_n) - S_Z^{(n)}(C_n)]} \right\}, \quad t \in [0, T] \tag{2.26}$$

with probability 1, *where* $C_n = \max\{W_i; W_i \leq \min(B_n, T)\}$ *and* δ_{C_n} *is the index of* C_n.

Theorem 1 is proved in Section 3. In the proof, it plays an important role that for sufficiently large n, $[S_Y^{(n)}(t) - S_Z^{(n)}(t)]$ has a positive lower bound for $t \in [0, T]$ with probability 1. This fact, ensured by (2.20) and (2.21) along with assumptions (A3) and (A4), leads to our Theorem 1(iii). One may note that in (2.8), $S_X^{(n)}(t)$ cannot be expressed in terms of $S_Y^{(n)}(t)$, $S_Z^{(n)}(t)$ and $Q^{(n)}(t)$, if $[S_Y^{(n)}(t) - S_Z^{(n)}(t)] = 0$ on a certain interval. In fact, this is precisely the reason that $S_X^{(n)}$ may have jumps at censored observations. Detailed discussion on this can be found in Mykland and Ren (1996) and is omitted in current paper for brevity. As mentioned earlier, the KM estimator (the self-consistent estimator for right censored data) only has jumps at non-censored observations for any n (Efron, 1967). But for doubly censored data, this is no longer the case (see Gu and Zhang, 1993, for examples). Our Theorem 1(iii) shows that on any compact set $[0, T]$, except the first jump point, the self-consistent estimator $S_X^{(n)}$ for doubly censored data only has jumps at non-censored observations for sufficiently large n with probability 1. In comparison, the structure of $S_X^{(n)}$ for doubly censored data is far more complicated. Nonetheless, our (2.26) generally provides a sufficiently useful tool for the study of the asymptotic properties of the statistics based on $S_X^{(n)}$. In fact, (2.26) plays a key role in the proof of our next theorem and in the study of the asymptotic properties of L-, M- and R-estimators for doubly censored data (Ren and Zhou, 1993, 1994).

Let

$$F = 1 - S_X, \qquad G = 1 - S_Y, \qquad H = 1 - S_Z, \tag{2.27}$$

$$F_n(t) = 1 - S_X^{(n)}(t), \qquad G_n(t) = 1 - S_Y^{(n)}(t), \qquad H_n(t) = 1 - S_Z^{(n)}(t) \tag{2.28}$$

and let f, g, h denote the density functions of F, G, H, respectively. (Note that under (A3) (see Remark 1), H does not have a density function. In such a case, h denotes the derivative of H in $(0, \infty)$). Then, the kernel-type density estimators of f, g, h, for doubly censored data are naturally given by

$$f_n(x) = \frac{1}{a(n)} \int_0^\infty K\left(\frac{x-y}{a(n)}\right) \mathrm{d}F_n(y), \tag{2.29}$$

$$g_n(x) = \frac{1}{a(n)} \int_0^\infty K\left(\frac{x-y}{a(n)}\right) \mathrm{d}G_n(y), \tag{2.30}$$

$$h_n(x) = \frac{1}{a(n)} \int_0^\infty K\left(\frac{x-y}{a(n)}\right) \mathrm{d}H_n(y), \tag{2.31}$$

respectively, where $a(n)$ is a sequence of positive numbers such that $a(n) \to 0$, as $n \to \infty$ and K is a density function. The strong uniform consistency of f_n, g_n, h_n and the asymptotic normality of $f_n(x)$ are established in the next theorem with the following conditions.

Assumption B.

(B1) f, g, h are continuous on $[0, \infty)$;

(B2) K is a continuous density function with support in $[-1, 1]$ and is of bounded variation;

(B3) $a(n) \to 0$ and $(\log n)/[n\,a(n)] \to 0$, as $n \to \infty$.

Assumption C.

(C1) f is bounded, $f(x) > 0$ and in the neighborhood of x, the second derivative of $f[S_Y - S_Z]$ exists and is bounded;

(C2) K is an even and bounded density function with support in $[-1, 1]$;

(C3) $n[a(n)]^3 \to 0$ and $na(n) \to \infty$, as $n \to \infty$;

(C4) There exist δ and Δ, $0 < \delta < \Delta < \infty$, such that $P\{Z \in (0, \delta)\} = 0$ and $P\{Z \leqslant \Delta\} = 1$,

Remark 2. The conditions (B1)–(B3) and (C1)–(C3) are usually required for the strong uniform consistency and the asymptotic normality of the kernel density estimators, respectively, for non-censored data (Silverman, 1978; Rosenblatt, 1971) or right censored data (Mielniczuk, 1986). (C4) is required by Chang (1990) for the weak convergence of $S_Y^{(n)}$ and $S_Z^{(n)}$. In Section 4, we will see that the weak convergence of $S_Y^{(n)}$ and $S_Z^{(n)}$ is used to derive the asymptotic normality of $f_n(x)$. One may see the comments by Chang (1990) about the condition (C4) in practical situations.

Theorem 2. (i) *If* (X, Y, Z) *satisfies Assumptions* A *and* B, *then for any* $A_n < T < \infty$, *we have that as* $n \to \infty$,

$$\sup_{t \in [0, T]} |f_n(t) - f(t)| \to 0 \quad \textit{with probability } 1, \tag{2.32}$$

$$\sup_{t \in [0, T]} |g_n(t) - g(t)| \to 0 \quad \textit{with probability } 1, \tag{2.33}$$

$$\sup_{t \in [0, T]} |h_n(t) - h(t)| \to 0 \quad \textit{with probability } 1; \tag{2.34}$$

(ii) *if (X, Y, Z) satisfies Assumptions* A *and* C, *then as $n \to \infty$,*

$$\sqrt{n\,a(n)}\,[f_n(x) - f(x)] \xrightarrow{D} N\left(0, \frac{f(x)}{[S_Y(x) - S_Z(x)]} \int K^2(u)\,du\right). \tag{2.35}$$

Theorem 2 is proved in Section 4. One may note that (2.32) is a doubly censored version of Theorem A of Silverman (1978), and that (2.33), (2.34) are similar to (ii) of Corollary 2 of Mielniczuk (1986) for right censored data. One may also note that (2.35) is a doubly censored version of Corollary 3(ii) of Mielniczuk (1986), who considered the right censored case.

The failure rate function of X is given by

$$\mathfrak{R}(t) = f(t)/S_X(t), \quad t \geqslant 0. \tag{2.36}$$

Naturally, the estimator of $\mathfrak{R}$ for doubly censored data is given by

$$\mathfrak{R}_n(t) = f_n(t)/S_X^{(n)}(t), \quad t \geqslant 0. \tag{2.37}$$

As a corollary of Theorem 2, from the strong consistency (given by (2.19)) and the weak convergence (Chang, 1990) of $S_X^{(n)}$, we have the strong uniform consistency and the asymptotic normality of $\mathfrak{R}_n$.

Corollary. (i) *If (X, Y, Z) satisfies Assumption* A *and* B, *then for any $A_n < T < \infty$, we have that with probability* 1,

$$\sup_{t \in [0, T]} |\mathfrak{R}_n(t) - \mathfrak{R}(t)| \to 0, \quad as\ n \to \infty; \tag{2.38}$$

(ii) *if (X, Y, Z) satisfies Assumption* A *and* C, *then as $n \to \infty$,*

$$\sqrt{n\,a(n)}\,[\mathfrak{R}_n(x) - \mathfrak{R}(x)] \xrightarrow{D} N\left(0, \frac{f(x)}{S_X^2(x)\,[S_Y(x) - S_Z(x)]} \int K^2(u)\,du\right). \tag{2.39}$$

3. Proof of Theorem 1

First, we note that by the continuity assumption (A4), we may, without loss of the generality, assume $0 < W_1 < W_2 < \cdots < W_n$. Ties among W_i's are neglected with probability one. Before proving Theorem 1, we establish the following lemma.

Lemma 3.1. *If $[S_Y^{(n)}(t) - S_Z^{(n)}(t)] > 0$ for $t \in [A_n, B_n)$, then $S_X^{(n)}$ is a step function on $[A_n, B_n)$ with the jump size at W_i given by*

$$\Delta_{W_i}^X = \begin{cases} \dfrac{1}{n[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]} & \text{if } \delta_i = 1, \\ 0 & \text{if } \delta_i \neq 1, \end{cases} \tag{3.1}$$

where $A_n < W_i < B_n$.

Proof. (i) From (2.13) and (2.14), we know that $S_Y^{(n)}$ and $S_Z^{(n)}$ are step functions on $[A_n, B_n)$ with the jump size at W_i given by

$$\Delta_{W_i}^Y = \begin{cases} 1/(nS_X^{(n)}(W_i)) & \text{if } \delta_i = 2, \\ 0 & \text{if } \delta_i \neq 2, \end{cases} \tag{3.2}$$

and

$$\Delta_{W_i}^Z = \begin{cases} 1/(n[1 - S_X^{(n)}(W_i)]) & \text{if } \delta_i = 3, \\ 0 & \text{if } \delta_i \neq 3, \end{cases} \tag{3.3}$$

respectively, and we know that $S_Y^{(n)}$ and $S_Z^{(n)}$ are constants on interval $[W_i, W_{i+1})$ for $W_i \in [A_n, B_n)$. From assumption $[S_Y^{(n)}(t) - S_Z^{(n)}(t)] > 0$ for $t \in [A_n, B_n)$ and (2.8), we have that for $t \in [A_n, B_n)$,

$$\begin{aligned} S_X^{(n)}(t) &= \frac{Q^{(n)}(t) - S_Z^{(n)}(t)}{S_Y^{(n)}(t) - S_Z^{(n)}(t)} \\ &= \frac{\frac{1}{n}\sum_{i=1}^n I\{W_i > t\} - \frac{1}{n}\sum_{i=1}^n [I\{\delta_i = 3, W_i > t\}(1 - S_X^{(n)}(W_i))^{-1}]}{S_Y^{(n)}(t) - S_Z^{(n)}(t)}. \end{aligned} \tag{3.4}$$

is a step function on $[A_n, B_n)$ with constant values on intervals $[W_i, W_{i+1})$ for $W_i \in [A_n, B_n)$, and that for $W_i \in [A_n, B_n)$, the jump size of $S_X^{(n)}$ at W_i is given by

$$\begin{aligned} \Delta_{W_i}^X &= S_X^{(n)}(W_{i-1}) - S_X^{(n)}(W_i) \\ &= \frac{Q^{(n)}(W_{i-1}) - S_Z^{(n)}(W_{i-1})}{S_Y^{(n)}(W_{i-1}) - S_Z^{(n)}(W_{i-1})} - \frac{Q^{(n)}(W_i) - S_Z^{(n)}(W_i)}{S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)}. \end{aligned} \tag{3.5}$$

We will investigate the value of $\Delta_{W_i}^X$ for different values of δ_i as follows.

(1) The case of $\delta_i = 1$. Since there are no jumps at W_i for $S_Y^{(n)}$ and $S_Z^{(n)}$, we have

$$S_Y^{(n)}(W_{i-1}) = S_Y^{(n)}(W_i) \quad \text{and} \quad S_Z^{(n)}(W_{i-1}) = S_Z^{(n)}(W_i).$$

Hence, from $[Q^{(n)}(W_{i-1}) - Q^{(n)}(W_i)] = 1/n$, we have

$$\Delta_{W_i}^X = \frac{1}{n[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]}.$$

(2) The case of $\delta_i = 2$. Since there is no jump at W_i for $S_Z^{(n)}$, we have

$$S_Z^{(n)}(W_{i-1}) = S_Z^{(n)}(W_i).$$

From (2.8) and (3.2), we have

$$\Delta^X_{W_i} = \frac{Q^{(n)}(W_{i-1}) - S^{(n)}_Z(W_i)}{S^{(n)}_Y(W_{i-1}) - S^{(n)}_Z(W_i)} - \frac{Q^{(n)}(W_i) - S^{(n)}_Z(W_i)}{S^{(n)}_Y(W_i) - S^{(n)}_Z(W_i)}$$

$$= \frac{[Q^{(n)}(W_{i-1}) - Q^{(n)}(W_i)]}{[S^{(n)}_Y(W_{i-1}) - S^{(n)}_Z(W_i)]} - \frac{[Q^{(n)}(W_i) - S^{(n)}_Z(W_i)]\Delta^Y_{W_i}}{[S^{(n)}_Y(W_{i-1}) - S^{(n)}_Z(W_i)][S^{(n)}_Y(W_i) - S^{(n)}_Z(W_i)]}$$

$$= \frac{1/n}{[S^{(n)}_Y(W_{i-1}) - S^{(n)}_Z(W_i)]} - \frac{S^{(n)}_X(W_i)\Delta^Y_{W_i}}{[S^{(n)}_Y(W_{i-1}) - S^{(n)}_Z(W_i)]} = 0.$$

(3) The case of $\delta_i = 3$. Since there is no jump at W_i for $S^{(n)}_Y$, we have

$S^{(n)}_Y(W_{i-1}) = S^{(n)}_Y(W_i)$.

From (2.8) and (3.3), we have

$$\Delta^X_{W_i} = \frac{Q^{(n)}(W_{i-1}) - S^{(n)}_Z(W_{i-1})}{S^{(n)}_Y(W_i) - S^{(n)}_Z(W_{i-1})} - \frac{Q^{(n)}(W_i) - S^{(n)}_Z(W_i)}{S^{(n)}_Y(W_i) - S^{(n)}_Z(W_i)}$$

$$= \frac{1/n}{[S^{(n)}_Y(W_i) - S^{(n)}_Z(W_{i-1})]} + \frac{Q^{(n)}(W_i) - S^{(n)}_Z(W_{i-1})}{[S^{(n)}_Y(W_i) - S^{(n)}_Z(W_{i-1})]} - \frac{Q^{(n)}(W_i) - S^{(n)}_Z(W_i)}{S^{(n)}_Y(W_i) - S^{(n)}_Z(W_i)}$$

$$= \frac{1/n}{[S^{(n)}_Y(W_i) - S^{(n)}_Z(W_{i-1})]} + \frac{Q^{(n)}(W_i) - S^{(n)}_Z(W_i)}{[S^{(n)}_Y(W_i) - S^{(n)}_Z(W_{i-1})]}$$

$$- \frac{\Delta^Z_{W_i}}{[S^{(n)}_Y(W_i) - S^{(n)}_Z(W_{i-1})]} - \frac{Q^{(n)}(W_i) - S^{(n)}_Z(W_i)}{S^{(n)}_Y(W_i) - S^{(n)}_Z(W_i)}$$

$$= \frac{1/n}{[S^{(n)}_Y(W_i) - S^{(n)}_Z(W_{i-1})]} - \frac{[1 - S^{(n)}_X(W_i)]\Delta^Z_{W_i}}{[S^{(n)}_Y(W_i) - S^{(n)}_Z(W_{i-1})]} = 0. \qquad \square$$

Proof of Theorem 1. (i) First we show that the smallest jump point A_n of $S^{(n)}_X$ must have index 1 or 3. From (2.15) and (2.16), we have

$$S^{(n)}_Y(A_n) - Q^{(n)}(A_n) = \frac{1}{n} - \frac{I\{\delta_{A_n} = 2\}}{nS^{(n)}_X(A_n)}. \tag{3.6}$$

From (2.18), we have that for $t < A_n$,

$$-\int_{(t,A_n]} (S^{(n)}_Y - S^{(n)}_Z)\,\mathrm{d}S^{(n)}_X = Q^{(n)}_1(t) - Q^{(n)}_1(A_n) + \frac{I\{\delta_{A_n} = 3\}}{n}$$

$$- \int_{(t,A_n]} (S^{(n)}_X(u-) - S^{(n)}_X(u)]\,\mathrm{d}S^{(n)}_Y(u) + \int_{(t,A_n]} [S^{(n)}_X(u-) - S^{(n)}_X(u)]\,\mathrm{d}S^{(n)}_Z(u),$$

which implies

$$[S^{(n)}_Y(A_n) - S^{(n)}_Z(A_n)]\Delta^X_{A_n} \geqslant 0. \tag{3.7}$$

From (2.8), we have

$$[Q^{(n)}(A_n) - S_Y^{(n)}(A_n)] = -[1 - S_X^{(n)}(A_n)][S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)]. \tag{3.8}$$

If $\delta_{A_n} = 2$, (3.6) and (3.8) imply

$$-\frac{\Delta_{A_n}^X}{nS_X^{(n)}(A_n)} = \Delta_{A_n}^X[S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)],$$

so that

$$\Delta_{A_n}^X\left\{\frac{1}{nS_X^{(n)}(A_n)} + [S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)]\right\} = 0.$$

Hence, from (3.7), we have $\Delta_{A_n}^X = 0$ if $\delta_{A_n} = 2$. This contradicts that A_n is the smallest jump point of $S_X^{(n)}$. Therefore, we must have $\delta_{A_n} = 1$ or 3. (2.22) and (2.23) follow from (3.6), (3.8) and (2.15).

(ii) From (2.15) and (2.17), we have

$$Q^{(n)}(B_n -)S_Z^{(n)}(B_n -) = I\{\delta_{B_n} = 1 \text{ or } 2\}/n, \tag{3.9}$$

which by (2.8), implies

$$S_X^{(n)}(B_n -) - [S_Y^{(n)}(B_n -) - S_Z^{(n)}(B_n\text{-})] = I\{\delta_{B_n} = 1 \text{ or } 2\}/n.$$

From $\Delta_{B_n}^X = S_X^{(n)}(B_n -) > 0$ and the assumption $[S_Y^{(n)}(B_n\text{-}) - S_Z^{(n)}(B_n -)] > 0$, we have $\delta_{B_n} = 1$ or 2 and (2.25). Noting that (2.16) and (2.17) and $\delta_{B_n} = 1$ or 2 imply

$$[S_Y^{(n)}(B_n) - S_Z^{(n)}(B_n)] = [S_Y^{(n)}(B_n -) - S_Z^{(n)}(B_n -)],$$

we have (2.24).

(iii) For any $A_n < T < \infty$, from Assumption A and (2.20) and (2.21), we have that for $t \in [0, T]$ and sufficiently large n,

$$[S_Y^{(n)}(t) - S_Z^{(n)}(t)] \geqslant c > 0 \quad \text{with probability 1}, \tag{3.10}$$

where c is a constant.

If $B_n \leqslant T$, from Lemma 3.1 and Theorem 1(ii)–(iii), we have that for $t \in [0, T]$,

$$S_X^{(n)}(t) = 1 - \left\{\frac{I\{A_n \leqslant t\}}{n[S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)]} + \frac{1}{n}\sum_{A_n < W_i < B_n}\frac{I\{\delta_i = 1, W_i \leqslant t\}}{[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]} + \frac{I\{B_n \leqslant t\}}{n[S_Y^{(n)}(B_n) - S_Z^{(n)}(B_n)]}\right\}. \tag{3.11}$$

If $B_n > T$, we have that for $t \in [0, C_n]$ and sufficiently large n,

$$[S_Y^{(n)}(t) - S_Z^{(n)}(t)] \geqslant c > 0, \quad \text{with probability 1},$$

where $C_n = \max\{W_i;\ W_i \leqslant T\}$ with index δ_{C_n}. In this case, we note that Lemma 3.1 still holds if B_n is replaced by C_n. Hence, we have that for $t \in [0, T]$,

$$S_X^{(n)}(t) = 1 - \left\{\frac{I\{A_n \leqslant t\}}{n[S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)]} + \frac{1}{n} \sum_{A_n < W_i \leqslant C_n} \frac{I\{\delta_i = 1, W_i \leqslant t\}}{[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]}\right\}. \quad (3.12)$$

(2.26) follows from (3.11) and (3.12). □

4. Proof of Theorem 2

From (iii) of Theorem 1, we have that for any $A_n < T < \infty$ and sufficiently large n,

$$\begin{aligned} F_n(t) = {} & \frac{I\{A_n \leqslant t\}}{n[S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)]} + \frac{1}{n} \sum_{A_n < W_i < C_n} \frac{I\{\delta_i = 1, W_i \leqslant t\}}{[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]} \\ & + \frac{I\{B_n \leqslant \min(t, T+1)\}}{n[S_Y^{(n)}(B_n) - S_Z^{(n)}(B_n)]} + \frac{I\{B_n > T+1, \delta_{C_n} = 1, C_n \leqslant t\}}{n[S_Y^{(n)}(C_n) - S_Z^{(n)}(C_n)]}, \\ & t \in [0, T+1] \end{aligned} \quad (4.1)$$

with probability 1, where $C_n = \max\{W_i;\ W_i \leqslant \min(B_n, T+1)\}$, From (2.16) and (2.17), we have

$$G_n(t) = \frac{1}{n} \sum_{i=1}^{n} \frac{I\{\delta_i = 2, W_i \leqslant \min(t, D_n)\}}{S_X^{(n)}(W_i)}, \quad t \geqslant 0, \quad (4.2)$$

where $D_n = \max\{W_i;\ W_i < B_n\}$, and

$$H_n(t) = 1 - \frac{1}{n} \sum_{i=1}^{n} \frac{I\{\delta_i = 3, W_i > \max(t, A_n)\}}{1 - S_X^{(n)}(W_i)}, \quad t \geqslant 0. \quad (4.3)$$

Define

$$\tilde{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} I\{W_i \leqslant t, \delta_i = 1\}, \quad (4.4)$$

$$\tilde{G}_n(t) = \frac{1}{n} \sum_{i=1}^{n} I\{W_i \leqslant t, \delta_i = 2\}, \quad (4.5)$$

$$\tilde{H}_n(t) = \frac{1}{n} \sum_{i=1}^{n} I\{W_i \leqslant t, \delta_i = 3\}, \quad (4.6)$$

and define

$$\tilde{f}_n(x) = \frac{1}{a(n)\,[S_Y(x) - S_Z(x)]} \int_0^{\infty} K\left(\frac{x-y}{a(n)}\right) \mathrm{d}\tilde{F}_n(y), \quad (4.7)$$

$$\tilde{g}_n(x) = \frac{1}{a(n)\,S_X(x)} \int_0^{\infty} K\left(\frac{x-y}{a(n)}\right) \mathrm{d}\tilde{G}_n(y), \quad (4.8)$$

$$\tilde{h}_n(x) = \frac{1}{a(n)\,[1 - S_X(x)]} \int_0^{\infty} K\left(\frac{x-y}{a(n)}\right) \mathrm{d}\tilde{H}_n(y). \quad (4.9)$$

We prove Theorem 2 by showing that $f_n - \tilde{f}_n$, $g_n - \tilde{g}_n$, $h_n - \tilde{h}_n$ converge to 0 uniformly on $[0, T]$ with probability 1, as $n \to \infty$.

Proof of Theorem 2. (i) Since (2.33) and (2.34) are established analogously as (2.32), we will only give the proof of (2.32).

We observe that for $x \in [0, T]$ and sufficiently large n,

$$f_n(x) = \frac{1}{a(n)} \int_0^\infty K\left(\frac{x-y}{a(n)}\right) \mathrm{d}F_n(y) = \frac{1}{a(n)} \int_0^{T+1} K\left(\frac{x-y}{a(n)}\right) \mathrm{d}F_n(y)$$
$$= \frac{K((x-A_n)/a(n))}{n\,a(n)\,[S_Y^{(n)}(A_n) - S_Z^{(n)}(A_n)]} + \frac{1}{n\,a(n)} \sum_{A_n < W_i < C_n} \frac{I\{\delta_i = 1\}\, K((x-W_i)/a(n))}{[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]}$$
$$+ \frac{I\{B_n \leqslant T+1\} K((x-B_n)/a(n))}{n\,a(n)[S_Y^{(n)}(B_n) - S_Z^{(n)}(B_n)]} + \frac{I\{B_n > T+1, \delta_{C_n} = 1\} K((x-C_n)/a(n))}{n\,a(n)[S_Y^{(n)}(C_n) - S_Z^{(n)}(C_n)]}, \tag{4.10}$$

with probability 1. Since from (2.15),

$$\tilde{F}_n(t) = \frac{1}{n} \sum_{i=1}^n I\{W_i \leqslant t, \delta_i = 1\} = \frac{1}{n} \sum_{A_n \leqslant W_i \leqslant B_n} I\{W_i \leqslant t, \delta_i = 1\},$$

we have

$$\tilde{f}_n(x) = \frac{1}{n\,a(n)} \sum_{A_n \leqslant W_i \leqslant B_n} \frac{I\{\delta_i = 1\}\, K((x-W_i)/a(n))}{[S_Y(x) - S_Z(x)]}$$
$$= \frac{1}{n\,a(n)} \sum_{A_n \leqslant W_i \leqslant C_n} \frac{I\{\delta_i = 1\}\, K((x-W_i)/a(n))}{[S_Y(x) - S_Z(x)]}$$

From assumption (A3), (A4), (B2), (B3) and (2.20), (2.21), we know that the first term and the last two terms in the last equation of (4.10) converge to zero with probability 1. Hence, we have that

$$\sup_{x \in [0,T]} |f_n(x) - \tilde{f}_n(x)| \leqslant \mathrm{o}_\mathrm{p}^{(n)}(1) + \left\{ \sup_{x \in [0,T]} |Z_n(x)| \right\} \left\{ \sup_{A_n \leqslant W_i \leqslant C_n,\, x \in [0,T]} M_i(x) \right\}, \tag{4.11}$$

where

$$Z_n(x) = \frac{1}{n\,a(n)} \sum_{A_n \leqslant W_i \leqslant C_n} I\{\delta_i = 1\}\, K((x-W_i)/a(n))),$$

$$M_i(x) = I\{|x - W_i| \leqslant a(n)\}\, |[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]^{-1} - [S_Y(x) - S_Z(x)]^{-1}|,$$

and $\mathrm{o}_\mathrm{p}^{(n)}(1)$ almost surely converges to 0 uniformly on $[0, T]$, as $n \to \infty$.

Note that for any $x \in [0, T]$,

$$M_i(x) \leqslant |[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]^{-1} - [S_Y(W_i) - S_Z(W_i)]^{-1}| + I\{|x - W_i| \leqslant a(n)\} |[S_Y(W_i) - S_Z(W_i)]^{-1} - [S_Y(x) - S_Z(x)]^{-1}| \leqslant |[S_Y^{(n)}(W_i) - S_Z^{(n)}(W_i)]^{-1} - [S_Y(W_i) - S_Z(W_i)]^{-1}| + M\,a(n), \tag{4.12}$$

where $M > 0$ is a constant. Hence, from (2.20), (2.21) and (B3), we have

$$\sup_{A_n \leqslant W_i \leqslant C_n,\, x \in [0, T]} M_i(x) \to 0 \quad \text{as } n \to \infty \tag{4.13}$$

with probability 1.

Note that

$$Z_n(x) = \frac{1}{a(n)} \int_0^\infty K\left(\frac{x - y}{a(n)}\right) d\tilde{F}_n(y) = [S_Y(x) - S_Z(x)] \tilde{f}_n(x), \tag{4.14}$$

and that $\tilde{F}_n$ is the empirical distribution function of

$$W(T) = W\, I\{\delta = 1\} + (W + T + 1) I\{\delta \neq 1\} \tag{4.15}$$

on $[0, T + 1]$. From Theorem A of Silverman (1978), we know that

$$Z_n(x) \to f(x) [S_Y(x) - S_Z(x)] \quad \text{as } n \to \infty, \tag{4.16}$$

$$\tilde{f}_n(x) \to f(x), \quad \text{as } n \to \infty \tag{4.17}$$

uniformly on $[0, T]$ with probability 1.

Hence, from (4.11)–(4.16), we have

$$\sup_{x \in [0, T]} |f_n(x) - \tilde{f}_n(x)| \to 0 \quad \text{as } n \to \infty \tag{4.18}$$

with probability 1. Therefore, (2.32) follows from (4.17) and (4.18).

(ii) From (C4) and Chang (1990), we know that $\sqrt{n}[S_Y^{(n)} - S_Y]$ and $\sqrt{n}[S_Z^{(n)} - S_Z]$ weakly converge to Gaussian processes on $[0, T]$ as $n \to \infty$. Hence, we have

$$\sqrt{n} \| S_Y^{(n)} - S_Y \|_T = O_p(1) \quad \text{as } n \to \infty$$

and

$$\sqrt{n} \| S_Z^{(n)} - S_Z \|_T = O_p(1) \quad \text{as } n \to \infty$$

where $\|\cdot\|_T$ denotes the supremum norm on $[0, T]$ and $O_p(1)$ is bounded in probability. Therefore, in (4.12) we have that for any $W_i \leqslant T + 1$,

$$M_i(x) \leqslant \frac{1}{\sqrt{n}} O_p(1) + M\,a(n), \tag{4.19}$$

which implies

$$\sqrt{n\,a(n)} \sup_{A_n \leqslant W_i \leqslant C_n} M_i(x) \leqslant \sqrt{a(n)}\,\mathrm{O_p}(1) + M\sqrt{n\,a^3(n)} \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty. \tag{4.20}$$

Note that for a single point x, (4.11) becomes

$$|f_n(x) - \tilde{f}_n(x)| \leqslant \mathrm{O_p}(1)/[n\,a(n)] + |Z_n(x)| \sup_{A_n \leqslant W_i \leqslant C_n} M_i(x). \tag{4.21}$$

From Rosenblatt (1971), we know

$$E\{Z_n(x) - f(x)\,[S_Y(x) - S_Z(x)]\}^2 \to 0 \quad \text{as } n \to \infty. \tag{4.22}$$

Hence, (4.20)–(4.22) imply

$$\sqrt{n\,a(n)}\,|f_n(x) - \tilde{f}_n(x)| \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty. \tag{4.23}$$

Rosenblatt (1971) showed that as $n \to \infty$,

$$\sqrt{n\,a(n)}\,(Z_n(x) - f(x)\,[S_Y(x) - S_Z(x)]) \xrightarrow{\mathrm{D}} N\left(0, f(x)\,[S_Y(x) - S_Z(x)] \int K^2(u)\,\mathrm{d}u\right), \tag{4.24}$$

if $n\,a^5(n) \to 0$, as $n \to \infty$. Therefore, (2.35) follows from (4.23), (4.14) and (4.24). □

Appendix

Proof of (2.15). Note that a solution $S_X^{(n)}$ of (2.5) is self-consistent and is a nonincreasing function with values in [0, 1]. From (2.8) and (2.13), we have that

$$Q^{(n)}(t) = S_Y^{(n)}(t) = 1 - [\text{the number of } W_i\text{'s} \leqslant t \text{ with } \delta_i = 2]/n$$

for any $t < A_n$, because $S_X^{(n)}(t) = 1$. This implies

$$[\text{the number of } W_i\text{'s} \leqslant t] = [\text{the number of } W_i\text{'s} \leqslant t \text{ with } \delta_i = 2].$$

Hence, we have $\delta_i = 2$, if $W_i < A_n$. Similarly, from (2.8) and (2.14), we have that for any $t \geqslant B_n$,

$$Q^{(n)}(t) = S_Z^{(n)}(t) = [\text{the number of } W_i\text{'s} > t \text{ with } \delta_i = 3]/n$$

because $S_Z^{(n)}(t) = 0$, so that

$$[\text{the number of } W_i\text{'s} > t] = [\text{the number of } W_i\text{'s} > t \text{ with } \delta_i = 3].$$

Hence, we have $\delta_i = 3$, if $W_i > B_n$. □

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