INVERTIBILITY OF THE GABOR FRAME OPERATOR ON THE WIENER AMALGAM SPACE

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Abstract. We use a generalization of Wiener's $1/f$ theorem to prove that for a Gabor frame with the generator in the Wiener amalgam space $W(L^\infty, \ell^1)(\mathbb{R}^d)$, the corresponding frame operator is invertible on this space. Therefore, for such a Gabor frame, the generator of the canonical dual belongs also to $W(L^\infty, \ell^1)(\mathbb{R}^d)$.

1. Introduction

For $\alpha, \beta > 0$ and $g \in L^2(\mathbb{R}^d)$, let $M_{\beta m} g(x) = e^{-2\pi i \beta m \cdot x} g(x)$ and $T_{\alpha n} g(x) = g(x - \alpha n)$. The collection $\mathcal{G}(g, \alpha, \beta) = \{g_{m,n} = M_{\beta m} T_{\alpha n} g,\ m, n \in \mathbb{Z}^d\} \subset L^2(\mathbb{R}^d)$ is called a Gabor frame if there exist positive constants $0 < A \le B < \infty$ such that for each $f \in L^2(\mathbb{R}^d)$ the following inequalities hold:

$$A\, \|f\|_{L^2}^2 \le \sum_{m,n \in \mathbb{Z}^d} |\langle f, g_{m,n}\rangle|^2 \le B\, \|f\|_{L^2}^2. \tag{1}$$

Equivalently, the frame condition can be restated in terms of the *frame operator* $S_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ associated to $\mathcal{G}(g, \alpha, \beta)$ and given by

$$S_g f = \sum_{m,n \in \mathbb{Z}^d} \langle f, g_{m,n}\rangle g_{m,n}, \quad f \in L^2(\mathbb{R}^d). \tag{2}$$

In particular, $\mathcal{G}(g, \alpha, \beta)$ is a frame for L^2 if and only if

$$A\|f\|_{L^2}^2 \le \langle S_g f, f\rangle \le B\|f\|_{L^2}^2, \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

When $\mathcal{G}(g, \alpha, \beta)$ is a frame for L^2 then with $\tilde{g} = S_g^{-1} g$ the following reconstruction formulas hold

$$f = \sum_{m,n \in \mathbb{Z}^d} \langle f, \tilde{g}_{m,n}\rangle g_{m,n} = \sum_{m,n \in \mathbb{Z}^d} \langle f, g_{m,n}\rangle \tilde{g}_{m,n}.$$

Moreover, $\mathcal{G}(\tilde{g}, \alpha, \beta)$ is also a Gabor frame for L^2 called the *canonical dual* (Gabor) frame. We refer to [12, 19, 21] for more on Gabor analysis.

A central question in Gabor analysis is to find conditions on $\alpha, \beta > 0$ and $g \in L^2$ such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2. Moreover, in many applications it is desirable to find a Gabor frame such that the generator g and its canonical dual $\tilde{g}$ have the same properties, e.g., same type of decay and/or smoothness. For example, it was proved that if $g \in \mathcal{S}(\mathbb{R}^d)$ the Schwartz class, and $\alpha, \beta > 0$ are such that

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$\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame, then $\tilde{g} \in \mathcal{S}$ [23]. Similarly, let $\phi(x) = e^{-x^2}$ and define the short time Fourier transform of a tempered distribution $f \in \mathcal{S}'$ by

$$V_\phi f(x, \omega) = \langle f, M_\omega T_x \phi \rangle = \int_{\mathbb{R}^d} f(t)\, \overline{\phi(t-x)}\, e^{-2\pi i x \cdot \omega}\, dt.$$

The space $\mathcal{M}^1(\mathbb{R}^d)$ of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathcal{M}^1} = \iint_{\mathbb{R}^{2d}} |V_\phi f(x, \omega)|\, dx\, d\omega < \infty$$

is known as the Feichtinger algebra [15]. In this context, Gröchenig and Leinart proved a deep result that shows that if $g \in \mathcal{M}^1(\mathbb{R}^d)$ and $\alpha, \beta > 0$ are such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame, then $\tilde{g} \in \mathcal{M}^1$ [18]. More specifically, using the so-called Janssen's representation of the Gabor frame operator, which converges absolutely in $\mathcal{B}(L^2(\mathbb{R}^d))$ whenever $g \in \mathcal{M}^1$, Gröchenig and Leinart, recasted the question into a non-commutative version of the celebrated $1/f$ Wiener's lemma ([33]) involving the twisted convolution. We refer to [16] for background on the twisted convolution.

Note that similarly to $\mathcal{S}$, $\mathcal{M}^1$ is also invariant under the Fourier transform [15, 19]. Moreover, it is trivially seen that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame if and only if $\mathcal{G}(\hat{g}, \beta, \alpha)$ is a Gabor frame. Therefore, the above results simply say that the generator of a Gabor frame and its canonical dual have the same time-frequency concentration. Furthermore, results involving time or frequency only conditions were proved in [3, 4]. Finally, we refer to [9, 11] for some related results.

In this paper, we prove a similar result that shows that if a Gabor frame $\mathcal{G}(g, \alpha, \beta)$ is generated by $g \in W(L^\infty, \ell^1_\nu)$ then the generator of its canonical dual $\tilde{g} \in W(L^\infty, \ell^1_\nu)$, where ν is an admissible weight (see Definition 2.2) and the space $W(L^\infty, \ell^1_\nu)$ is a weighted Wiener amalgam space. Some particular cases of our results were first obtained by Walnut with some extra conditions [31]. The unweighted amalgam space $W(L^\infty, \ell^1)$ was introduced by N. Wiener in connection with the Tauberian Theorems [33]. The precise definition of $W(L^\infty, \ell^1_\nu)$ is given as follows.

For $\alpha > 0$ let $Q_\alpha = [0, \alpha)^d$ and χ_{Q_α} be the characteristic function of Q_α. Let also $\nu : \mathbb{Z}^d \to [1, \infty)$ be a weight function. A function $f \in W(L^\infty, \ell^1_\nu)$ if and only if

$$\|f\|_{W(L^\infty, \ell^1_\nu)} = \sum_{n \in \mathbb{Z}^d} \|f \cdot T_{n\alpha} \chi_{Q_\alpha}\|_{L^\infty} \nu(n) < \infty. \tag{3}$$

Moreover, equipped with this norm $W(L^\infty, \ell^1_\nu)$ is a Banach space, whose definition is independent of α in the sense of equivalent norm. Furthermore, the following embeddings can be easily established: $\mathcal{S} \subset \mathcal{M}^1 \subset W(L^\infty, \ell^1_\nu) \subset W(L^\infty, \ell^1) \subset L^2$.

We wish to point out that, the condition $g \in W(L^\infty, \ell^1)$ is not enough for the Gabor frame operator to admit an absolutely convergent Janssen's representation ([19, Section 7.2]). Therefore, our result does not follow from [18]. To prove our result, we rely instead on another representation of the Gabor frame operator: the Walnut's representation [31]. In particular, our proof is derived using this representation of the frame operator together with a far-reaching generalization of the $1/f$ Wiener's lemma due to Baskakov [7, 8]. This particular extension of the lemma turns out to be the most suitable for us among it's numerous analogs, see [1, 6, 17, 18, 22, 24, 28, 29, 30], etc. Another analog that is suitable for us and pertains to the almost periodic situation will appear in [2]. We observe that Kurbatov [25, 26] seems to be the first to use this type of result in the context of amalgam spaces. We also refer to [14] for relevant results.

Our paper is organized as follows. In Section 2 we state the precise version of the $1/f$ Wiener's lemma that is suited to our result. Moreover, we introduce the two main tools used in proving our result: the Walnut's representation of the Gabor frame operator, and the bracket product. In Section 3 we state and prove our main result and, furthermore, outline a second and different proof. The second approach, however, relies on a conjecture that we have not yet been able to prove.

2. Preliminaries

2.1. **Wiener's lemma for Fourier series of operators.** In this section we present a reformulation of a non-commutative Wiener's lemma proved by Baskakov in [7, 8]. We begin by introducing a notion of a Fourier series of a linear operator with respect to a representation of a compact Abelian group.

Although the results of this section hold for an arbitrary compact Abelian group $\mathbb{G}$, we restrict our attention to $\mathbb{G} = \mathbb{T}^d_{1/\beta} \simeq Q_{1/\beta} = [0, 1/\beta)^d$, which is specifically tailored for our application. We use an additive form for the group operation on $\mathbb{G}$. For an isometric strongly continuous representation $U : \mathbb{G} \to \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the algebra of all bounded linear operators on a (complex) Banach space X, we define $\tilde{U} : \mathbb{G} \to \mathcal{B}(\mathcal{B}(X))$ via

$$\tilde{U}(\theta)A = U(\theta)AU(-\theta), \quad \theta \in \mathbb{G},\ A \in \mathcal{B}(X).$$

Following [7], the Fourier series of an operator $A \in \mathcal{B}(X)$ with respect to the representation U is, by definition, the Fourier series of the function $\hat{A} : \mathbb{G} \to \mathcal{B}(X)$ given by

$$\hat{A}(\theta) = \tilde{U}(\theta)A, \quad \theta \in \mathbb{G}.$$

Recall that this Fourier series is

$$\hat{A}(\theta)f \simeq \sum_{k\in\mathbb{Z}^d} e^{2\pi i\beta\theta\cdot k} A_k f, \quad f \in X, \tag{4}$$

and the Fourier coefficients $A_k \in \mathcal{B}(X)$ are given by

$$A_k f = \beta^d \int_{Q_{1/\beta}} e^{-2\pi i\beta\theta\cdot k} \hat{A}(\theta) f d\theta = \int_{[0,1)^d} e^{-2\pi i\theta\cdot k} \hat{A}(\frac{\theta}{\beta}) f d\theta. \tag{5}$$

Observe that the Fourier coefficients are eigen-vectors of the corresponding representation, i.e.

$$\tilde{U}(\theta)A_k = e^{2\pi i\beta\theta\cdot k} A_k,\ k \in \mathbb{Z}^d. \tag{6}$$

Example 2.1. We are especially interested in the case when $X = \mathcal{H}$ is the Hilbert space $L^2(\mathbb{R}^d)$ and the representation $U : \mathbb{G} \to \mathcal{B}(\mathcal{H})$ is defined by

$$U(\theta)f(x) = M_{\beta\theta}f(x) = e^{2\pi i\beta\theta\cdot x} f(x), \quad f \in \mathcal{H}.$$

Then the operators $T_{\frac{n}{\beta}}$, $n \in \mathbb{Z}^d$, are eigen-vectors for the representation $\tilde{U}$:

$$\tilde{U} T_{\frac{n}{\beta}} f(x) = U(\theta) T_{\frac{n}{\beta}} U(-\theta) f(x) = e^{2\pi i \beta \theta \cdot n} f(x - \frac{n}{\beta}), \quad f \in \mathcal{H},\ n \in \mathbb{Z}^d.$$

This implies that any operator $A \in \mathcal{B}(\mathcal{H})$ has Fourier coefficients of the form

$$A_n = \mathfrak{G}_n T_{\frac{n}{\beta}}, \quad n \in \mathbb{Z}^d,$$

where $\mathfrak{G}_n$ commute with $M_{\beta\theta}$ for all $\theta \in Q_{1/\beta}$. Therefore, and following [25], $\mathfrak{G}_n$ is an operator of multiplication by a uniquely determined function $G_n \in L^\infty(\mathbb{R}^d)$ and

$$\|A_n\| = \|\mathfrak{G}_n\| = \|G_n\|_{L^\infty}, \quad n \in \mathbb{Z}^d.$$

Observe that this construction remains partly valid when $\mathcal{H}$ is a closed subspace of the Hilbert space $L^2(\mathbb{R}^d)$ invariant with respect to modulations M_θ, $\theta \in \mathbb{R}$, and translations $T_{\frac{n}{\beta}}$, $n \in \mathbb{Z}^d$. In this case, however, we can no longer infer that the operators $\mathfrak{G}_n$, $n \in \mathbb{Z}^d$, *uniquely* determine a function $G_n \in L^\infty(\mathbb{R}^d)$ such that $(\mathfrak{G}_n f)(x) = G_n(x) f(x)$. Hence, we can only guarantee that

$$\|A_n\| = \|\mathfrak{G}_n\| \le \|G_n\|_{L^\infty}, \quad n \in \mathbb{Z}^d.$$

As usual when Wiener's lemma is discussed, we are interested in linear operators whose Fourier series are summable or summable with a weight.

Definition 2.2. An *admissible weight* is a function $\nu : \mathbb{Z}^d \to [1, \infty)$ such that

(1) ν is an even function, that is, $\nu(-n) = \nu(n)$, for all $n \in \mathbb{Z}^d$,
(2) $\nu(k+n) \le \nu(k)\nu(n)$, for all $k, n \in \mathbb{Z}^d$, and
(3) $\lim\limits_{k\to\infty} k^{-1} \ln \nu(kn) = 0$, for all $n \in \mathbb{Z}^d$.

For an admissible weight ν, we consider a Banach algebra

$$\mathcal{B}_\nu(X) = \{A \in \mathcal{B}(X) : \sum_{k\in\mathbb{Z}^d} \|A_k\| \, \nu(k) < \infty\}$$

of linear operators with ν-summable Fourier series. If $\nu \equiv 1$ we get the algebra $\mathcal{B}_1$ of operators with summable Fourier series.

The result below follows immediately from [7, Theorem 2 and Remark] or [2].

Theorem 2.3. *Let ν be an admissible weight and $A \in \mathcal{B}_\nu(X)$ be an invertible operator. Then $A^{-1} \in \mathcal{B}_\nu(X)$. In particular, if A is invertible and $A \in \mathcal{B}_1$, then $A^{-1} \in \mathcal{B}_1$.*

The next corollary, which follows immediately from Theorem 2.3 and Example 2.1, plays a key role in establishing our main result.

Corollary 2.4. *Let $\beta > 0$, ν be an admissible weight, and $\mathcal{H} = L^2(\mathbb{R}^d)$. Assume that an invertible operator $S \in \mathcal{B}(\mathcal{H})$ has a ν-summable Fourier series, that is,*

$$S = \sum_{n\in\mathbb{Z}^d} \mathfrak{G}_n T_{\frac{n}{\beta}} = \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}},$$

where $\mathfrak{G}_n \in \mathcal{B}(\mathcal{H})$ is the operator of multiplication by a function $G_n \in L^\infty(\mathbb{R}^d)$ and $\sum\limits_{n\in\mathbb{Z}^d} \|\mathfrak{G}_n T_{\frac{n}{\beta}}\| \nu(n) = \sum\limits_{n\in\mathbb{Z}^d} \|G_n\|_{L^\infty} \nu(n) < \infty$. Then the inverse operator $S^{-1} \in \mathcal{B}(\mathcal{H})$

has also a ν-summable Fourier series, that is, there exists a sequence of functions $\tilde{G}_n \in L^\infty(\mathbb{R}^d)$ such that

$$S^{-1} = \sum_{n\in\mathbb{Z}^d} \tilde{\mathfrak{G}}_n T_{\frac{n}{\beta}} = \sum_{n\in\mathbb{Z}^d} \tilde{G}_n \cdot T_{\frac{n}{\beta}}, \ \textit{and} \ \sum_{n\in\mathbb{Z}^d} \|\tilde{G}_n\|_{L^\infty}\nu(n) < \infty.$$

The following result is a different version of Corollary 2.4 that deals with operators defined on closed subspaces of L^2.

Corollary 2.5. *Let $\alpha > 0$, ν be an admissible weight, and $\mathcal{H}$ be a closed subspace of the Hilbert space $L^2(\mathbb{R}^d)$ invariant with respect to modulations M_θ, $\theta \in \mathbb{R}$, and translations $T_{\alpha n}$, $n \in \mathbb{Z}^d$. Assume that an invertible operator $S \in \mathcal{B}(\mathcal{H})$ has a ν-summable Fourier series, that is,*

$$S = \sum_{n\in\mathbb{Z}^d} \mathfrak{G}_n T_{\alpha n} \quad \textit{and} \quad \sum_{n\in\mathbb{Z}^d} \|\mathfrak{G}_n\|\nu(n) < \infty,$$

where the operators $\mathfrak{G}_n \in \mathcal{B}(\mathcal{H})$ commute with $M_{\frac{\theta}{\alpha}}$ for all $\theta \in Q_\alpha$. Then the inverse operator $S^{-1} \in \mathcal{B}(\mathcal{H})$ also has a ν-summable Fourier series, that is, there exists a sequence of operators $\tilde{\mathfrak{G}}_n \in \mathcal{B}(\mathcal{H})$ commuting with $M_{\frac{\theta}{\alpha}}$ for all $\theta \in Q_\alpha$ and such that

$$S^{-1} = \sum_{n\in\mathbb{Z}^d} \tilde{\mathfrak{G}}_n T_{\alpha n} \quad \textit{and} \quad \sum_{n\in\mathbb{Z}^d} \|\tilde{\mathfrak{G}}_n\|\nu(n) < \infty.$$

Remark 2.6. Observe that in Corollary 2.5, even though $\mathfrak{G}_n, \tilde{\mathfrak{G}}_n \in \mathcal{B}(\mathcal{H})$ are operators of multiplication by functions $G_n, \tilde{G}_n \in L^\infty(\mathbb{R}^d)$, respectively, we can no longer guarantee that

$$\sum_{n\in\mathbb{Z}^d} \|G_n\|_{L^\infty}\nu(n) < \infty \text{ and } \sum_{n\in\mathbb{Z}^d} \|\tilde{G}_n\|_{L^\infty}\nu(n) < \infty.$$

In the following theorem we use Corollary 2.4 to establish the boundedness of operators $S \in \mathcal{B}_\nu(\mathcal{H})$ on the Wiener amalgam space $W(L^\infty, \ell^1_\nu)$.

Theorem 2.7. *Let $\beta > 0$, ν be an admissible weight, and $\mathcal{H} = L^2(\mathbb{R}^d)$. Assume that $S = \sum_{n\in\mathbb{Z}^d} \mathfrak{G}_n T_{\frac{n}{\beta}} \in \mathcal{B}_\nu(\mathcal{H})$ has a ν-summable Fourier series. Then S defines a bounded operator from $W(L^\infty, \ell^1_\nu)$ to $W(L^\infty, \ell^1_\nu)$.*

Proof. Since $S \in \mathcal{B}_\nu$,

$$S = \sum_{n\in\mathbb{Z}^d} \mathfrak{G}_n T_{\frac{n}{\beta}} = \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}},$$

where $\mathfrak{G}_n \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{Z}^d$, are the operators of multiplication by functions $G_n \in L^\infty(\mathbb{R}^d)$ and $\sum\limits_{n\in\mathbb{Z}^d} \|\mathfrak{G}_n T_{\frac{n}{\beta}}\|\nu(n) = \sum\limits_{n\in\mathbb{Z}^d} \|G_n\|_{L^\infty}\nu(n) < \infty$. Hence,

$$\begin{aligned}
\|Sf\|_{W(L^\infty,\ell^1_\nu)} &= \left\| \sum_{n\in\mathbb{Z}^d} (T_{n/\beta} f) G_n \right\|_{W(L^\infty,\ell^1_\nu)} \\
&= \sum_{k\in\mathbb{Z}^d} \left\| \sum_{n\in\mathbb{Z}^d} \chi_{k+Q_{1/\beta}} (T_{n/\beta} f) G_n \right\|_\infty \nu(k) \\
&\le \sum_{k,n\in\mathbb{Z}^d} \left\| \chi_{k+Q_{1/\beta}} (T_{n/\beta} f) \right\|_\infty \nu(k) \, \|G_n\|_\infty \\
&\le \sum_{k,n\in\mathbb{Z}^d} \left\| \chi_{n+k+Q_{1/\beta}} f \right\|_\infty \nu(n+k) \, \|G_n\|_\infty \, \nu(n) \\
&\le \|f\|_{W(L^\infty,\ell^1_\nu)} \sum_{n\in\mathbb{Z}^d} \|G_n\|_\infty \, \nu(n) \le \infty,
\end{aligned}$$

and the proof is complete. □

2.2. **Bracket product and Walnut representation.** Unless stated otherwise, in all that follows we assume that $\alpha, \beta > 0$, ν is an admissible weight, $g \in W(L^\infty, \ell^1_\nu)$ is such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2, and $\tilde{g} = S_g^{-1} g \in L^2$ is the generator of the canonical dual frame.

Let us recall a few properties of the bracket product widely used in the study of shift invariant systems [5, 10, 27].

For $f, h \in L^2(\mathbb{R}^d)$ and $\alpha > 0$ the α-*bracket product* of f and h is the $\alpha-$periodic function, which is a periodization of $f \cdot \bar{h} \in L^1(\mathbb{R}^d)$:

$$[f, h]_\alpha(x) = \sum_{k\in\mathbb{Z}^d} (f \cdot \overline{h})(x - \alpha k), \tag{7}$$

$x \in Q_\alpha = [0, \alpha)^d$. Note that the series in (7) converges for $a.e.\, x \in Q_\alpha$. Observe also that the (formal) Fourier series of this periodic function is

$$[f, h]_\alpha(x) \simeq \alpha^{-d} \sum_{n\in\mathbb{Z}^d} \langle f, M_{\frac{n}{\alpha}} h \rangle e^{\frac{2\pi i n \cdot x}{\alpha}}. \tag{8}$$

For $g \in W(L^\infty, \ell^1_\nu)$, let

$$G_n(x) = [g, T_{\frac{n}{\beta}} g]_\alpha(x) = \sum_{k\in\mathbb{Z}^d} (g \cdot T_{\frac{n}{\beta}} \overline{g})(x - \alpha k). \tag{9}$$

For the dual generator $\tilde{g} = S_g^{-1} g$ we let $\tilde{G}_n$ be

$$\tilde{G}_n(x) = [\tilde{g}, T_{\frac{n}{\beta}} \tilde{g}]_\alpha(x) = \sum_{k\in\mathbb{Z}^d} (\tilde{g} \cdot T_{\frac{n}{\beta}} \overline{\tilde{g}})(x - \alpha k), \tag{10}$$

which is well-defined for $a.e.\, x \in Q_\alpha$ since $\tilde{g} \cdot T_{\frac{n}{\beta}} \overline{\tilde{g}} \in L^1$.

It can be shown [19, Lemma 6.3.1], see also [20, Lemma 5.2] and [31, Lemma 2.1], that if $g \in W(L^\infty, \ell^1_\nu)$, there exists a constant C which depends only on α, β and d such that

$$\sum_{n\in\mathbb{Z}^d} \|G_n\|_{L^\infty(Q_\alpha)}\nu(n) \le C\|g\|^2_{W(L^\infty,\ell^1_\nu)} < \infty. \tag{11}$$

and

$$\sum_{n\in\mathbb{Z}^d} \left\|[g, T_{\alpha n} g]_{1/\beta}\right\|_{L^\infty(Q_{1/\beta})} \nu(n) \le C\|g\|^2_{W(L^\infty,\ell^1_\nu)} < \infty. \tag{12}$$

In Lemma 2.8 below, we shall present a converse to the above statement that will play a key role in obtaining our main result. To prove it we introduce the *Walnut representation* of the Gabor frame operators S_g and $S_{\tilde g}$. Following [19, Proposition 7.1.1], we have

$$\langle S_g f,\ h\rangle = \langle \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}} f,\ h\rangle, \tag{13}$$

$$\langle S_{\tilde g} f,\ h\rangle = \langle \beta^{-d} \sum_{n\in\mathbb{Z}^d} \tilde G_n \cdot T_{\frac{n}{\beta}} f,\ h\rangle, \tag{14}$$

for all f, h bounded and compactly supported. In fact, because $g \in W(L^\infty, \ell^1_\nu)$ the operator S_g has a *strong* Walnut representation [19, Theorem 6.3.2]

$$S_g f = \beta^{-d} \sum_{n\in\mathbb{Z}^d} \mathfrak{G}_n T_{\frac{n}{\beta}} f = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}} f, \quad f \in L^2(\mathbb{R}^d), \tag{15}$$

where $\mathfrak{G}_n \in \mathcal{B}(L^2(\mathbb{R}^d))$ is the operator corresponding to multiplication by the bounded function G_n given in (9). Moreover, observe that $\|\mathfrak{G}_n\| = \|G_n\|_{L^\infty(Q_\alpha)}$ and, therefore, (11) implies $\sum_n \|\mathfrak{G}_n\|_{op}\nu(n) = \sum_n \|G_n\|_{L^\infty(Q_\alpha)}\nu(n) < \infty$.

By Example 2.1 and the above inequality, the Walnut representation (15) is the Fourier series of $S_g \in \mathcal{B}_\nu(L^2(\mathbb{R}^d))$ with respect to the representation

$$U(\theta) f(x) = e^{2\pi i\beta\theta\cdot x} f(x), \quad \theta \in Q_{1/\beta},\ f \in L^2(\mathbb{R}^d).$$

Similarly, the weak Walnut representation (14) implies that

$$S_{\tilde g} \simeq \beta^{-d} \sum_{n\in\mathbb{Z}^d} \widetilde{\mathfrak{G}}_n T_{\frac{n}{\beta}},$$

where $\widetilde{\mathfrak{G}}_n$ is the operator of multiplication by $\tilde G_n$, $n \in \mathbb{Z}^d$, is the Fourier series of $S_{\tilde g}$.

Lemma 2.8. *Assume that $g \in L^2(\mathbb{R}^d)$ is such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2 and*

$$\sum_{n\in\mathbb{Z}^d} \|G_n\|_{L^\infty(Q_\alpha)}\nu(n) < \infty. \tag{16}$$

Then the frame operator S_g is a bounded operator from $W(L^\infty, \ell^1_\nu)$ to $W(L^\infty, \ell^1_\nu)$.

Proof. Since (16) implies $S_g \in \mathcal{B}_\nu$, the result follows immediately from Theorem 2.7. $\square$

Remark 2.9. For a different proof of Lemma 2.8 we refer to [20], and [31, Theorem 3.1]. A similar result is also proved in [9, Theorem 7.2]. Following that proof it can be shown that condition (16) in Lemma 2.8 is not only sufficient but also necessary.

3. Main results

We are now ready to prove our main result.

Theorem 3.1. *Let $\alpha, \beta > 0$, $g \in W(L^\infty, \ell^1_\nu)$, be such that $\mathcal{G}(g,\alpha,\beta)$ and $\mathcal{G}(\tilde{g},\alpha,\beta)$ be canonical dual Gabor frames for $L^2(\mathbb{R}^d)$, where $\tilde{g} = S_g^{-1}g$. Then $\tilde{g} \in W(L^\infty, \ell^1_\nu)$.*

Proof. As mentioned above, since $g \in W(L^\infty, \ell^1_\nu)$, the Gabor frame operator S_g has a ν-summable Fourier series (15). By Corollary 2.4, its inverse $S_g^{-1} = S_{\tilde{g}}$, [19, Lemma 5.1.6], also has a ν-summable Fourier series, i.e.,

$$S_{\tilde{g}} = \beta^{-d} \sum_{n\in\mathbb{Z}^d} \widetilde{\mathfrak{G}}_n T_{\frac{n}{\beta}},$$

and

$$\sum_{n\in\mathbb{Z}^d} \|\tilde{G}_n\|_{L^\infty(Q_\alpha)}\nu(n) < \infty.$$

It remains to apply Lemma 2.8 to conclude that $\tilde{g} \in W(L^\infty, \ell^1_\nu)$. $\square$

Remark 3.2. It is known that for a Gabor frame $\mathcal{G}(g,\alpha,\beta)$ of $L^2(\mathbb{R}^d)$ the system $\mathcal{G}(\gamma,\alpha,\beta)$ is a dual frame if and only if $\gamma = \tilde{g} + h$ where $\tilde{g}$ is the canonical dual, and $h \in L^2(\mathbb{R}^d)$ is such that $\langle h, M_{\frac{n}{\alpha}} T_{\frac{m}{\beta}} g\rangle = 0$ for all $m, n \in \mathbb{Z}^d$, e.g., see, [19, Lemma 7.6.1], and [32]. Therefore, in view of Theorem 3.1, one can ask whether for a given Gabor frame $\mathcal{G}(g,\alpha,\beta)$ with $g \in W(L^\infty, \ell^1)$, all the dual frames belong to the same space. In general, the answer to this question is no, as shown by the following example.

Let $g = \chi_{[0,1]}$ be the generator of the Gabor frame $\mathcal{G}(g, 1/2, 1)$ and $\tilde{g}$ be the generator of its canonical dual. For an arbitrary sequence $(a_k) \in \ell^2\backslash\ell^1$, define $h \in L^2$ via

$$h(x) = \sum_{k\in\mathbb{Z}} a_k \chi_{[k,k+1)}(x) e^{2\pi i x}.$$

Then $\langle h, M_{2m}T_n g\rangle = 0$ for all $m, n \in \mathbb{Z}$ and, therefore, $\tilde{g} + h$ is a dual generator for g. However, by construction, $h \notin W(L^\infty, \ell^1)$, and hence $\tilde{g} + h \notin W(L^\infty, \ell^1)$.

We recall that if $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, then $\mathcal{G}(g^\dagger,\alpha,\beta)$ is a tight frame, where $g^\dagger = S_g^{-1/2}g$. Indeed, for all $f \in L^2(\mathbb{R}^d)$, we have

$$f = S_g^{-1/2} S_g S_g^{-1/2} f = \sum_{k,l} \langle f, g^\dagger_{k,l}\rangle g^\dagger_{k,l}.$$

The next result proves that if $g \in W(L^\infty, \ell^1_\nu)$, then $g^\dagger \in W(L^\infty, \ell^1_\nu)$. More specifically we have

Corollary 3.3. *Let $\alpha, \beta > 0$ and $g \in W(L^\infty, \ell^1_\nu)$ be such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Then $g^\dagger \in W(L^\infty, \ell^1_\nu)$.*

Proof. Let $g \in W(L^\infty, \ell^1_\nu)$ and $\alpha, \beta > 0$ be such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Note that $S_g \in \mathcal{B}_\nu(L^2(\mathbb{R}^d))$ is a positive definite operator on $L^2(\mathbb{R}^d)$. Therefore, we can use the Riesz-Dunford functional calculus [13, Ch. VII] to get

$$S_g^{-1/2} = \frac{1}{2\pi i}\int_\Gamma \lambda^{-1/2}(S_g - \lambda I)^{-1} d\lambda,$$

where Γ is a positively oriented contour in the right complex half-plane surrounding the spectrum of S_g. By Corollary 2.4 the above integral converges in the norm of $\mathcal{B}_\nu$ and we get $S_g^{-1/2} \in \mathcal{B}_\nu(L^2(\mathbb{R}^d))$. Consequently, Theorem 2.7 can be used to conclude that $g^\dagger = S_g^{-1/2} g \in W(L^\infty, \ell^1_\nu)$. $\square$

Remark 3.4. As mentioned in the Introduction, Theorem 3.1 and Corollary 3.3 were first proved under extra assumptions on g, α and β in [31, Corollary 3.5].

We wish to conclude by outlining a *possible* alternative approach to the proof of Theorem 3.1 which does not use Lemma 2.8. Instead, this proof relies upon the following propositions, the first of which, to our knowledge has not been proved before despite its simplicity.

Proposition 3.5. *Let $g \in W(L^\infty, \ell^1_\nu)$ and $\mathcal{G}(g,\alpha,\beta)$ and $\mathcal{G}(\tilde{g},\alpha,\beta)$ be canonical dual Gabor frames for $L^2(\mathbb{R}^d)$. Then for almost every $x \in Q_{\frac{1}{\beta}}$ and all $k \in \mathbb{Z}^d$*

$$[\tilde{g}, T_{\alpha k} g]_{\frac{1}{\beta}}(x) = \beta^{-d} \sum_{n\in\mathbb{Z}^d} \overline{[g, T_{\alpha n} g]_{\frac{1}{\beta}}}(x - \alpha k)\, [\tilde{g}, T_{\alpha(k+n)}\tilde{g}]_{\frac{1}{\beta}}(x) \tag{17}$$

and we have the following norm estimates

$$\sum_{k\in\mathbb{Z}^d} \|[\tilde{g}, T_{\alpha k} g]_{\frac{1}{\beta}}\|_{L^\infty}\nu(k) \le \beta^{-d} \sum_{n\in\mathbb{Z}^d} \|[g, T_{\alpha n} g]_{\frac{1}{\beta}}\|_{L^\infty}\nu(n) \sum_{n\in\mathbb{Z}^d} \|[\tilde{g}, T_{\alpha n}\tilde{g}]_{\frac{1}{\beta}}\|_{L^\infty}\nu(n), \tag{18}$$

whenever the right-hand side is finite.

Remark 3.6. From Theorem 3.1 and (12), we know that the right-hand side of (18) is always finite under assumptions of Proposition 3.5. However, we cannot use this fact if we want to give an alternative proof of Theorem 3.1.

Proof. Because $\mathcal{G}(g,\alpha,\beta)$ and $\mathcal{G}(\tilde{g},\alpha,\beta)$ are dual frames we have

$$g = \sum_{r,s\in\mathbb{Z}^d} \left\langle g,\ M_{\beta s} T_{\alpha r} g\right\rangle M_{\beta s} T_{\alpha r}\tilde{g}$$

and so

$$T_{\alpha k} g = \sum_{r,s\in\mathbb{Z}^d} \left\langle g,\ M_{\beta s} T_{\alpha r} g\right\rangle e^{-2\pi i\alpha\beta s\cdot k}\, M_{\beta s} T_{\alpha(r+k)}\tilde{g}.$$

Therefore, for $a.e.\, x \in Q_{\frac{1}{\beta}}$

$$\begin{aligned}
[\tilde{g}, T_{\alpha k} g]_{\frac{1}{\beta}}(x) &= \sum_{n\in\mathbb{Z}^d} (\tilde{g}\cdot T_{\alpha k}\overline{g})(x-\tfrac{n}{\beta}) \\
&= \sum_{r,s\in\mathbb{Z}^d} \overline{\langle g,\ M_{\beta s}T_{\alpha r} g\rangle}\, e^{2\pi i\alpha\beta s\cdot k}\, e^{-2\pi i\beta s\cdot x}\, [\tilde{g}, T_{\alpha(k+r)}\tilde{g}]_{\frac{1}{\beta}}(x) \\
&= \sum_{r\in\mathbb{Z}^d} [\tilde{g}, T_{\alpha(k+r)}\tilde{g}]_{\frac{1}{\beta}}(x) \sum_{s\in\mathbb{Z}^d} \overline{\langle g,\ M_{\beta s}T_{\alpha r} g\rangle\, e^{2\pi i\beta s\cdot(x-\alpha k)}} \\
&= \beta^{-d} \sum_{r\in\mathbb{Z}^d} \overline{[g, T_{\alpha r} g]_{\frac{1}{\beta}}}(x-\alpha k)\, [\tilde{g}, T_{\alpha(k+r)}\tilde{g}]_{\frac{1}{\beta}}(x),
\end{aligned}$$

where the last equation follows from Carleson's theorem since for $g \in W(L^\infty, \ell^1)$ the Fourier series of $[g, T_{\alpha r} g]_{\frac{1}{\beta}}(x-\alpha k) = \beta^d \sum_{s\in\mathbb{Z}^d} \overline{\langle g,\ M_{\beta s}T_{\alpha r} g\rangle}\, e^{2\pi i\beta s\cdot(x-\alpha k)}$ with L^2 convergence. Consequently,

$$\|[\tilde{g}, T_{\alpha k} g]_{\frac{1}{\beta}}\|_{L^\infty} \le \beta^{-d} \sum_{r\in\mathbb{Z}^d} \|[g, T_{\alpha r} g]_{\frac{1}{\beta}}\|_{L^\infty}\, \|[\tilde{g}, T_{\alpha(k+r)}\tilde{g}]_{\frac{1}{\beta}}\|_{L^\infty},$$

$$\sum_{k\in\mathbb{Z}^d} \|[\tilde{g}, T_{\alpha k} g]_{\frac{1}{\beta}}\|_{L^\infty}\nu(k) \le \beta^{-d} \sum_{r\in\mathbb{Z}^d} \|[g, T_{\alpha r} g]_{\frac{1}{\beta}}\|_{L^\infty}\nu(r) \sum_{r\in\mathbb{Z}^d} \|[\tilde{g}, T_{\alpha r}\tilde{g}]_{\frac{1}{\beta}}\|_{L^\infty}\nu(r),$$

and the proof is complete. □

Proposition 3.7. *Let $g \in W(L^\infty, \ell^1_\nu)$ and $\alpha, \beta > 0$ be such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Assume that $\mathcal{G}(\tilde{g},\alpha,\beta)$ is its canonical dual frame. Then $\sum_{r\in\mathbb{Z}^d} \|[\tilde{g}, T_{\frac{r}{\beta}}\tilde{g}]_\alpha\|_{L^\infty}\nu(r) < \infty$.*

Proof. The result follows immediately from Corollary 2.4, just as in the proof of Theorem 3.1. □

Note that Proposition 3.7 implies that

$$\|[\tilde{g},\tilde{g}]_\alpha\|_{L^\infty} = \left\| \sum_{k\in\mathbb{Z}^d} |\tilde{g}(\cdot - \alpha k)|^2 \right\|_{L^\infty} \le B < \infty$$

and it follows that $\tilde{g} \in L^\infty(\mathbb{R}^d)$.

Conjecture 3.8. Let $\alpha, \beta > 0$ and $\tilde{g} \in L^2(\mathbb{R}^d)$ be such that

$$\sum_{r\in\mathbb{Z}^d} \left\| [\tilde{g}, T_{\frac{r}{\beta}}\tilde{g}]_\alpha \right\|_{L^\infty} \nu(r) < \infty.$$

We conjecture that in this case

$$\sum_{n\in\mathbb{Z}^d} \left\| [\tilde{g}, T_{\alpha n}\tilde{g}]_{\frac{1}{\beta}} \right\|_{L^\infty} \nu(n) < \infty.$$

Remark 2.6 indicates a major obstacle in proving Conjecture 3.8. However, if the conjecture is true, we can give the proof of Theorem 3.1 as follows.

Let $m_k(x) = \beta^{-d}[\tilde{g}, T_{\alpha k} g]_{\frac{1}{\beta}}(x)$, $k \in \mathbb{Z}^d$. These functions are well-defined because $g, \tilde{g} \in L^2(\mathbb{R}^d)$ and so $\tilde{g} \cdot T_{\alpha k}\overline{g} \in L^1(\mathbb{R}^d)$. Consequently, $m_k \in L^\infty(Q_{\frac{1}{\beta}})$ and is $\frac{1}{\beta}-$periodic. Moreover, $\hat{m}_k(l) = C_g \tilde{g}(\alpha k, \beta l)$, $k, l \in \mathbb{Z}^d$, are the Gabor coefficients of $\tilde{g}$ with respect to the frame $\mathcal{G}(g, \alpha, \beta)$ [20]. By Proposition 3.5 and Conjecture 3.8,

$$\sum_{k\in\mathbb{Z}^d} \|m_k\|_{L^\infty}\nu(k) \leq \beta^{-d} \sum_{n\in\mathbb{Z}^d} \|[g, T_{\alpha n} g]_{\frac{1}{\beta}}\|_{L^\infty}\nu(n) \sum_{n\in\mathbb{Z}^d} \|[\tilde{g}, T_{\alpha n}\tilde{g}]_{\frac{1}{\beta}}\|_{L^\infty}\nu(n) < \infty.$$

Hence, it follows from [20, Theorem 4.4] that $\tilde{g} \in W\left(L^\infty, \ell^1_\nu\right)$.

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References

[1] R. Balan, *A noncommutative Wiener Lemma and a faithful tracial state on Banach algebra of time-frequency operators,* to appear in Trans. Amer. Math. Soc. (2007).

[2] R. Balan and I. Krishtal, *An almost periodic noncommutative Wiener's Lemma and its applications to time-frequency analysis,* preprint.

[3] H. Bölcskei, *A necessary and sufficient condition for dual Weyl-Heisenberg frames to be compactly supported,* J. Fourier Anal. Appl. **5** (1999), no. 5, 409–419.

[4] H. Bölcskei and A. J. E. M. Janssen, *Gabor frames, unimodularity, and window decay,* J. Fourier Anal. Appl. **6** (2000), no. 3, 255–276.

[5] C. de Boor, R. DeVore, and A. Ron, *The structure of shift invariant spaces and applications to approximation theory,* J. Funct. Anal. **119** (1994), 37–78.

[6] A.G. Baskakov, *Wiener's theorem and asymptotic estimates for elements of inverse matrices,* Funct. Anal. Appl., **24**(1990), 222–224.

[7] A.G. Baskakov, *Asymptotic estimates for elements of matrices of inverse operators, and harmonic analysis,* (Russian) Sibirsk. Mat. Zh. **38** (1997), no. 1, 14–28; translation in Siberian Math. J. **38** (1997), no. 1, 10–22.

[8] A.G. Baskakov, Estimates for the elements of inverse matrices, and the spectral analysis of linear operators, (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **61** (1997), no. 6, 3–26; translation in *Izv Math.* **61** (1997), no. 6, 1113–1135.

[9] P. G. Casazza, O. Christensen, and A.J.E.M. Janssen, *Weil-Heisenberg frames, translation invariant systems and the Walnut representation*, J. Funct. Anal. **180** (2001), no. 1, 85–147.

[10] P. G. Casazza and M. C. Lammers, *Bracket products for Weyl-Heisenberg frames,* in: Advances in Gabor analysis, H. G. Feichtinger and T. Strohmer (eds), 71–98, Appl. Numer. Harmon. Anal., Birkhuser Boston, MA, 2003.

[11] O. Christensen, *Pairs of dual Gabor frame generators with compact support and desired frequency localization,* Appl. Comput. Harmon. Anal. **20** (2006), no. 3, 403–410.

[12] I. Daubechies, "Ten lectures on wavelets," Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[13] N. Dunford, and J.T. Schwartz, "Linear operators. Part I. General theory." Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.

[14] B. Farrell and T. Strohmer, *Wiener's Lemma for the Heisenberg Group and a Class of Pseudodifferential Operators,* preprint (2007).

[15] H. G. Feichtinger, *On a new Segal algebra,* Monatsh. Math. **92** (1981), no. 4, 269–289.

[16] G. B. Folland, "Harmonic Analysis in Phase Space," Princeton Univ. Press, Princeton, NJ, 1990.

[17] I. Gohberg, M. A. Kaashoek, H.J. Woerderman, *The band method for positive and strictly contractive extension problems: an alternative version and new applications, Integral Equations Operator Theory* , **12** (1989), no. 3, 343–382.

[18] K. Gröchenig and M. Leinert, *Wiener's lemma for twisted convolution and Gabor frames,* J. Amer. Math. Soc. **17** (2003), no. 1, 1–18.

[19] K. Gröchenig, "Foundations of Time-Frequency Analysis," Birkhäuser, Boston, 2001.

[20] K. Gröchenig, C. Heil, and K. Okoudjou, *Gabor analysis in weighted amalgam spaces,* Sampl. Theory Signal Image Process. **1** (2002), no. 3, 225–259.

[21] C. Heil and D. Walnut, *Continuous and discrete wavelet transforms,* SIAM Rev. **31** (1989), 628–666.

[22] S. Jaffard, *Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **7** (1990), no. 5, 461–476.

[23] A. J. E. M. Janssen, *Duality and biorthogonality for Weyl-Heisenberg frames,* J. Fourier Anal. Appl. **1** (1995), no. 4, 403–436.

[24] V.G. Kurbatov, *Algebras of difference and integral operators,* Funct. Anal. Appl., **24** (1990), no. 2, 156–158.

[25] V. G. Kurbatov, "Functional Differential Operators and Equations," volume 473 of Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, Boston, London, 1999.

[26] V. Kurbatov, *Some algebras of operators majorized by convolution,* Funct. Differ. Equ. **8** (2001), no. 3-4, 323–333. International Conference on Differential and Functional Differential Equations, (Moscow, 1999).

[27] A. Ron and Z. Shen, *Weyl-Heisenberg frames and Riesz bases in* $L^2(\mathbb{R}^d)$*,* Duke Math. J. **89** (1997), 237–282.

[28] J. Sjöstrand, *Wiener type algebras of pseudodifferential operators,* Séminaire sur les Équations aux Dérivées Partielles, 1994–1995, Exp. No. IV, 21 pp., École Polytech., Palaiseau, 1995.

[29] Q. Sun, *Wiener's lemma for infinite matrices with polynomial off-diagonal decay, C. R. Math. Acad. Sci. Paris* **340** (2005), no. 8, 567–570.

[30] M.A. Šubin [Shubin], *Almost periodic functions and partial differential operators,* (Russian) Uspehi Mat. Nauk, **33** (1978), no. 2 (200), 3–47; translation in Russian Math. Surveys, **33** (1978), no. 2, 1–52.

[31] D. F. Walnut, *Continuity properties of the Gabor frame operator,* J. Math. Anal. Appl. **165** (1992), no. 2, 479–504.

[32] J. Wexler and S. Raz, *Discrete Gabor expansions,* Signal Processing **21** (1990), 207–221.

[33] N. Wiener, *Tauberian theorems*, Ann. of Math., **33**, 1–100, 1932.

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