MULTI-WINDOW GABOR FRAMES IN AMALGAM SPACES

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Abstract. We show that multi-window Gabor frames with windows in the Wiener algebra $W(L^\infty, \ell^1)$ are Banach frames for all Wiener amalgam spaces. As a by-product of our results we prove the canonical dual of a Gabor frame with a continuous generator in the Wiener algebra also belongs to this space. Our proofs are mostly based on recent non-commutative versions of Wiener's $1/f$ lemma.

1. Introduction

A Gabor system is a collection of functions $\mathcal{G}(g,\Lambda) = \left\{ \pi(\lambda)g \,\middle|\, \lambda \in \Lambda \right\}$, where $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ is a lattice, $g \in L^2(\mathbb{R}^d)$, and the *time-frequency shifts* $\pi(\lambda)g = \pi(x,\omega)g$ of g are given by

$$\pi(x,\omega)g(y) = e^{2\pi i\omega\cdot y}g(y-x), \qquad y \in \mathbb{R}^d,\ \lambda = (x,\omega) \in \Lambda.$$

This system is called a *frame* if $\|f\|_2^2 \approx \sum_\lambda |\langle f, \pi(\lambda)g\rangle|^2$. In this case, there exists a *dual Gabor system* $\mathcal{G}(\widetilde{g},\Lambda) = \left\{ \pi(\lambda)\widetilde{g} \,\middle|\, \lambda \in \Lambda \right\}$ providing the L^2-expansions

$$f = \sum_\lambda \langle f, \pi(\lambda)g\rangle\, \pi(\lambda)\widetilde{g} = \sum_\lambda \langle f, \pi(\lambda)\widetilde{g}\rangle\, \pi(\lambda)g. \tag{1}$$

It is known that under suitable assumptions on g and $\widetilde{g}$ equations in (1) extend to L^p spaces [3, 18, 21, 22]. To some extent, these results parallel the theory of Gabor expansions on modulation spaces [15]. However, since modulation spaces are defined in terms of time-frequency concentration – and are indeed characterized by the *size* of the numbers $\langle f, \pi(\lambda)g\rangle$ – Gabor expansions are also available in a more irregular context, where Λ is not a lattice. In contrast, the theory of Gabor expansions in L^p spaces relies on the strict algebraic structure of Λ. Indeed, as shown in [31], the Poisson summation formula implies that the frame operator $Sf := \sum_\lambda \langle f, \pi(\lambda)g\rangle\, \pi(\lambda)g$ can be written as

$$Sf(x) = \frac{1}{\beta^d} \sum_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \left(\overline{g(x - j/\beta - \alpha k)}g(x-\alpha k)\right) f(x - j/\beta). \tag{2}$$

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This expression allows one to transfer spatial information about g to boundedness properties of S and is at the core of the L^p-theory of Gabor expansions.

One often has explicit information only about g, while the existence of $\widetilde{g}$ is merely inferred from the frame inequality. It is then important to know whether certain good properties of g are also inherited by $\widetilde{g}$, so as to deduce the validity of (1) in various function spaces. The key technical point is showing that S is invertible not only in L^2 but also in other relevant spaces. This was proved for modulation spaces in [24, 20] and for L^p spaces in [28]. In this latter case the analysis relies on the fact that S^{-1} is the frame operator associated with the dual Gabor system $\mathcal{G}(\widetilde{g}, \Lambda)$ and thus admits an expansion like the one in (2).

The objective of this article is to extend the L^p-theory of Gabor expansions to multi-window Gabor systems (see [2, 25]),

$$\mathcal{G}(\Lambda^1, \dots, \Lambda^n, g^1, \dots, g^n) = \left\{ \pi(\lambda^i) g^i \,\middle|\, \lambda^i \in \Lambda^i, 1 \leq i \leq n \right\},$$

where $\Lambda^1, \dots, \Lambda^n \subseteq \mathbb{R}^{2d}$ are lattices $\Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$ and $g^1, \dots, g^n : \mathbb{R}^d \to \mathbb{C}$. The challenge in doing so is that, in contrast to the case of a single lattice Λ, the corresponding dual system does not consist of lattice time-frequency translates of a certain family of functions $\widetilde{g^1}, \dots, \widetilde{g^n}$. One of the main technical contributions of this paper is to show that, nevertheless, S^{-1} admits a generalized expansion

$$S^{-1} f(x) = \sum_k G_k(x) f(x - x_k), \tag{3}$$

where the family of points $\{x_k\}_k$ may not be contained in a lattice. We then prove that certain spatial localization properties of $g^1, \dots, g^n$ imply corresponding localization properties for the family of functions $\{G_k\}_k$ and deduce that S^{-1} is bounded on L^p-spaces. Moreover, the result extends to the more general context of Wiener amalgam spaces, which are spaces of functions that belong locally to L^q and globally to L^p.

To prove these results, we study a Banach algebra of operators admitting an expansion as in (3) with a suitable summability condition. We then resort to a recent Wiener-type result on non-commutative almost-periodic Fourier series [4] to prove that this algebra is spectral within the class of bounded operators on L^p. This means that if an operator from that algebra is invertible on L^p, then the inverse operator necessarily belongs to the algebra. This approach is now common in time-frequency analysis [1, 4, 5, 6, 7, 11, 15, 20, 24, 26, 27] but its application to spaces that are not characterized by time-frequency decay is rather subtle. As a by-product, we obtain consequences that are new even for the case of one generator. In particular, we prove that if all the functions g^i are continuous, so is every function in the dual system. This answers positively a question that was inspired by [28].

This paper is organized as follows. In Section 2 we define Wiener amalgam spaces and recall their characterization via Gabor frames. In Section 3 we present the main technical result of this paper: a spectral invariance theorem for a sub-algebra of weighted-shift operators in $B(L^p(\mathbb{R}^d))$. In Section 4, we use the result of the previous section to extend the theory of multi-window Gabor frames to the class of Wiener amalgam spaces. In particular, this last section contains a Wiener-type lemma for multi-window Gabor frames.

2. Amalgam spaces and Gabor expansions

Given $x, \omega \in \mathbb{R}^d$, the translation and modulation operators act on a function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$T_x f(y) := f(y - x), \qquad M_\omega f(y) := e^{2\pi i \omega \cdot y} f(y),$$

where $\omega \cdot y$ is the usual dot product. The time-frequency shift associated with the point $\lambda = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ is the operator $\pi(\lambda) = \pi(x, \omega) := M_\omega T_x$.

Given two non-negative functions f, g, we write $f \lesssim g$ if $f \leq Cg$, for some constant $C > 0$. If E is a Banach space, we denote by $B(E)$ the Banach algebra of all bounded linear operators on E.

We use the following normalization of the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{C}$:

$$\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \omega \cdot x} dx.$$

2.1. **Definition and properties of the amalgam spaces.**

A function $w : \mathbb{R}^d \to (0, +\infty)$ is called a *weight* if it is continuous and symmetric (i.e. $w(x) = w(-x)$). A weight w is *submultiplicative* if

$$w(x + y) \leq w(x) w(y), \quad x, y \in \mathbb{R}^d.$$

Prototypical examples are given by the polynomial weights $w(x) = (1+|x|)^s$, which are submultiplicative if $s \geq 0$. The main results in this article require to consider an extra condition on the weights. A submultiplicative weight w is called *admissible* if $w(0) = 1$, and it satisfies the *Gelfand-Raikov-Shilov* condition,

$$\lim_{k \to \infty} w(kx)^{1/k} = 1, \quad x \in \mathbb{R}^d.$$

It follows that any admissible weight w satisfies $w(x) \geq 1$, $x \in \mathbb{R}^d$.

Given a submultiplicative weight w, a second weight $v : \mathbb{R}^d \to (0, \infty)$ is called *w-moderate* if there exists a constant $C_v > 0$ such that,

$$v(x + y) \leq C_v w(x) v(y), \quad x, y \in \mathbb{R}^d. \tag{4}$$

For polynomial weights $v(x) = (1 + |x|)^t$, $w(x) = (1 + |x|)^s$, v is w-moderate if $|t| \leq s$. If v is w-moderate, it follows from (4) and the symmetry of w that $1/v$ is also w-moderate (with the same constant).

In what follows we shall use a fixed submultiplicative weight w, and consider classes of function spaces related to various w-moderate weights v. For

$1 \le p, q \le \infty$, we define the *Wiener amalgam space* $W(L^p, L^q_v)$ as the class of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that,

$$\|f\|_{W(L^p,L^q_v)} := \left(\sum_{k\in\mathbb{Z}^d} \|f\|^q_{L^p([0,1)^d+k)} v(k)^q \right)^{1/q} < \infty, \tag{5}$$

with the usual modifications when $q = +\infty$. As with Lebesgue spaces, we identify two functions if they coincide almost everywhere. For a study of this class of spaces in a much broader context see [13, 14, 17]. We only point out that, as a consequence of the assumptions on the weights v and w, it can be shown that the partition $\{[0,1)^d + k : k \in \mathbb{Z}^d\}$ in (5) can be replaced by more general coverings yielding an equivalent norm.

Weighted amalgam spaces are *solid*, that is, if $f \in W(L^p, L^q_v)$ and $m \in L^\infty(\mathbb{R}^d)$, then $mf \in W(L^p, L^q_v)$ and

$$\|mf\|_{W(L^p,L^q_v)} \le \|m\|_{L^\infty(\mathbb{R}^d)} \|f\|_{W(L^p,L^q_v)}. \tag{6}$$

In addition, using the fact that v is w-moderate, it follows that $W(L^p, L^q_v)$ is closed under translations and

$$\|T_x f\|_{W(L^p,L^q_v)} \le C_v w(x) \|f\|_{W(L^p,L^q_v)}, \tag{7}$$

where C_v is the constant in (4).

The *Köthe-dual* of $W(L^p, L^q_v)$ is the space of all measurable functions $g : \mathbb{R}^d \to \mathbb{C}$ such that $g \cdot W(L^p, L^q_v) \subseteq L^1(\mathbb{R}^d)$. It is equal to $W(L^{p'}, L^{q'}_{1/v})$, where $1/p + 1/p' = 1/q + 1/q' = 1$ for all $1 \le p, q \le \infty$. In particular, the pairing

$$\langle \cdot, \cdot \rangle : W(L^p, L^q_v) \times W(L^{p'}, L^{q'}_{1/v}) \to \mathbb{C}, \qquad \langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx,$$

is bounded. The linear functionals arising from integration against functions in $W(L^{p'}, L^{q'}_{1/v})$ determine a topology in $W(L^p, L^q_v)$ that will be denoted by $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$.

2.2. **Gabor expansions on amalgam spaces.** We now recall the theory of Gabor expansions on Wiener amalgam spaces as developed in [16, 18, 21, 22]. Let $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ be a separable lattice, and $\alpha, \beta > 0$. The family of sequence spaces corresponding to amalgam spaces via Gabor frames is defined as follows. For a weight v and $1 \le p, q \le \infty$, let $\mathcal{F}L^p([0, 1/\beta)^d)$ be the image of $L^p([0, 1/\beta)^d)$ under the discrete Fourier transform, i.e., a sequence $c \equiv \left\{ c_j \,\middle|\, j \in \beta\mathbb{Z}^d \right\} \subseteq \mathbb{C}$ belongs to $\mathcal{F}L^p([0, 1/\beta)^d)$ if there exists a (unique) function $f \in L^p([0, 1/\beta)^d)$ such that,

$$c_j = \hat{f}(j) = \beta^d \int_{[0,1/\beta)^d} f(x) e^{-2\pi i j x} dx, \qquad j \in \beta\mathbb{Z}^d.$$

The space $\mathcal{F}L^p([0,1/\beta)^d)$ is given the norm $\|c\|_{\mathcal{F}L^p([0,1/\beta)^d)} := \|f\|_{L^p([0,1/\beta)^d)}$. Let $S_v^{p,q}(\Lambda)$ be the set of all sequences $c \equiv \{ c_\lambda \mid \lambda \in \Lambda \} \subseteq \mathbb{C}$ such that, for each $k \in \alpha\mathbb{Z}^d$, the sequence $(c_{k,j})_{j\in\beta\mathbb{Z}^d}$ belongs to $\mathcal{F}L^p([0,1/\beta)^d)$ and

$$\|c\|_{S_v^{p,q}(\Lambda)} := \left(\sum_{k\in\alpha\mathbb{Z}^d} \left\|(c_{k,j})_{j\in\beta\mathbb{Z}^d}\right\|^q_{\mathcal{F}L^p([0,1/\beta)^d)} v(k)^q \right)^{1/q} < \infty,$$

with the usual modifications when $q = \infty$. When $1 < p < \infty$ this is simply,

$$\|c\|_{S_v^{p,q}(\Lambda)} := \left(\sum_{k\in\alpha\mathbb{Z}^d} \left\| \sum_{j\in\beta\mathbb{Z}^d} c_{k,j} e^{2\pi i j \cdot} \right\|^q_{L^p([0,1/\beta)^d)} v(k)^q \right)^{1/q} < \infty,$$

and the usual modifications hold for $q = \infty$.

The following Theorem from [22] describes the analysis and synthesis operators, clarifies their precise meaning and gives their mapping properties.

Theorem 1. [22, Theorem 3.2]. *Let w be a submultiplicative weight, v a w-moderate weight, $g \in W(L^\infty, L^1_w)$, and $1 \le p, q \le \infty$. Then the following properties hold.*

(a) *The analysis (coefficient) operator,*

$$C_{g,\Lambda} : W(L^p, L^q_v) \to S_v^{p,q}(\Lambda), \qquad C_{g,\Lambda}(f) := (\langle f, \pi(\lambda)g\rangle)_{\lambda\in\Lambda}$$

is bounded with a bound that only depends on $\alpha, \beta, \|g\|_{W(L^\infty,L^1_w)}$, and the constant C_v in (4).

(b) *Let $c \in S_v^{p,q}(\Lambda)$ and $m_k \in L^p([0,1/\beta)^d)$ be the unique functions such that $\widehat{m_k}(j) = c_{k,j}$. Then the series*

$$R_{g,\Lambda}(c) := \sum_{k\in\alpha\mathbb{Z}^d} m_k T_k g,$$

converges unconditionally in the $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$-topology and, moreover, unconditionally in the norm topology of $W(L^p, L^q_v)$ if $p, q < \infty$.

(c) *The synthesis (reconstruction) operator $R_{g,\Lambda} : S_v^{p,q}(\Lambda) \to W(L^p, L^q_v)$ is bounded with a bound that depends only on $\alpha, \beta, \|g\|_{W(L^\infty,L^1_w)}$, and the constant C_v in* (4).

The definition of the operator $R_{g,\Lambda}$ is rather abstract. As shown in [16], the convergence can be made explicit by means of a summability method.

For $g \in W(L^\infty, L^1_w)$, a sequence $c \in S_v^{p,q}(\Lambda)$, and $N, M \ge 0$ let us consider the partial sums

$$R_{N,M}(c)(x) := \sum_{|k|_\infty \le \alpha N} \sum_{|j|_\infty \le \beta M} c_{k,j} e^{2\pi i j x} g(x-k).$$

In the conditions "$|k|_\infty \leq N, |j|_\infty \leq M$" above we consider elements $(k, j) \in \Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$; it is important that we use the max norm. We also consider the regularized partial sums,

$$\sigma_{N,M}(c)(x) := \sum_{|k|_\infty \leq \alpha N} \sum_{|j|_\infty \leq \beta M} r_{j,M} c_{k,j} e^{2\pi i j x} g(x-k),$$

where the *regularizing weights* are given by,

$$r_{j,M} := \prod_{h=1}^{d} \left(1 - \frac{|j_h|}{\beta(M+1)}\right). \tag{8}$$

We then have the following convergence result [16, 22].

Theorem 2. *Let w be a submultiplicative weight, v a w-moderate weight, $g \in W(L^\infty, L^1_w)$, and $1 \leq p, q \leq \infty$. Then the following properties hold.*

(a) *If $1 < p < \infty$ and $q < \infty$, then*
$$R_{N,M}(c) \to R_{g,\Lambda}(c), \qquad \text{as } N, M \to \infty,$$
in the norm of $W(L^p, L^q_v)$.

(b) *For each $c \in S^{p,q}_v(\Lambda)$,*
$$\sigma_{N,M}(c) \to R_{g,\Lambda}(c), \qquad \text{as } N, M \to \infty,$$
in the $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$-topology and also in the norm of $W(L^p, L^q_v)$ if $p, q < \infty$.

Remark 1. *A more refined convergence statement, with more general summability methods, can be found in* [16]. *We will only need the norm and weak convergence of Gabor expansions but we point out that the problem of pointwise summability has also been extensively studied* [16, 18, 21, 22, 32].

Proof. Part (a) is proved in [22, Proposition 4.6]. The case $p < +\infty$ of (b) is proved in [16, Theorem 4], where only unweighted amalgam spaces are considered. The same proof extends with simple modifications to the weighted case and weak*-convergence for $p = \infty$. □

We now present a representation of Gabor frame operators that will be essential for the results to come. For proofs see [31] or [22, Theorem 4.2 and Lemma 5.2] for the weighted version.

Theorem 3. *Let w be a submultiplicative weight, v a w-moderate weight, $g, h \in W(L^\infty, L^1_w)$ and $1 \leq p, q \leq +\infty$. Then the operator $R_{h,\Lambda} C_{g,\Lambda} : W(L^p, L^q_v) \to W(L^p, L^q_v)$ can be written as*

$$R_{h,\Lambda} C_{g,\Lambda} f = \beta^{-d} \sum_{j \in \mathbb{Z}^d} G_j T_{\frac{j}{\beta}} f, \tag{9}$$

where,

$$G_j(x) := \sum_{k \in \mathbb{Z}^d} \overline{g(x - j/\beta - \alpha k)} h(x - \alpha k), \qquad x \in \mathbb{R}^d. \tag{10}$$

In addition, the functions $G_j : \mathbb{R}^d \to \mathbb{C}$ *satisfy*

$$\sum_{j\in\mathbb{Z}^d} \|G_j\|_\infty w(j/\beta) \lesssim \|g\|_{W(L^\infty,L^1_w)} \|h\|_{W(L^\infty,L^1_w)} < +\infty. \tag{11}$$

As a consequence, the series in (9) *converges absolutely in the norm of* $W(L^p, L^q_v)$.

3. The algebra of L^∞-weighted shifts

3.1. **L^∞-weighted shifts.** Representation (9) motivates the following introduction of a Banach *-algebra of operators on function spaces that will be the key technical object of this paper. For an admissible weight w we let $\mathcal{A}_w$ be the set of all families $\mathcal{M} = (m_x)_{x\in\mathbb{R}^d} \in \ell^1_w(\mathbb{R}^d, L^\infty(\mathbb{R}^d))$ with the standard Banach space norm

$$\|\mathcal{M}\|_{\mathcal{A}_w} = \sum_{x\in\mathbb{R}^d} \|m_x\|_{L^\infty(\mathbb{R}^d)} w(x) < +\infty. \tag{12}$$

The algebra structure and the involution on $\mathcal{A}_w$, however, will be non-standard. They will come from the identification of $\mathcal{A}_w$ with the class of operators on function spaces of the form

$$f \mapsto \sum_{x\in\mathbb{R}^d} m_x f(\cdot - x). \tag{13}$$

Observe that due to (12) the family $\mathcal{M} = (m_x)_{x\in\mathbb{R}^d}$ has countable support and also that the operator in (13) is well-defined and bounded on all $L^p(\mathbb{R}^d)$, $p \in [1,\infty]$ (recall that the admissibility of w implies that $w \geq 1$).

With a slight abuse of notation, given a function $m \in L^\infty(\mathbb{R}^d)$ we also denote by m the multiplication operator $f \mapsto mf$. It is then convenient to write $\mathcal{M} \in \mathcal{A}_w$ as

$$\mathcal{M} = \sum_{x\in\mathbb{R}^d} m_x T_x, \quad (m_x)_{x\in\mathbb{R}^d} \in \ell^1_w(\mathbb{R}^d, L^\infty(\mathbb{R}^d)),$$

and endow $\mathcal{A}_w$ with the product and involution inherited from $B(L^2(\mathbb{R}^d))$. More precisely, the product on $\mathcal{A}_w$ is given by

$$\left(\sum_x m_x T_x\right)\left(\sum_x n_x T_x\right) = \sum_x \left(\sum_y m_y n_{x-y}(\cdot - y)\right) T_x,$$

and the involution – by

$$\left(\sum_x m_x T_x\right)^* = \sum_x \overline{m_x(\cdot + x)} T_{-x} = \sum_x \overline{m_{-x}(\cdot - x)} T_x.$$

It is straightforward to verify that with this structure $\mathcal{A}_w$ is, indeed, a Banach *-algebra which embeds continuously into $B(L^2(\mathbb{R}^d))$. We shall establish a number of other continuity properties of the operators defined by families in $\mathcal{A}_w$ in Proposition 1 below. These will be useful in dealing with Gabor expansions on amalgam spaces.

Observe that the identification of families in $\mathcal{A}_w$ and operators on $B(L^p(\mathbb{R}^d))$ given by the operator in (13) is one-to-one; this follows from the characterization of $\mathcal{A}_w$ in the following subsection but can also be proved directly. Because of this we shall no longer distinguish between the families in $\mathcal{A}_w$ and operators generated by them. We will write $\mathcal{A}_w \subset B(L^p(\mathbb{R}^d))$ if we need to highlight that we treat members of $\mathcal{A}_w$ as operators on $L^p(\mathbb{R}^d)$. We also point out that for $m \in L^\infty(\mathbb{R}^d)$ and $x, w \in \mathbb{R}^d$

$$M_\omega m T_x M_{-\omega} = e^{2\pi i \omega \cdot x} m T_x. \tag{14}$$

Proposition 1. *Let $1 \leq p, q \leq +\infty$ and let v be a w-moderate weight. Then the following statements hold.*

(a) $\mathcal{A}_w \hookrightarrow B(W(L^p, L^q_v))$. *More precisely, every $\mathcal{M} = \sum_x m_x T_x \in \mathcal{A}_w$ defines a bounded operator on $W(L^p, L^q_v)$ given by the formula*

$$\mathcal{M}(f) := \sum_x m_x f(\cdot - x).$$

The series defining $\mathcal{M} : W(L^p, L^q_v) \to W(L^p, L^q_v)$ converges absolutely in the norm of $W(L^p, L^q_v)$ and $\|\mathcal{M}\|_{B(W(L^p, L^q_v))} \leq C_v \|\mathcal{M}\|_{\mathcal{A}_w}$, where C_v is the constant in (4).

(b) *For every $\mathcal{M} \in \mathcal{A}_w$, $f \in W(L^p, L^q_v)$ and $g \in W(L^{p'}, L^{q'}_{1/v})$,*

$$\langle \mathcal{M}(f), g \rangle = \langle f, \mathcal{M}^*(g) \rangle .$$

(c) *For every $\mathcal{M} \in \mathcal{A}_w$, the operator $\mathcal{M} : W(L^p, L^q_v) \to W(L^p, L^q_v)$ is continuous in the $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$-topology.*

Proof. Part (a) follows immediately from (6) and (7). Part (b) follows from the fact that the involution in $\mathcal{A}_w$ coincides with taking adjoint. The interchange of summation and integration is justified by the absolute convergence in part (a). Part (c) follows immediately from (b). □

3.2. **Spectral invariance.** We now present our main technical result, and note that similar and more general results appear in [8, 9, 29]. Nonetheless, we include a self-contained proof of this result, since most of our subsequent results are based on it. The key ingredient in the proof is the identification of the algebra $\mathcal{A}_w$ with a class of almost periodic elements associated with a certain group representation. We give a brief account of the theory as required for our purposes. For a more general presentation see [4] and references therein.

For $y \in \mathbb{R}^d$ and $\mathcal{M} \in B(L^p(\mathbb{R}^d))$, $p \in [1, \infty]$, let $\rho(y)\mathcal{M} := M_y \mathcal{M} M_{-y}$, i.e.,

$$\rho(y)\mathcal{M} f(x) = e^{2\pi i y \cdot x} (\mathcal{M} g)(x) \ , \ g(x) = e^{-2\pi i y \cdot x} f(x)$$

The map $\rho : \mathbb{R}^d \to B(B(L^p(\mathbb{R}^d)))$ defines an isometric representation of $\mathbb{R}^d$ on the algebra $B(L^p(\mathbb{R}^d))$. This means that ρ is a representation of $\mathbb{R}^d$ on the Banach space $B(L^p(\mathbb{R}^d))$ and, in addition, for each $y \in \mathbb{R}^d$, $\rho(y)$ is an algebra automorphism and an isometry.

A continuous map $Y : \mathbb{R}^d \to B(L^p(\mathbb{R}^d))$ is *almost-periodic in the sense of Bohr* if for every $\varepsilon > 0$ there is a compact $K = K_\varepsilon \subset \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$

$$(x + K) \cap \{y \in \mathbb{R}^d \mid \|Y(g+y) - Y(g)\| < \varepsilon,\ \forall g \in \mathbb{R}^d\} \neq \emptyset$$

Then Y extends uniquely to a continuous map on the Bohr compactification $\hat{\mathbb{R}}^d_c$ of $\mathbb{R}^d$, also denoted by Y. Thus, now $Y : \hat{\mathbb{R}}^d_c \to B(L^p(\mathbb{R}^d))$, where $\hat{\mathbb{R}}^d_c$ represents the topological dual group (i.e. the group of characters) of $\mathbb{R}^d$ when $\mathbb{R}^d$ is endowed with the discrete topology. The normalized Haar measure on $\hat{\mathbb{R}}^d_c$ is denoted by $\bar{\mu}(dy)$.

For each $\mathcal{M} \in B(L^p(\mathbb{R}^d))$, we consider the map,

$$\widehat{\mathcal{M}} : \mathbb{R}^d \to B(L^p(\mathbb{R}^d)), \qquad \widehat{\mathcal{M}}(y) := \rho(y)\mathcal{M} = M_y \mathcal{M} M_{-y}. \tag{15}$$

An operator $\mathcal{M} \in B(L^p(\mathbb{R}^d))$ is said to be ρ-*almost periodic* if the map $\widehat{\mathcal{M}}$ is continuous and almost-periodic in the sense of Bohr. For every ρ-almost periodic operator $\mathcal{M}$, the function $\widehat{\mathcal{M}}$ admits a $B(L^p(\mathbb{R}^d))$-valued Fourier series,

$$\widehat{\mathcal{M}}(y) \sim \sum_{x \in \mathbb{R}^d} e^{2\pi i y \cdot x} C_x(\mathcal{M}), \qquad (y \in \mathbb{R}^d). \tag{16}$$

The coefficients $C_x(\mathcal{M}) \in B(L^p(\mathbb{R}^d))$ in (16) are uniquely determined by $\mathcal{M}$ via

(17)
$$C_x(\mathcal{M}) = \int_{\hat{\mathbb{R}}^d_c} \widehat{\mathcal{M}}(y) e^{-2\pi i y \cdot x} \bar{\mu}(dy) = \lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} \widehat{\mathcal{M}}(y) e^{-2\pi i y \cdot x} dy$$

and, therefore, satisfy

$$\rho(y) C_x(\mathcal{M}) = e^{2\pi i y \cdot x} C_x(\mathcal{M}). \tag{18}$$

Hence, they are eigenvectors of ρ (see [4] for details).

Within the class of ρ-almost periodic operators we consider $AP^p_w(\rho)$, the subclass of those operators for which the Fourier series in (16) is w-summable, where w is an admissible weight. More precisely, a ρ-almost periodic operator $\mathcal{M}$ belongs to $AP^p_w(\rho)$ if its Fourier coefficients with respect to ρ satisfy

$$\|\mathcal{M}\|_{AP^p_w(\rho)} := \sum_{x \in \mathbb{R}^d} \|C_x(\mathcal{M})\|_{B(L^p(\mathbb{R}^d))} w(x) < +\infty. \tag{19}$$

Since $w \geq 1$, for operators in $AP^p_w(\rho)$ the series in (16) converges absolutely in the norm of $B(L^p(\mathbb{R}^d))$ to $\widehat{\mathcal{M}}(y)$:

$$\widehat{\mathcal{M}}(y) = \sum_{x \in \mathbb{R}^d} e^{2\pi i y \cdot x} C_x(\mathcal{M}), \qquad y \in \mathbb{R}^d, \tag{20}$$

where each $C_x \in B(L^p(\mathbb{R}^d))$ satisfies (17) and, hence, (18). In particular, for $y = 0$, it follows that each $\mathcal{M} \in AP^p_w(\rho)$ can be written as

$$\mathcal{M} = \sum_{x\in\mathbb{R}^d} C_x(\mathcal{M}). \tag{21}$$

Conversely, if $\mathcal{M}$ is given by (21), with the coefficients C_x satisfying (19) and (18), it follows from the theory of almost-periodic series that $\mathcal{M} \in AP^p_w(\rho)$ and C_x satisfy (17).

A special case of [4, Theorem 3.2] proves the spectral invariance of $AP^p_w(\rho) \hookrightarrow B(L^p(\mathbb{R}^d))$, $p \in [1,\infty]$. Our goal here is to establish connection between $\mathcal{A}_w$ and $AP^p_w(\rho)$ and prove a spectral invariance result for $\mathcal{A}_w$.

To achieve this goal we first characterize the eigenvectors C_x of the representation ρ.

Lemma 1. *For any $1 \le p \le \infty$ and any $m \in L^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $C_x = mT_x$ is an eigenvector of $\rho : \mathbb{R}^d \to B(L^p(\mathbb{R}^d))$. For $1 \le p < \infty$ these are the only eigenvectors.*

Proof. If $C_x = mT_x$, then, according to (14), it satisfies (19).

The converse works only for $1 \le p < \infty$. Suppose that $C_x \in B(L^p(\mathbb{R}^d))$ satisfies (19). Using (14) once again we have,

$$\rho(y)(C_xT_{-x}) = e^{2\pi iy\cdot x}C_x e^{-2\pi iy\cdot x}T_{-x} = C_xT_{-x}.$$

It follows that C_xT_{-x} commutes with every modulation M_y. Hence, C_xT_{-x} must be a multiplication operator m, so $C_x = mT_x$. □

For $p = \infty$ there are eigenvectors of ρ which are not of the form mT_x. An example of such an eigenvector is given in [29, Section 5.1.11]. Hence, one would need additional conditions to conclude that $C_x = mT_x$ for some $m \in L^\infty(\mathbb{R}^d)$.

From the discussion above, $AP^p_w(\rho)$ consists of all the operators $\mathcal{M} = \sum\limits_{x\in\mathbb{R}^d} C_x$, with C_x satisfying (19) and (18). In addition, by the previous lemma, for $1 \le p < \infty$ an operator C_x satisfies (18) if and only if it is of the form $C_x = mT_x$, for some function $m \in L^\infty(\mathbb{R}^d)$. In this case, $\|C_x\|_{B(L^2(\mathbb{R}^d))} = \|m\|_\infty$ and, thus, (19) reduces to (12). Hence we obtain

Proposition 2. *For $p \in [1,\infty)$ the class $\mathcal{A}_w \subset B(L^p(\mathbb{R}^d))$ coincides with $AP^p_w(\rho)$, the class of ρ-almost periodic elements, having w-summable Fourier coefficients.*

For $p = \infty$, the two classes are different. Nevertheless, the results we have obtained so far are sufficient to prove our main technical result.

Theorem 4. *Let w be an admissible weight. Then, the embedding $\mathcal{A}_w \hookrightarrow B(L^p(\mathbb{R}^d))$, $p \in [1,\infty]$ is spectral. In other words, if $\mathcal{M} \in \mathcal{A}_w$ defines an invertible operator $\sum_x m_xT_x \in B(L^p(\mathbb{R}^d))$ for some $p \in [1,\infty]$, then $\mathcal{M}^{-1} \in \mathcal{A}_w$.*

Proof. For $1 \le p < \infty$ the result follows from Proposition 2 and [4, Theorem 3.2]. This last result states that $AP_w^p(\rho)$ is spectral.

For $p = \infty$ we follow a different path. Given an operator

$$\mathcal{M} = \sum_{x \in \mathbb{R}^d} m_x T_x \in \mathcal{A}_w \subset B(L^\infty(\mathbb{R}^d))$$

with $\sum_{x\in\mathbb{R}^d} w(x)\|m_x\|_{L^\infty(\mathbb{R}^d)} < \infty$, we consider the operator

$$\mathcal{N} = \sum_{x\in\mathbb{R}^d} T_x(m_{-x})T_x = \sum_{x\in\mathbb{R}^d} m_{-x}(\cdot - x)T_x \in \mathcal{A}_w \subset B(L^1(\mathbb{R}^d)),$$

which is well defined since $\|T_x(m_{-x})\|_{L^\infty(\mathbb{R}^d)} = \|m_{-x}\|_{L^\infty(\mathbb{R}^d)}$. By direct computation, the transpose $\mathcal{N}'$ (Banach adjoint) of $\mathcal{N} : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is precisely $\mathcal{M} : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$. Thus, $\mathcal{M} = \mathcal{N}'$ and by [30, Theorem 3, Chapter 20] it follows that $\mathcal{N}$ is invertible when $\mathcal{M}$ is invertible. Now, by spectrality of $\mathcal{A}_w$ in $B(L^1(\mathbb{R}^d))$ (as obtained earlier) and [30, Theorem 8(ii), Chapter 15], we obtain that $\mathcal{M}^{-1} = (\mathcal{N}^{-1})' \in \mathcal{A}_w$, that is $\mathcal{M}^{-1} = \sum\limits_{x\in\mathbb{R}^d} n_x T_x$ for some bounded functions n_x such that $\sum\limits_{x\in\mathbb{R}^d} w(x)\|n_x\|_{L^\infty(\mathbb{R}^d)} < \infty$. □

Remark 2. *In* [28] *two of us used a special case of Theorem 4 for ρ-periodic (rather than ρ-almost periodic) operators in $B(L^2(\mathbb{R}^d))$. In* [28, Example 2.1]*, however, we neglected to mention this restriction and erroneously implied that all of the operators in $B(L^2(\mathbb{R}^d))$ were ρ-periodic.*

The following spectral invariance property follows from Theorem 4, where we denote by $\sigma_p(\mathcal{M})$ and $\sigma_{\mathcal{A}_w}(\mathcal{M})$ the spectra of the operator $\mathcal{M} \in \mathcal{A}_w$ in the algebras $B(L^p(\mathbb{R}^d))$, $p \in [1, \infty]$, and $\mathcal{A}_w$, respectively.

Corollary 1. *Consider $\mathcal{M} = \sum_x m_x T_x \in \mathcal{A}_w$. Then $\sigma_p(\mathcal{M}) = \sigma_{\mathcal{A}_w}(\mathcal{M})$ for all $p \in [1, \infty]$.*

We conclude the section with the following very important result.

Theorem 5. *Assume that $\mathcal{M} \in \mathcal{A}_w$ satisfies $\mathcal{M}^* = \mathcal{M} = \sum_x m_x T_x$ and $A_r\|f\|_r \le \|\mathcal{M}f\|_r$ for some $A_r > 0$ and all $f \in L^r(\mathbb{R}^d)$ for some $r \in [1, \infty]$. Then $\mathcal{M}^{-1} \in \mathcal{A}_w$.*

Moreover, suppose that $E \subseteq W(L^p, L^q_v)$, $1 \le p, q \le +\infty$, is a closed subspace (in the norm of $W(L^p, L^q_v)$) such that $\mathcal{M}E \subseteq E$. Then $\mathcal{M}^{-1}E \subseteq E$ and, as a consequence, $\mathcal{M}E = E$.

Proof. From Corollary 1 we deduce that $\sigma_{\mathcal{A}_w}(\mathcal{M}) = \sigma_r(\mathcal{M}) = \sigma_2(\mathcal{M}) \subset \mathbb{R}$ since $\mathcal{M} \in B(L^2(\mathbb{R}^d))$ is self-adjoint. Recall that in Banach algebras every boundary point of the spectrum belongs to the approximative spectrum. The boundedness below condition, however, implies that 0 does not belong to the approximative spectrum of $\mathcal{M} \in B(L^r(\mathbb{R}^d))$. Hence, $0 \notin \sigma_r(\mathcal{M})$ and, by Theorem 4, $\mathcal{M}^{-1} \in \mathcal{A}_w$.

To prove the second part, let $\mathcal{A}_w(E)$ be the subalgebra of $\mathcal{A}_w$ formed by all those operators S such that $SE \subseteq E$. Since E is closed in $W(L^p, L^q_v)$

and $\mathcal{A}_w \hookrightarrow B(W(L^p, L^q_v))$ by Proposition 1, it follows that $\mathcal{A}_w(E)$ is a closed subalgebra of $\mathcal{A}_w$ (we do not claim that it is closed under the involution). From the first part of the proof it follows that the set $\mathbb{C} \setminus \sigma_{\mathcal{A}_w}(\mathcal{M})$ is connected. Consequently, (see for example [12, Theorem VII 5.4]), $\sigma_{\mathcal{A}_w(E)}(\mathcal{M}) = \sigma_{\mathcal{A}_w}(\mathcal{M})$. Finally, $0 \notin \sigma_{\mathcal{A}_w}(\mathcal{M}) = \sigma_{\mathcal{A}_w(E)}(\mathcal{M})$ which proves that $\mathcal{M}^{-1} \in \mathcal{A}_w(E)$, as desired. □

4. Dual Gabor frames on amalgam spaces

4.1. Multi-window Gabor frames.

Let $\Lambda = \Lambda^1 \times \ldots \times \Lambda^n$ be the Cartesian product of separable lattices $\Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$ and let $g^1, \ldots, g^n \in W(L^\infty, L^1_w)$. We consider the (multi-window) Gabor system

$$\mathcal{G} = \left\{ g^i_{\lambda^i} := \pi(\lambda^i) g^i \,\middle|\, \lambda^i \in \Lambda^i, 1 \leq i \leq n \right\}.$$

We view the system $\mathcal{G}$ as an indexed set, so that $\mathcal{G}$ might contain repeated elements. The frame operator of the system $\mathcal{G}$ is given by,

$$S_{\mathcal{G}} = S_{g^1,\Lambda^1} + \ldots S_{g^n,\Lambda^n},$$

where $S_{g^i,\Lambda^i} = R_{g^i,\Lambda^i} C_{g^i,\Lambda^i}$ (see Section 2.2). For $1 \leq p, q \leq +\infty$ and a w-moderate weight v, we define the space $S^{p,q}_v(\Lambda) := S^{p,q}_v(\Lambda^1) \times \ldots \times S^{p,q}_v(\Lambda^n)$ endowed with the norm,

$$\|c = (c^1, \ldots, c^n)\|_{S^{p,q}_v(\Lambda)} := \sum_{i=1}^{n} \|c^i\|_{S^{p,q}_v(\Lambda^i)}.$$

The analysis map is $W(L^p, L^q_v) \ni f \mapsto C_{\mathcal{G}}(f) := (C_{g^i,\Lambda^i}(f))_{1 \leq i \leq n} \in S^{p,q}_v(\Lambda)$, while the synthesis map is $S^{p,q}_v \ni c \mapsto R_{\mathcal{G}}(c) := \sum_{i=1}^n R_{g^i,\Lambda^i}(c^i) \in W(L^p, L^q_v)$. With these definitions, the boundedness results in Theorem 1 extend immediately to the multi-window case. The frame expansions are more complicated, however, since the dual system of a frame of the form of $\mathcal{G}$ may not be a multi-window Gabor frame. We now investigate this matter.

4.2. Invertibility of the frame operator and expansions.

Theorem 6. *Let w be an admissible weight, $g^1, \ldots, g^n \in W(L^\infty, L^1_w)$, and $\Lambda = \Lambda^1 \times \ldots \times \Lambda^n$, with $\Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$ separable lattices. Suppose that the Gabor system*

$$\mathcal{G} = \left\{ g^i_{\lambda^i} := \pi(\lambda^i) g^i \,\middle|\, \lambda^i \in \Lambda^i, 1 \leq i \leq n \right\},$$

is such that its frame operator $S_{\mathcal{G}}$ is bounded below in $L^r(\mathbb{R}^d)$ for some $r \in [1, \infty]$, i.e.

$$A_r \|f\|_r \leq \|S_{\mathcal{G}} f\|_r, \ A_r > 0, \quad \text{for all } f \in L^r(\mathbb{R}^d).$$

Then the frame operator $S_{\mathcal{G}}$ is invertible on $W(L^p, L^q_v)$ for all $1 \leq p, q \leq \infty$ and every w-moderate weight v. Moreover, the inverse operator $S_{\mathcal{G}}^{-1} : W(L^p, L^q_v) \to W(L^p, L^q_v)$ is continuous both in $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$ and the norm topologies.

Proof. For each $1 \leq i \leq n$, the frame operator $S_{g^i,\Lambda^i} = R_{g^i,\Lambda^i} C_{g^i,\Lambda^i}$ belongs to the algebra $\mathcal{A}_w$ as a consequence of the Walnut representation in Theorem 3. Hence, $S_{\mathcal{G}} = S_{g^1,\Lambda^1} + \dots S_{g^n,\Lambda^n} \in \mathcal{A}_w$. Since $S_{\mathcal{G}}$ is bounded below in $L^r(\mathbb{R}^d)$, Theorem 5 implies that $S_{\mathcal{G}}{}^{-1} \in \mathcal{A}_w$. The conclusion now follows from Proposition 1. □

We now derive the corresponding Gabor expansions.

Theorem 7. *Under the conditions of Theorem 6, define the dual atoms by* $\tilde{g}^i_{\lambda^i} := S_{\mathcal{G}}{}^{-1}(g^i_{\lambda^i})$. *Let* $1 \leq p, q \leq \infty$ *and* v *be a* w*-moderate weight. Then the following expansions hold.*

(a) *For every* $f \in W(L^p, L^q_v)$,

$$f = \lim_{N,M\to\infty} \sum_{i=1}^{n} \sum_{|k|_\infty \leq N} \sum_{|j|_\infty \leq M} r_{\beta_i j,M} \left\langle f, \tilde{g}^i_{(\alpha_i k, \beta_i j)} \right\rangle g^i_{(\alpha_i k, \beta_i j)}$$

$$= \lim_{N,M\to\infty} \sum_{i=1}^{n} \sum_{|k|_\infty \leq N} \sum_{|j|_\infty \leq M} r_{\beta_i j,M} \left\langle f, g^i_{(\alpha_i k, \beta_i j)} \right\rangle \tilde{g}^i_{(\alpha_i k, \beta_i j)},$$

where the regularizing weights $r_{\beta_i j,M}$ *are given in* (8) *and the series converge in the* $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$*-topology. For* $p, q < +\infty$ *the series also converge in the norm of* $W(L^p, L^q_v)$.

(b) *If* $1 < p < \infty$ *and* $q < \infty$*, for every* $f \in W(L^p, L^q_v)$,

$$f = \lim_{N,M\to\infty} \sum_{i=1}^{n} \sum_{|k|_\infty \leq N} \sum_{|j|_\infty \leq M} \left\langle f, \tilde{g}^i_{(\alpha_i k, \beta_i j)} \right\rangle g^i_{(\alpha_i k, \beta_i j)}$$

$$= \lim_{N,M\to\infty} \sum_{i=1}^{n} \sum_{|k|_\infty \leq N} \sum_{|j|_\infty \leq M} \left\langle f, g^i_{(\alpha_i k, \beta_i j)} \right\rangle \tilde{g}^i_{(\alpha_i k, \beta_i j)},$$

where the series converge in the norm of $W(L^p, L^q_v)$.

Remark 3. *A more refined convergence statement, including more sophisticated summability methods can be obtained using the results in* [16].

Proof. Theorem 2 implies that for all $f \in W(L^p, L^q_v)$,

$$S_{\mathcal{G}}(f) = \lim_{N,M\to\infty} \sum_{i=1}^{n} \sum_{|k|_\infty \leq N} \sum_{|j|_\infty \leq M} r_{\beta_i j,M} \left\langle f, g^i_{(\alpha_i k, \beta_i j)} \right\rangle g^i_{(\alpha_i k, \beta_i j)}, \tag{22}$$

with the kind of convergence required in (a). Since $S_{\mathcal{G}}{}^{-1} \in \mathcal{A}_w$, Proposition 1 implies that $S_{\mathcal{G}}{}^{-1} : W(L^p, L^q_v) \to W(L^p, L^q_v)$ is continuous both in the norm and $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$-topology. Consequently, we can apply $S_{\mathcal{G}}{}^{-1}$ to both sides of (22) to obtain the first expansion in (a). The second one follows by applying (22) to the function $S_{\mathcal{G}}{}^{-1}(f)$ and using Proposition 1 to get,

$$\left\langle S_{\mathcal{G}}{}^{-1}(f), g^i_{\lambda^i} \right\rangle = \left\langle f, S_{\mathcal{G}}{}^{-1}(g^i_{\lambda^i}) \right\rangle = \left\langle f, \tilde{g}^i_{\lambda^i} \right\rangle.$$

The statement in (b) follows similarly, this time using the corresponding statement in Theorem 2. □

4.3. **Continuity of dual generators.** We now apply Theorem 5 to Gabor expansions.

Theorem 8. *In the conditions of Theorem 6, let $1 \leq p, q \leq \infty$ and let v be a w-moderate weight. Let $E \subseteq W(L^p, L^q_v)$ be a closed subspace (in the norm of $W(L^p, L^q_v)$) such that $S_{\mathcal{G}} E \subseteq E$. Suppose that the atoms $g^1, \ldots, g^n \in E$. Then the dual atoms, $\tilde{g}^i_{\lambda^i} = S_{\mathcal{G}}{}^{-1}(g^i_{\lambda^i}) \in E$.*

Proof. As seen in the proof of Theorem 6, $S_{\mathcal{G}} \in \mathcal{A}_w$. Hence, the conclusion follows from Theorem 5. □

As an application of Theorem 8 we obtain the following corollary, which was one of our main goals. The case $n = 1$ was an open problem in [28].

Corollary 2. *In the conditions of Theorem 6, if all the atoms $g^1, \ldots, g^n$ are continuous functions, so are all the dual atoms $\tilde{g}^i_{\lambda^i} = S_{\mathcal{G}}{}^{-1}(g^i_{\lambda^i})$.*

Proof. We apply Theorem 8 to the subspace $W(C_0, L^1_w)$ formed by the functions of $W(L^\infty, L^1_w)$ that are continuous. To this end we need to observe that $S_{\mathcal{G}} W(C_0, L^1_w) \subseteq W(C_0, L^1_w)$. Since $S_{\mathcal{G}} = S_{g^1,\Lambda^1} + \ldots S_{g^n,\Lambda^n}$, it suffices to show that each S_{g^i,Λ^i} maps $W(C_0, L^1_w)$ into $W(C_0, L^1_w)$.

Let $f \in W(C_0, L^1_w)$. The Walnut representation of S_{g^i,Λ^i} in Theorem 3 gives $S_{g^i,\Lambda^i}(f) = \beta_i^{-d} \sum_j G^i_j T_{j/\beta_i} f$ with absolute convergence in the norm of $W(L^\infty, L^1_w)$. Hence it suffices to observe that each of the functions G^i_j is continuous. According to Theorem 3 these are given by

$$G^i_j(x) := \sum_{k \in \mathbb{Z}^d} \overline{g^i(x - j/\beta_i - \alpha_i k)} g^i(x - \alpha_i k).$$

Since the function g^i is continuous it suffices to note that in the last series the convergence is locally uniform. This is an easy consequence of the fact that $\|g^i\|_{W(L^\infty, L^1_w)} < \infty$. □

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