SOME SPECTRAL PROPERTIES OF PSEUDO-DIFFERENTIAL OPERATORS ON THE SIERPIŃSKI GASKET

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ABSTRACT. We prove versions of the strong Szëgo limit theorem for certain classes of pseudodifferential operators defined on the Sierpiński gasket. Our results used in a fundamental way the existence of localized eigenfunctions for the Laplacian on this fractal.

1. INTRODUCTION

The results of this paper have their origin in a celebrated theorem of Szegö, who proved that if P_n is projection onto the span of $\{e^{im\theta}, 0 \leq m \leq n\}$ in L^2 of the unit circle and $[f]$ is multiplication by a positive $C^{1+\alpha}$ function for $\alpha > 0$ then $(n+1)^{-1} \log \det P_n[f]P_n$ converges to $\int_0^{2\pi} \log f(\theta) \, d\theta/2\pi$. Equivalently, $(n+1)^{-1} \operatorname{Trace} \log P_n[f]P_n$ has the same limit. The literature which has grown from this result is vast, see [14]; here we are interested in the generalizations that replace $[f]$ with a pseudodifferential operator defined on fractal sets, in which case the Szegö limit theorem may be viewed as a way to obtain asymptotic behavior of the operator using only its symbol. In the setting of Riemannian manifolds there are important results of this latter type due to Widom [17] and Guillemin [5].

In [11] a standard generalization of the classical Szegö theorem was proved in the setting where the underlying space is the Sierpiński Gasket and P_Λ is the projection onto the eigenspace with eigenvalues less than Λ of the Laplacian obtained using analysis on fractals. In this situation the projection is not Toeplitz, so most classical techniques fail and the proofs in [11] rely instead on the fact that most eigenfunctions are localized. The present work continues this development to consider the case where $[f]$ is replaced by a pseudodifferential operator. Our results are of three related types. Our main result, Theorem 8, gives asymptotics when one considers $\operatorname{Trace} F(P_n p(x, -\Delta)P_n)$ where $p(x, -\Delta)$ is the pseudodifferential operator and F is continuous on an interval containing the spectrum of $P_n p(x, -\Delta)P_n$. Theorem 13, which is in some sense a special case of the main result but for which we give a different proof, considers the classical case of $\log \det P_n p(x, -\Delta)P_n$. Theorem 17 gives the asymptotics of clusters of eigenvalues of a pseudodifferential operator in

terms of the symbol. In all cases the proofs rely on the predominance of localized eigenfunctions in the Laplacian spectrum on the Sierpiński Gasket.

2. BACKGROUND

2.1. **Analysis on the Sierpiński gasket.** The Sierpiński gasket X is the unique non-empty compact fixed set of the iterated function system $\{F_j = \frac{1}{2}(x - a_j) + a_j\}$, $j = 1, 2, 3$, where $\{a_j\}$ are not co-linear in $\mathbb{R}^2$. For $w = w_1 \cdots w_N$ a word of length N with all $w_j \in \{1, 2, 3\}$ let $F_w = F_{w_1} \circ \cdots \circ F_{w_N}$ and call $F_w(X)$ an N-cell. We will sometimes write the decomposition into N-cells as $X = \cup_{i=1}^{3^N} C_i$ where each $C_i = F_w(X)$ for a word w of length N.

Equip X with the unique probability measure μ for which an N-cell has measure 3^{-N} and the symmetric self-similar resistance form $\mathcal{E}$ in the sense of Kigami [10]. The latter is a Dirichlet form on $L^2(\mu)$ with domain $\mathcal{F} \subset C(X)$ and by the theory of such forms (see [4]) there is a negative definite self-adjoint Laplacian Δ defined by $\mathcal{E}(u, v) = \int (-\Delta u) v \, d\mu$ for all $v \in \mathcal{F}$ such that $v(a_j) = 0$ for $j = 1, 2, 3$. This Laplacian is often called the Dirichlet Laplacian to distinguish it from the Neumann Laplacian which is defined in the same way but omitting the condition $v(a_j) = 0$.

A complete description of the spectrum of Δ was given in [3] using the spectral decimation formula of [13]. The eigenfunctions of Δ are described in [2] (see [16, 11, 12, 1] or [15] for proofs). Without going into details, the features needed for our work are as follows. The spectrum is discrete and decomposes naturally into three sets called the 2-series, 5-series and 6-series eigenvalues, which further decompose according to the *generation of birth*, which is a number $j \in \mathbb{N}$. All 2-series eigenvalues have $j = 1$ and multiplicity 1, each $j \in \mathbb{N}$ occurs in the 5-series and the corresponding eigenspace has multiplicity $(3^{j-1} + 3)/2$, and each $j \geq 2$ occurs in the 6-series with multiplicity $(3^j - 3)/2$. Moreover the 5 and 6-series eigenvalues are localized. Suppose $j > N \geq 1$ and $\{C_i\}_{i=1}^{3^N}$ is the N-cell decomposition as above. If λ is a 5-series eigenvalue with generation of birth j and eigenspace E_j then there is a basis for E_j in which $(3^{j-1} - 3^N)/2$ eigenfunctions are localized at level N, meaning that they are each supported on a single N-cell, and the remaining $(3^N - 3)/2$ are not localized at level N; the number localized on a single N-cell is $(3^{j-N-1} - 1)/2$. If λ is a 6-series eigenvalue with generation of birth j and eigenspace E_j then there is a basis for E_j in which $(3^j - 3^{N+1})/2$ of the eigenfunctions are localized at level N, the remaining $(3^{N+1} - 3)/2$ eigenfunctions are not localized at level N, and the number supported on a single N-cell is $(3^{j-N} - 3)/2$ (see, for example [11]).

2.2. **Pseudo-differential operators on the Sierpiński gasket.** We recall some of the theory developed in [9] and use it to define pseudo-differential operators on the Sierpiński gasket. Let Δ be the Dirichlet Laplacian. Then the spectrum $\mathrm{sp}(-\Delta)$ of $-\Delta$ consists of finite-multiplicity eigenvalues that accumulate only at ∞ ([10, 15]). Arranging them as $\mathrm{sp}(-\Delta) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots\}$ with $\lim_{n \to \infty} \lambda_n = \infty$, let $\{\varphi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mu)$ such that φ_n is an eigenfunction with eigenvalue λ_n for all $n \geq 1$. The set D of finite linear combinations of φ_n is dense in $L^2(\mu)$.

If $p : (0, \infty) \to \mathbb{C}$ is a measurable then

$$p(-\Delta)u = \sum_n p(\lambda_n)\langle u, \varphi_n \rangle \varphi_n$$

for $u \in D$ gives a densely defined operator on $L^2(\mu)$ called a *constant coefficient pseudo-differential operator*. If p is bounded, then $p(-\Delta)$ extends to a bounded linear operator on $L^2(\mu)$ by the spectral theorem.

If p is a 0-symbol in the sense of [9, Definition 3.1] then $p(-\Delta)$ is a pseudo-differential operator of order 0. Moreover, it is a singular integral operator on $L^2(\mu)$ ([9, Theorem 3.6]) and therefore extends to a bounded operator on $L^q(\mu)$ for all $q \in (1, \infty)$.

If $p : X \times (0, \infty) \to \mathbb{C}$ is measurable we define a *variable coefficient pseudo-differential operator* $p(x, -\Delta)$ as in [9, Definition 9.2]:

$$(2.1) \qquad p(x, -\Delta)u(x) = \sum_n \int_X p(x, \lambda_n) P_{\lambda_n}(x, y) u(y) d\mu(y)$$

for $u \in D$, where $\{P_{\lambda_n}\}_{n \in \mathbb{N}}$ is the spectral resolution of $-\Delta$. If p is a 0-symbol in the sense of [9, Definition 9.1], then $p(x, -\Delta)$ extends to a bounded linear operator on $L^q(\mu)$ for all $q \in (1, \infty)$ ([9, Theorem 9.3 and Theorem 9.6]). If $p : X \to \mathbb{R}_+$ is bounded and *independent* of λ, then the operator that p determines on $L^2(\mu)$ is multiplication by $p(x)$. In this case, we write $[p]$ for this operator, following the notation in [11].

We describe next the relationship between the spectrum of a constant coefficient pseudo-differential operator $p(-\Delta)$ and $\mathrm{sp}(-\Delta)$ for a continuous map p. Of course, the result is valid for all the spaces considered in [9], not only the Sierpiński gasket.

Proposition 1. *If $p : (0, \infty) \to \mathbb{C}$ is continuous then* $\mathrm{sp}\, p(-\Delta) = \overline{p(\mathrm{sp}(-\Delta))}$.

Proof. Let λ be in the resolvent $\rho(p(-\Delta))$ of $p(-\Delta)$. Then $\lambda I - p(-\Delta)$ has bounded inverse, so there is $C > 0$ such that $\|(\lambda I - p(-\Delta))^{-1}u\| \leq C\|u\|$ for all $u \in L^2(\mu)$. Let $v \in \mathrm{dom}\, p(-\Delta)$. Then $(\lambda I - p(-\Delta))v \in L^2(\mu)$ and $\|(\lambda I - p(-\Delta))v\| \geq \frac{1}{C}\|v\|$. In particular, if $v = \varphi_n$ is an element of our $L^2(\mu)$ basis of eigenfunctions of $-\Delta$ we obtain $\|(\lambda I - p(-\Delta))\varphi_n\| \geq 1/C$. Since $p(-\Delta)\varphi_n = p(\lambda_n)\varphi_n$ and $\|\varphi_n\| = 1$ for all $n \in \mathbb{N}$, it follows that $\lambda \notin \overline{p(\mathrm{sp}(-\Delta))}$.

For the converse, suppose $z \notin \overline{p(\mathrm{sp}(-\Delta))}$. Then there is $K > 0$ such that $|p(\lambda) - z| \geq K > 0$ for all $\lambda \in \mathrm{sp}(-\Delta)$ and thus $|p(\lambda) - z|^{-1} \leq K^{-1}$ for all $\lambda \in \mathrm{sp}(-\Delta)$. Therefore $(p(-\Delta) - z)^{-1}$ is bounded on $L^2(\mu)$ and $z \in \rho(p(-\Delta))$. $\square$

Corollary 2. *If $p : (0, \infty) \to \mathbb{R}$ is continuous and $\lim_{\lambda \to \infty} p(\lambda) = \infty$ then* $\mathrm{sp}\, p(-\Delta) = p(\mathrm{sp}(-\Delta))$.

Proof. The hypothesis implies that the only accumulation point for $\{p(\lambda_n)\}_{n \in \mathbb{N}}$ is ∞ so we can apply Proposition 1. $\square$

3. Szegö Limit Theorems for pseudo-differential operators on the Sierpiński gasket

Let X, μ and Δ be Section 2. We also follow the notation of Section 4 of [11]: for $\Lambda > 0$, let E_Λ be the span of all eigenfunctions corresponding to eigenvalues λ of $-\Delta$ with $\lambda \leq \Lambda$, let P_Λ be the orthogonal projection onto E_Λ, and set d_Λ to be the dimension of E_Λ. For an eigenvalue λ of $-\Delta$, write E_λ for the eigenspace of λ, P_λ for the orthogonal projection onto E_λ, and d_λ for the dimension of E_λ.

3.1. **Trace-type Szegö limit theorems.** Fix a measurable map $p : X \times (0, \infty) \to \mathbb{R}$ and let $p(x, -\Delta)$ be the densely defined operator as in (2.1). We assume, unless otherwise stated, that $p(\cdot, \lambda)$ is continuous for all eigenvalues λ of $-\Delta$ and that there is a continuous map $q : X \mapsto \mathbb{R}$ such that the following limit exists and is uniform in x:

$$(3.1) \qquad \lim_{\lambda \in \mathrm{sp}(-\Delta), \lambda \to \infty} p(x, \lambda) = q(x).$$

In this section we study the asymptotic behavior of $\mathrm{Tr}\, F(P_\Lambda p(x, -\Delta) P_\Lambda)$ as $\Lambda \to \infty$ for continuous functions F. Our results are clearly true if $\|p\|_\infty = 0$; henceforth we assume that $\|p\|_\infty > 0$.

Lemma 3. *The eigenvalues of $P_\Lambda p(x, -\Delta) P_\Lambda$ are contained in a bounded interval $[A, B]$ for all $\Lambda > 0$.*

Proof. Let $\varepsilon > 0$. Then there is some $\overline{\Lambda}$ such that if $\lambda \in \mathrm{sp}(-\Delta)$ and $\lambda > \overline{\Lambda}$, then $q(x) - \varepsilon < p(x, \lambda) < q(x) + \varepsilon$. Since

$$\langle P_\lambda p(x, -\Delta) P_\lambda \varphi, \varphi \rangle = \int p(x, \lambda) \varphi(x)^2 d\mu(x)$$

for all $\varphi \in E_\lambda$, it follows that, as operators, $P_\lambda [q - \varepsilon] P_\lambda \leq P_\lambda p(x, -\Delta) P_\lambda \leq P_\lambda [q + \varepsilon] P_\lambda$. Now $[q + \varepsilon]$ and $[q - \varepsilon]$ are bounded on $L^2(\mu)$, $P_{\overline{\Lambda}} L^2(\mu)$ is finite dimensional, and

$$P_\Lambda L^2(\mu) = \oplus_{\lambda \leq \Lambda} P_\lambda L^2(\mu) = \left(\oplus_{\lambda \leq \overline{\Lambda}} P_\lambda L^2(\mu) \right) \oplus \left(\oplus_{\overline{\Lambda} < \lambda \leq \Lambda} P_\lambda L^2(\mu) \right),$$

so the assertion of the lemma holds with A the minimum of the smallest eigenvalue of $[q - \varepsilon]$ and the smallest eigenvalue of $P_{\overline{\Lambda}} p(x, -\Delta) P_{\overline{\Lambda}}$, and B the maximum of the L^2 norm of $[q + \varepsilon]$ and the largest eigenvalue of $P_{\overline{\Lambda}} p(x, -\Delta) P_{\overline{\Lambda}}$. $\square$

Lemma 4. *Let $\Lambda > 0$. Let $p : X \times (0, \infty) \to \mathbb{R}$ be a bounded measurable function such that $p(\cdot, \lambda_n)$ is continuous for all $n \in \mathbb{N}$. Then the map on $C[A, B]$ defined by*

$$F \mapsto \frac{1}{d_\Lambda} \mathrm{Tr}\, F(P_\Lambda p(x, -\Delta) P_\Lambda)$$

is a continuous non-negative functional, where A and B are as in Lemma 3.

Proof. The map is clearly linear and non-negative. Continuity follows immediately from the fact that $P_\Lambda L^2(X, \mu)$ is finite dimensional. $\square$

In preparation for our main result, Theorem 8, we prove a version for an increasing sequence $\{\lambda_j\}$ of 6-series eigenvalues where λ_j has generation of birth j. E_j is the eigenspace corresponding to λ_j and has dimension $d_j = (3^j - 3)/2$. For each $1 \leq N < j$ we have an orthonormal basis $\{u_k\}_{k=1}^{d_j} = \{\overline{u}_k\}_{k=1}^{d_j^N} \cup \{v_k\}_{k=1}^{\alpha^N}$ for E_j, where the $\overline{u}_k$ are localized at scale N and the v_k are not localized at scale N. Then $d_j^N = (3^j - 3^{N+1})/2$, $\alpha^N = (3^{N+1} - 3)/2$ and the number of eigenvalues supported on a single N-cell is $m_j^N = (3^{j-N} - 3)/2$. As remarked in Section 3 of [11], an analogous construction may be done for a 5-series eigenfunction with $d_j = (3^{j-1} + 3)$, $d_j^N = (3^{j-1} - 3^N)/2$, $\alpha^N = (3^N - 3)/2$, and $m_j^N = (3^{j-N-1} - 1)/2$. It follows that the results that we prove for 6-series eigenvalues in Lemma 5 and Theorem 6 below are also true for the 5-series eigenvalues with essentially the same proofs.

Let P_j denote the projection onto E_j. The matrix $\Gamma_j := P_j p(x, -\Delta) P_j$ is a $d_j \times d_j$ matrix whose entries are given by

$$\gamma_j(m, l) := \int p(x, \lambda_j) u_m(x) u_l(x) d\mu(x).$$

Our first version of a Szegö theorem is for the operator of multiplication by a simple function.

Lemma 5. *Let $N \geq 1$ be fixed, and suppose $f = \sum_{i=1}^{3^N} a_i \chi_{C_i}$ is a simple function. Then for all $k \geq 0$*

$$(3.2) \qquad \lim_{j \to \infty} \frac{\text{Tr}(P_j[f]P_j)^k}{d_j} = \int f(x)^k d\mu(x).$$

Proof. The case $k = 0$ is trivial since $\text{Tr}(I_{E_j}) = d_j$. Let $k > 0$ and fix $j > N$. The matrix $P_j[f]P_j$ has the following structure with respect to the basis $\{u_m\}_{m=1}^{d_j}$:

$$\left[\begin{array}{cc} R_j & 0 \\ 0 & N_j \end{array} \right],$$

where R_j is a $d_j^N \times d_j^N$-matrix corresponding to the localized eigenvectors and N_j is an $\alpha^N \times \alpha^N$-matrix corresponding to the non-localized eigenvalues (see also equation (11) of [11] – notice, however, the small typographical error in [11] where $\star$ should be 0). Moreover, the matrix R_j consists of 3^N diagonal blocks; each block is of the form $a_i I_{m_j^N}$, $i = 1, \ldots, 3^N$. We have that

$$\text{Tr}(P_j[f]P_j])^k = \text{Tr}(R_j)^k + \text{Tr}(N_j)^k.$$

We can compute the first term explicitly:

$$\text{Tr}(R_j)^k = \sum_{i=1}^{3^N} m_j^N a_i^k = d_j^N \sum_{i=1}^{3^N} \frac{a_i^k}{3^N} = d_j^N \int f(x)^k d\mu(x).$$

For the second term we use that each element in N_j is smaller in absolute value than $\|f\|_\infty$. Hence $|\text{Tr}(N_j)^k| \leq (\alpha^N)^k \|f\|_\infty^k$. Using the fact that $d_j - d_j^N = \alpha^N$ we then obtain that

$$\left| \frac{\text{Tr}(P_j[f]P_j)^k}{d_j} - \int f(x)^k d\mu(x) \right| \leq \frac{\alpha^N}{d_j} \int |f(x)|^k d\mu(x) + \frac{(\alpha^N)^k}{d_j} \|f\|_\infty.$$

Since $\lim_{j \to \infty} (\alpha^N)^k / d_j = 0$ for all $k \geq 0$ and f is bounded on the compact set X the result follows. $\qquad \square$

We use this lemma to prove the following Szegö theorem for pseudodifferential operators for a sequence of 6-series eigenvalues, extending [11, Theorem 1].

Theorem 6. *Let $p : X \times (0, \infty) \to \mathbb{R}$ be a bounded measurable function such that $p(\cdot, \lambda_j)$ is continuous for all $j \in \mathbb{N}$. Assume that $\lim_{j \to \infty} p(x, \lambda_j) = q(x)$ is uniform in x. Then*

$$(3.3) \qquad \lim_{j \to \infty} \frac{1}{d_j} \text{Tr}(P_j p(x, -\Delta) P_j)^k = \int_X q(x)^k d\mu(x)$$

for all $k \geq 0$. Consequently,

$$(3.4) \qquad \lim_{j \to \infty} \frac{1}{d_j} \text{Tr} \, F(P_j p(x, -\Delta) P_j) = \int_X F(q(x)) d\mu(x)$$

for any continuous F supported on $[A, B]$, where A and B are as in Lemma 3.

Proof. We prove (3.3) by induction. The case $k = 0$ is trivial since $(P_j p(x, -\Delta) P_j)^0 = I_{E_j}$ and $\mathrm{Tr}(I_{E_j}) = d_j$. Let $k \geq 1$ and assume that (3.3) is true for all $0 \leq m < k$. We claim that it suffices to prove the result in the case when there is $C > 0$ such that $p(x, \lambda) \geq C$ for all $(x, \lambda) \in X \times (0, \infty)$. Indeed, $p(x, \lambda) + 2\|p\|_\infty \geq \|p\|_\infty > 0$ and, using the induction hypothesis, the result for $p(x, \lambda)$ follows if one proves (3.3) for $p(x, \lambda) + 2\|p\|_\infty$.

Let $\varepsilon > 0$ and assume that $\varepsilon < C/2$. Since the function $\lambda \mapsto \lambda^k$ is uniformly continuous on $[A, B]$, there is $0 < \delta < \varepsilon$ such that if $|\lambda - \lambda'| < \delta$ then $|\lambda^k - (\lambda')^k| < \varepsilon$. Since q is continuous, we can find $N \geq 1$ and a simple function $f_N = \sum_{i=1}^{3^N} a_k \chi_{C_i}$, where $\{C_i\}_{i=1}^{3^N}$ is the decomposition of X into N-cells, such that $\|q(\cdot) - f_N(\cdot)\|_\infty < \delta/2$. Therefore, $|\int q(x)^k d\mu(x) - \int f_N(x)^k d\mu(x)| < \varepsilon$ since $\mu(X) = 1$. Moreover, we can find $J \geq 1$ such that $\|p(\cdot, \lambda_j) - q(\cdot)\|_\infty < \delta/2$ for all $j \geq J$. It follows that

$$(3.5) \qquad \frac{C}{2} \leq f_N(x) - \delta \leq p(x, \lambda_j) \leq f_N(x) + \delta$$

for all $x \in X$ and $j \geq J$. By increasing J, if necessary, we may assume by Lemma 5 that

$$\left| \frac{\mathrm{Tr}(P_j[f_N]P_j)^k}{d_j} - \int (f(x_N))^k d\mu(x) \right| < \varepsilon$$

for all $j \geq J$.

Let $j \geq J$. Equation (3.5) implies that $0 \leq P_j[f_N - \delta]P_j \leq P_j p(x, -\Delta) P_j \leq P_j[f_N + \delta]P_j$. We conclude that $|\sigma_m^j - \sigma_{m,N}^j| < \delta$, $m = 1, \ldots, d_j$, where σ_m^j are the eigenvalues of $P_j p(x, -\Delta) P_j$ and $\sigma_{m,N}^j$ are the eigenvalues of $P_j[f_N]P_j$. Therefore $|(\sigma_m^j)^k - (\sigma_{m,N}^j)^k| < \varepsilon$ for all $m = 1, \ldots, d_j$. Since $\mathrm{Tr}(P_j p(x, -\Delta) P_j)^k = \sum_m (\sigma_m^j)^k$ and $\mathrm{Tr}(P_j[f_N]P_j)^k = \sum_m (\sigma_{m,N}^j)^k$, we have that

$$\left| \frac{\mathrm{Tr}(P_j p(x, -\Delta) P_j)^k}{d_j} - \int_X q(x)^k d\mu(x) \right| \leq \left| \frac{\mathrm{Tr}(P_j p(x, -\Delta) P_j)^k}{d_j} - \frac{\mathrm{Tr}(P_j[f_N]P_j)^k}{d_j} \right|$$

$$+ \left| \frac{\mathrm{Tr}(P_j[f_N]P_j)^k}{d_j} - \int_X f_N(x)^k d\mu(x) \right|$$

$$+ |\int_X f_N(x)^k d\mu(x) - \int_X q(x)^k d\mu(x)|$$

$$< \left| \frac{\sum_{m=1}^{d_j} ((\sigma_m^j)^k - (\sigma_{m,N}^j)^k)}{d_j} \right| + 2\varepsilon$$

$$< 3\varepsilon.$$

This finishes the proof of (3.3).

For the last statement of the theorem, a proof similar to that of Lemma 4 shows

$$F \mapsto \frac{1}{d_j} \mathrm{Tr}\, F(P_j p(x, -\Delta) P_j)$$

is a non-negative functional on the set of continuous functions supported on $[A, B]$ for all $j \geq 1$. The Stone-Weierstrass theorem implies the result. $\qquad \square$

Remark 7. The multiplication operator $[f]$ is the special case $p(x, \lambda) = f(x)$ for a continuous function $f : X \to \mathbb{R}$. Then for F continuous with support on

$[-\|[f]\|, \|[f]\|]$

$$\lim_{j\to\infty} \frac{1}{d_j} \operatorname{Tr} F(P_j[f]P_j) = \int_X F(f(x))d\mu(x).$$

Spectral multipliers occur in the special case where $p : (0, \infty) \to \mathbb{R}_+$ is a bounded measurable function such that $\lim_{n\to\infty} p(\lambda_n) = q$ for some constant q. In this situation if F is continuous with support on $[-\|p(-\Delta)\|, \|p(-\Delta)\|]$ then

$$\lim_{j\to\infty} \frac{1}{d_j} \operatorname{Tr} F(P_j p(-\Delta)P_j) = F(q).$$

The main goal of this section is to prove the following Szegö-type theorem for pseudo-differential operators on the Sierpiński gasket. It is an analogue of classical results like [6, Theorem 29.1.7], which seem to have originated in [17], see also [5].

Theorem 8. *Let* $p : X \times (0, \infty) \to \mathbb{R}$ *be a bounded measurable function such that* $p(\cdot, \lambda_n)$ *is continuous for all* $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} p(x, \lambda_n) = q(x)$ *is uniform in* x. *Then, for any continuous function* F *supported on* $[A, B]$, *we have that*

(3.6)
$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(x, -\Delta)P_\Lambda) = \int_X F(q(x))d\mu(x),$$

where the interval $[A, B]$ *is chosen as in Lemma 3.*

Remark 9.

(a) If, in addition to the hypotheses of Theorem 8, p is a 0-symbol in the sense of [9, Definition 9.1], then for any continuous F on $[-\|p(x, -\Delta)\|, \|p(x, -\Delta)\|]$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(x, -\Delta)P_\Lambda) = \int_X F(q(x))d\mu(x).$$

(b) Spectral mulitpliers are the special case where $p : (0, \infty) \to \mathbb{R}$ is a bounded measurable map such that $\lim_{j\to\infty} p(\lambda_j) = q$. Then for any continuous F supported on $[-\|p(-\Delta)\|, \|p(-\Delta)\|]$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(-\Delta)P_\Lambda) = F(q).$$

(c) Multiplication by a continuous $f : X \to \mathbb{R}$ is the special case $p(x, \lambda) = f(x)$. Then for any continuous F supported on $[-\|[f]\|, \|[f]\|]$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda[f]P_\Lambda) = \int_X F(f(x))d\mu(x).$$

If $\min f(x) > 0$ then the above formula can be obtained from [11, Theorem 3]. Specifically, the authors of [11] proved that if $F > 0$ is continuous, then

$$\lim_{j\to\infty} \frac{\sum_{m=1}^{d_j} F(\sigma_m^{(j)})}{d_j} = \int F(f(x))d\mu(x),$$

where $\sigma_m^{(j)}$ are the eigenvalues of $P_j[f]P_j$. One can obtain our result by taking $F(x) = x^k$ and using the fact that $\operatorname{Tr}(P_j[f]P_j)^k = \sum_{m=1}^{d_j}(\sigma_m^{(j)})^k$.

Proof of Theorem 8 . We prove first that

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr}(P_\Lambda p(x,-\Delta)P_\Lambda)^k = \int_X q(x)^k d\mu(x) \tag{3.7}$$

for all $k \geq 0$. It is easy to prove the formula for $k = 0$. Let $k \geq 1$ and let $\varepsilon > 0$. Clearly

$$\operatorname{Tr}(P_\Lambda p(x,-\Delta)P_\Lambda)^k = \sum_{\lambda \leq \Lambda} \operatorname{Tr}(P_\lambda p(x,-\Delta)P_\lambda)^k$$

and $d_\Lambda = \sum_{\lambda \leq \Lambda} d_\lambda$. The proof of Theorem 6 and its equivalent for 5-series imply that there is $J > 1$ such that if λ is a 6-series or a 5-series eigenvalue with generation of birth at least J, then

$$\left| \frac{\operatorname{Tr}(P_\lambda p(x,-\Delta)P_\lambda)^k}{d_\lambda} - \int q(x)^k d\mu(x) \right| < \varepsilon. \tag{3.8}$$

We will write in this proof $\#\lambda$ for the generation of birth of the eigenvalue λ. For each $\Lambda > 0$ we let $\Gamma_J(\Lambda)$ be the set of eigenvalues $\lambda \leq \Lambda$ such that $\#\lambda > J$ and $\tilde{\Gamma}_J(\Lambda)$ be the set of eigenvalues $\lambda \leq \Lambda$ such that $\#\lambda \leq J$. Notice that $\Gamma_J(\Lambda)$ consists only of 5- and 6-series eigenvalues.

Now fix $\Lambda_1 > 0$ such that $\Gamma_J(\Lambda_1) \neq \emptyset$ and $\tilde{\Gamma}_J(\Lambda_1) \neq \emptyset$. Then, for all $\Lambda > \Lambda_1$,

$$\left| \frac{\operatorname{Tr}(P_\Lambda p(x,-\Delta)P_\Lambda)^k}{d_\Lambda} - \int q(x)^k d\mu(x) \right|$$

$$\leq \frac{\sum_{\lambda\in\tilde{\Gamma}_J(\Lambda)} |\operatorname{Tr}(P_\lambda p(x,-\Delta)P_\lambda)^k - d_\lambda \int q(x)^k d\mu(x)|}{d_\Lambda}$$

$$+ \frac{\sum_{\lambda\in\Gamma_J(\Lambda)} |\operatorname{Tr}(P_\lambda p(x,-\Delta)P_\lambda)^k - d_\lambda \int q(x)^k d\mu(x)|}{d_\Lambda}$$

$$= I + II. \tag{3.9}$$

The proof of Theorem 2 of [11] implies that $\lim_{\Lambda\to\infty} \sum_{\lambda\in\tilde{\Gamma}_J(\Lambda)} d_\lambda/d_\Lambda = 0$ (see inequality (22) and the one following it from [11]). Hence, there is $\Lambda_2 > \Lambda_1$ such that, if $\Lambda > \Lambda_2$, we have that

$$\frac{\sum_{\lambda\in\tilde{\Gamma}_J(\Lambda)} d_\lambda}{d_\Lambda} < \frac{\varepsilon}{\|p\|_\infty^k + \|q\|_\infty^k}.$$

Since $\operatorname{Tr}(P_\lambda p(x,-\Delta)P_\lambda)^k \leq d_\lambda \|p\|_\infty^k$ for all λ and $\mu(X) = 1$ we obtain for $\Lambda > \Lambda_2$

$$I \leq (\|p\|_\infty^k + \|q\|_\infty^k)\frac{\sum_{\lambda\in\tilde{\Gamma}_J(\Lambda)} d_\lambda}{d_\Lambda} < \varepsilon.$$

Finally, by (3.8), for $\Lambda > \Lambda_2$:

$$II < \varepsilon \frac{\sum_{\lambda\in\Gamma_J(\Lambda)} d_\lambda}{d_\Lambda} < \varepsilon.$$

Substitution into (3.9) gives

$$\left| \frac{\operatorname{Tr}(P_\Lambda p(x,-\Delta)P_\Lambda)^k}{d_\Lambda} - \int q(x)^k d\mu(x) \right| < 2\varepsilon.$$

and an application of the Stone-Weierstrass theorem completes the proof. $\square$

We conclude this subsection by showing that the hypothesis of uniform convergence of $\lim_{j\to\infty} p(x,\lambda_j) = q(x)$ in Theorem 6 may be relaxed if we assume some smoothness conditions on p. The result gives convergence of (3.4) along a subsequence of $\{\lambda_j\}_{j\in\mathbb{N}}$.

Proposition 10. *Let $\{\lambda_j\}_{j\in\mathbb{N}}$ be an increasing sequence of 6-series eigenvalues such that λ_j has generation of birth j, for all $j \geq 1$. Assume that $p(\cdot,\lambda_j) \in \mathrm{Dom}(\Delta)$ for all $j \in \mathbb{N}$. Assume that*

$$\lim_{j\to\infty} p(x,\lambda_j) = q(x) \quad \text{for all } x \in X$$

and that both $p(\cdot,\lambda_j)$ and $\Delta_x p(\cdot,\lambda_j)$ are bounded uniformly in j. Then there is a subsequence $\{\lambda_{k_j}\}$ of $\{\lambda_j\}$ such that

$$\lim_{j\to\infty} \frac{1}{d_{k_j}} F(P_{k_j} p(x,-\Delta) P_{k_j}) = \int_X F(q(x)) d\mu(x)$$

for all continuous functions F supported on $[A, B]$.

Proof. Recall from the proof of [8, Lemma 3.3] that there is a constant $C' > 0$ such that for any $f \in \mathrm{dom}(\Delta)$ and for any $x, y \in X$,

$$|f(x) - f(y)| \leq C' R(x,y) \left(\sup_{z\in X} |\Delta f(z)| + \max_{p,q\in\partial X} |f(p) - f(q)| \right).$$

Since, by our hypothesis, $p(x,\lambda_j)$ and $\Delta_x p(x,\lambda_j)$ are bounded uniformly in j, using the above estimate, we can find a constant $C > 0$ such that

$$|p(x,\lambda_j) - p(y,\lambda_j)| \leq C R(x,y)$$

for all $j \in \mathbb{N}$. Hence the sequence $\{p(\cdot,\lambda_j)\}$ is equicontinuous. Since the sequence is also uniformly bounded, the Arzelá-Ascoli theorem implies that there is a subsequence $\{p(\cdot,\lambda_{k_j})\}$ such that

$$\lim_{j\to\infty} p(x,\lambda_{k_j}) = q(x)$$

uniformly in x. Therefore we can apply Theorem 6 to the subsequence $\{p(x,\lambda_{k_j})\}$ to obtain the conclusion. $\square$

3.2. **Determinant Szegö-type limit theorems.** Proposition 11 below is a generalization of [11, Theorem 1], and can be proved using Theorem 6. Specifically, the result would follow if we set $F(x) = \log(x)$ and consider a positive map p. We provide an alternative proof following the steps of the proof of [11, Theorem 1]. We hope that our argument will shed light on some of the technical details that are sketched in the above mentioned proof. We still assume that $\{\lambda_j\}_{j\geq 1}$ is a fixed increasing sequence of 6-series or 5-series eigenvalues such that λ_j has generation of birth j for all $j \in \mathbb{N}$.

Proposition 11. *Assume that $p : X \times (0,\infty) \to \mathbb{R}_+$ is a measurable function such that $p(\cdot,\lambda_j)$ is continuous for all $j \in \mathbb{N}$ and such that there is $C > 0$ so that $p(x,\lambda_j) \geq C$ for all (x,λ_j). Assume that*

$$(3.10) \qquad \lim_{j\to\infty} p(x,\lambda_j) = q(x)$$

exists and the limit is uniform in x. Then

$$(3.11) \qquad \lim_{j\to\infty} \frac{1}{d_j} \log \det P_j p(x,-\Delta) P_j = \int_X \log q(x) d\mu(x).$$

Proof. First we notice that since the limit (3.10) is uniform in x, the function q is continuous on X and $q(x) \geq C > 0$ for all $x \in X$. Hence $\log q(x)$ is integrable.

Let $\varepsilon > 0$ and assume that $\varepsilon < \min(1, C/4)$. There exists $N \geq 1$ and a simple function $f_N = \sum_{i=1}^{3^N} a_i \chi_{C_i}$, where $\{C_i\}_{i=1}^{3^N}$ is a decomposition of X into cells of order N as in Section 2.1, such that $\|q(\cdot) - f_N(\cdot)\|_\infty < (1/2)\min(\varepsilon, \varepsilon C/2)$. Therefore $f_N(x) \geq 3C/4 > 0$ for all $x \in X$. Since $\log$ is a continuous function, by increasing N, if necessary, we can also assume that

$$|\int \log q(x) d\mu(x) - \int_X \log f_N(x) d\mu(x)| < \varepsilon.$$

Since we assume that the limit in (3.10) is uniform, there is $J \geq 1$ such that $\frac{\alpha^N}{d_j}\log\|f_N\| < \varepsilon$ and $\|p(x, \lambda_j) - q(x)\|_\infty < (1/2)\min(\varepsilon, \varepsilon C/2)$ for all $j \geq J$. Hence $\|p(\cdot, \lambda_j) - f_N(\cdot)\|_\infty < \min(\varepsilon, \varepsilon C/2)$ and

(3.12)
$$1 - \varepsilon < \frac{p(x, \lambda_j)}{f_N(x)} < 1 + \varepsilon$$

for all $x \in X$ and $j \geq J$.

Let $j \geq J$ be fixed. Recall that the operator $P_j p(x, -\Delta)P_j$ has a block structure

$$\Gamma_j = \begin{bmatrix} R_j & * \\ * & N_j \end{bmatrix}$$

with respect to the basis $\{u_k\}_{k=1}^{d_j}$. The entries of Γ_j are given by

$$\gamma_j(i, k) = \int_X p(x, \lambda_j) u_i(x) u_k(x) d\mu(x).$$

Similarly, the operators $P_j[f_N(1 - \varepsilon)]P_j$ and $P_j[f_N(1 + \varepsilon)]P_j$ have block structures (see also [11])

$$\Gamma_{j,N}(\pm\varepsilon) = \begin{bmatrix} R_{j,N}(\pm\varepsilon) & 0 \\ 0 & N_{j,N}(\pm\varepsilon) \end{bmatrix}$$

respectively. The blocks R_j, $R_{j,N}(-\varepsilon)$, and $R_{j,N}(\varepsilon)$ are $d_j^N \times d_j^N$ blocks corresponding to the "localized" part, while N_j, $N_{j,N}(-\varepsilon)$, and $N_{j,N}(\varepsilon)$ correspond to the "nonlocalized" part. The inequality (3.12) implies that

$$0 \leq \langle \Gamma_{j,N}(-\varepsilon)g, g \rangle \leq \langle \Gamma_j g, g \rangle \leq \langle \Gamma_{j,N}(\varepsilon)g, g \rangle$$

for all $g \in E_j$. Thus, as operators, $0 \leq \Gamma_{j,N}(-\varepsilon) \leq \Gamma_j \leq \Gamma_{j,N}(\varepsilon)$. [7, Corollary 7.7.4] implies that

$$\det \Gamma_{j,N}(-\varepsilon) \leq \det \Gamma_j \leq \det \Gamma_{j,N}(\varepsilon).$$

Hence

$$\log \det \Gamma_{j,N}(-\varepsilon) \leq \log \det \Gamma_j \leq \log \det \Gamma_{j,N}(\varepsilon).$$

The block $R_{j,N}(\varepsilon)$ consists of 3^N blocks of the form $a_i(1 + \varepsilon)I_{m_j^N}$, $i = 1, \ldots, 3^N$. Hence

$$\log \det \Gamma_{j,N}(\varepsilon) = d_j^N \int \log f_N(x) d\mu(x) + d_j^N \log(1 + \epsilon) + \log \det N_{j,N}(\varepsilon).$$

An estimate as in [11] shows that $|\log \det N_{j,N}(\varepsilon)| \leq \alpha^N \log\|f_N\|_\infty + \alpha^N \log(1 + \varepsilon)$. Similarly,

$$\log \det \Gamma_{j,N}(-\varepsilon) = d_j^N \int \log f_N(x) d\mu(x) + d_j^N \log(1 - \epsilon) + \log \det N_{j,N}(-\varepsilon)$$

and $|\log \det N_{j,N}(\varepsilon)| \leq \alpha^N \log \|f_N\|_\infty + \alpha^N \log(1 - \varepsilon)$. Using the fact that $d_j = d_j^N + \alpha^N$, we obtain that

$$\left| \frac{\log \det P_j p(x, -\Delta) P_j}{d_j} - \int \log q(x) d\mu(x) \right| \leq \left| \int \log q(x) d\mu(x) - \int \log f_N(x) d\mu(x) \right|$$
$$+ \frac{\alpha^N}{d_j} (\|\log f_N\|_1 + \log \|f_N\|_\infty) + \varepsilon.$$

This implies the conclusion. $\qquad\square$

The proof of Proposition 10 can be easily adapted to the proof of the following corollary.

Corollary 12. *Assume that $p : X \times (0, \infty) \to \mathbb{R}_+$ is a measurable function such that $p(\cdot, \lambda_j)$ is continuous for all $j \in \mathbb{N}$ and such that there is $C > 0$ so that $p(x, \lambda_j) \geq C$ for all (x, λ_j). Moreover, assume that $p(\cdot, \lambda) \in \mathrm{Dom}(\Delta)$ for all $\lambda \in \mathrm{sp}(-\Delta)$ and that*

$$\lim_{j \to \infty} p(x, \lambda_j) = q(x) \quad \text{for all } x \in X$$

and that both $p(x, \lambda)$ and $\Delta_x p(x, \lambda)$ are bounded on $X \times \mathrm{sp}(-\Delta)$. Then there is a subsequence $\{\lambda_{k_j}\}$ of $\{\lambda_j\}$ such that

$$\lim_{j \to \infty} \frac{1}{d_{k_j}} \log \det P_{k_j} p(x, -\Delta) P_{k_j} = \int_X \log q(x) d\mu(x).$$

More generally, we have:

Proposition 13. *Assume that $p : X \times (0, \infty) \to \mathbb{R}_+$ is a measurable function such that $p(\cdot, \lambda)$ is continuous for all $\lambda \in \mathrm{sp}(-\Delta)$ and such that there is $C > 0$ so that $p(x, \lambda) \geq C$ for all $(x, \lambda) \in X \times \mathrm{sp}(-\Delta)$. Assume that*

$$(3.13) \qquad\qquad \lim_{\lambda \to \infty} p(x, \lambda) = q(x)$$

exists and the limit is uniform in x. Then

$$(3.14) \qquad \lim_{\Lambda \to \infty} \frac{1}{d_\Lambda} \log \det P_\Lambda p(x, -\Delta) P_\Lambda = \int_X \log q(x) d\mu(x).$$

Proof. The proof is similar to the proof of Theorem 8 and we omit the details. $\quad\square$

4. Examples and Applications

In this section we consider some applications of our main results. Our examples show how one can extract asymptotic behaviour of operators from elementary integrals, but we also give a result on spectral clustering for generalized Schrödinger operators following a line of argument due to Widom [17].

4.1. **Examples.** Our results can be used to obtain asymptotics of Riesz and Bessel operators, as well as some specific generalized Schrödinger operators.

Example 14. Let $p : (0, \infty) \to \mathbb{R}$ be $p(\lambda) = 1 + \lambda^{-\beta}$, where β is a positive real number. Then $p(-\Delta) = I + (-\Delta)^{-\beta}$. We know from [8] that $p(-\Delta)$ is bounded on $L^p(\mu)$ for all $p > 1$. Let F be a continuous function supported on $(-\|p(-\Delta)\|, \|p(-\Delta)\|)$. Since $\lim_{\lambda \to \infty} p(\lambda) = 1$, we have that

$$\lim_{j \to \infty} \frac{\mathrm{Tr}\, F(P_j p(-\Delta) P_j)}{d_j} = F(1)$$

for any increasing sequence $\{\lambda_j\}$ of 6-series or 5-series eigenvalues such that λ_j has generation of birth j. Moreover

$$\lim_{\Lambda \to \infty} \frac{\operatorname{Tr} F(P_\Lambda p(-\Delta) P_\Lambda)}{d_\Lambda} = F(1).$$

Since $p(\lambda) \geq 1$ for all $\lambda > 0$, we can apply Proposition 11 and Proposition 13 and obtain that

$$\lim_{j \to \infty} \frac{1}{d_j} \log \det P_j p(-\Delta) P_j = 0$$

and

$$\lim_{\Lambda \to \infty} \frac{1}{d_\Lambda} \log \det P_\Lambda p(-\Delta) P_\Lambda = 0.$$

Let $p : (0, \infty) \to \mathbb{R}$ be $p(\lambda) = 1 + (1 + \lambda)^{-\beta}$, where β is a positive real number. Then the corresponding operator is the Bessel operator given by $p(-\Delta) = I + (I - \Delta)^{-\beta}$. All the conclusions of the previous examples hold for this operator as well.

Example 15. Let $p(x, \lambda) = q(\lambda) + \chi(x)$, where q is a bounded measurable function on $(0, \infty)$ such that $\lim_{\lambda \to \infty} q(\lambda) = l$ exists and χ is continuous on X. Then $p(x, -\Delta)$ is a generalized Schrödinger operator. Since it is the sum of two bounded operators it is bounded on $L^2(\mu)$. Let F be a continuous function supported on $(-\|p(x, -\Delta)\|, \|p(x, -\Delta)\|)$. Then, if $\{\lambda_j\}_{j \geq 1}$ is an increasing sequence of 6-series or, respectively, 5-series eigenvalues, we have both of the following equalities:

$$\lim_{j \to \infty} \frac{\operatorname{Tr} F(P_j p(x, -\Delta) P_j)}{d_j} = \int F(l + \chi(x)) d\mu(x) = \lim_{\Lambda \to \infty} \frac{\operatorname{Tr} F(P_\Lambda p(x, -\Delta) P_\Lambda)}{d_\Lambda}.$$

In particular, if $l = 0$, we have that

$$\lim_{j \to \infty} \frac{\operatorname{Tr} F(P_j p(x, -\Delta) P_j)}{d_j} = \int F(\chi(x)) d\mu(x) = \lim_{\Lambda \to \infty} \frac{\operatorname{Tr} F(P_\Lambda p(x, -\Delta) P_\Lambda)}{d_\Lambda}.$$

If we further assume $p(x, \lambda) \geq C > 0$ for all $(x, \lambda) \in X \times (0, \infty)$ then

$$\lim_{j \to \infty} \frac{1}{d_j} \log \det P_j p(x, -\Delta) P_j = \int \log(l + \chi(x)) d\mu(x)$$

and the same is true if we replace P_j with P_Λ and d_j with d_Λ, so in particular if $l = 0$,

$$\lim_{j \to \infty} \frac{1}{d_j} \log \det P_j p(x, -\Delta) P_j = \int \log(\chi(x)) d\mu(x) = \lim_{\Lambda \to \infty} \frac{1}{d_\Lambda} \log \det P_\Lambda p(x, -\Delta) P_\Lambda.$$

4.2. **Application: Asymptotics of eigenvalue clusters for general Schrödinger operators.** Let $p : (0, \infty) \to \mathbb{R}$ be a measurable function and let χ be a real-valued bounded measurable function on X. We call the operator $H = p(-\Delta) + [\chi]$ a *generalized Schrödinger operator* with *potential* χ. We study the asymptotic behavior of spectra of generalized Schrödinger operators with continuous potentials and continuous p, generalizing some results of [12].

We begin with a lemma that is a generalization of the key [12, Lemma 1].

Lemma 16. *Let $p : (0, \infty) \to \mathbb{R}$ be a continuous function such that there is $A \in \mathbb{R}$ with $p(\lambda) \geq A$ for all $\lambda \geq \lambda_1$, where λ_1 is the smallest positive eigenvalue of $-\Delta$. For $i = 1, 2$, let χ_i be real-valued bounded measurable functions on X. Let $H_i = p(-\Delta) + [\chi_i]$ denote the corresponding generalized Schrödinger operators. For $n \geq 1$, the nth eigenvalues ν_n^i of H_i, $i = 1, 2$, satisfy the following inequality:*

$$|\nu_n^1 - \nu_n^2| \leq \|\chi_1 - \chi_2\|_{L^\infty}.$$

Proof. The hypothesis implies that

$$\langle H_i f, f \rangle \geq (A + \min_X \chi_i) \|f\|_2^2$$

for all $f \in D$, $i = 1, 2$. Hence H_i is bounded from below, $i = 1, 2$. The remainder of the proof is identical to the proof of [12, Lemma 1]. □

Assume that $p : (0, \infty) \to \mathbb{R}$ is a continuous function, that there is $\overline{\lambda} > 0$ such that p is increasing on $[\overline{\lambda}, \infty)$ and

(4.1) $$|p(\lambda) - p(\lambda')| \geq c|\lambda - \lambda'|^\beta$$

for all $\lambda, \lambda' \geq \overline{\lambda}$ and some constants $c > 0$ and $\beta > 0$. Let χ be a continuous function on X and $H = p(-\Delta) + [\chi]$ be the corresponding Schrödinger operator. Let $\{\lambda_j\}$ be a sequence of 6-series eigenvalues of $-\Delta$ such that the separation between λ_j and the next higher and lower eigenvalues of $-\Delta$ grows exponentially in j. For example, if λ_1 is any 6-series eigenvalue with generation of birth 1 and if $\lambda_j = 5^{j-1}\lambda_1$, then the sequence $\{\lambda_j\}_{j \geq 1}$ satisfies our assumption ([12], [15, Chapter 3]). Let $\tilde{\Lambda}_j$ be the portion of the spectrum of H lying in $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$. Lemma 16 implies that, for large j, $\tilde{\Lambda}_j$ contains exactly d_j eigenvalues $\{\nu_i^j\}_{i=1}^{d_j}$. We call this the $p(\lambda_j)$ *cluster* of the eigenvalues of H. If we translate by $p(\lambda_j)$ units to the left we obtain clusters that lie in a fixed interval. Define the characteristic measure of the $p(\lambda_j)$ cluster of H by

$$\Psi_j(\lambda) = \frac{1}{d_j} \sum_{i=1}^{d_j} \delta(\lambda - (\nu_i^j - p(\lambda_j))).$$

and observe that integrating the function x^k against this measure yields

(4.2) $$\langle \Psi_j, x^k \rangle = \frac{1}{d_j} \mathrm{Tr}\big(\overline{P}_j(p(-\Delta) + [\chi] - p(\lambda_j))\overline{P}_j\big)^k,$$

for all $k \geq 0$, where $\overline{P}_j$ is the spectral projection for $p(-\Delta) + [\chi]$ associated with the $p(\lambda_j)$ cluster. This allows us to analyze the weak limit of Ψ_j using Theorem 8.

Theorem 17. *The sequence $\{\Psi_j\}_{j \geq 1}$ converges weakly to the pullback of the measure μ under χ defined for all continuous functions f supported on $[\min \chi, \max \chi]$ by*

$$\langle \Psi_0, f \rangle = \int_X f(\chi(x)) d\mu(x).$$

Lemma 18. *Assume that $N > 0$ and that $\chi_N = \sum_{i=1}^N a_i \chi_{C_i}$ is a simple function, where $\{C_i\}_{i=1}^{3^N}$ is a partition of X into N-cells. Let $H_N = p(-\Delta) + [\chi_N]$ be the corresponding generalized Schrödinger operator, $\tilde{\Lambda}_j^N$ the $p(\lambda_j)$ cluster of H_N, and let $\overline{P}_j^N$ be the spectral projection for H_N associated with the $p(\lambda_j)$ cluster. Then*

$$\lim_{j \to \infty} \frac{\mathrm{Tr}\big(\overline{P}_j^N (p(-\Delta) + [\chi_N] - p(\lambda_j))\overline{P}_j^N\big)^k}{d_j} = \lim_{j \to \infty} \frac{(P_j[\chi_N]P_j)^k}{d_j} = \int_X \chi_N(x)^k d\mu(x),$$

for all $k \geq 0$.

Proof. Consider the first equality. If $\overline{u}$ is an eigenfunction of the basis of E_j that is localized in a single N-cell C_i, then $H_N \overline{u} = (p(\lambda_j) + a_i)\overline{u}$. Thus $\lambda_i^j := p(\lambda_j) + a_i$ is an

eigenvalue of H_N with multiplicity at least m_j^N. Doing this identifies d_j^N eigenvalues in $\tilde{\Lambda}_j^N$ and we let A_j^N be the set of the remaining α^N eigenvalues. Hence

$$\mathrm{Tr}\big(\overline{P}_j^N(p(-\Delta) + [\chi_N] - p(\lambda_j))\overline{P}_j^N\big)^k = m_j^N \sum_{i=1}^{3^N} a_i^k + \sum_{\nu \in A_j^N} \nu^k.$$

Recall from the proof of Lemma 5 that $\mathrm{Tr}(P_j[\chi_N]P_j)^k = m_j^N \sum_{i=1}^{3^N} a_i^k + \mathrm{Tr}(N_j)^k$. Since both $\sum_{\nu \in A_j^N} \nu^k$ and $\mathrm{Tr}(N_j)^k$ are bounded by a constant times $(\alpha^N)^k$, the equality of the two limits follows. The second equality is from Theorem 8. $\qquad\square$

Proof of Theorem 17. Since χ is continuous, it can be approximated uniformly by a sequence of simple functions $\chi_N = \sum_{i=1}^{3^N} a_i \chi_{C_i}$, of the type previously described. We can choose χ_N such that $\min \chi \leq \min \chi_N$ and $\max \chi_N \leq \max \chi$ for all N from which $\tilde{\Lambda}_j^N$ is contained in $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$. Moreover there is J_1 depending only $\|\chi\|_\infty$ such that $j \geq J_1$ implies $\tilde{\Lambda}_j^N$ contains d_j eigenvalues $\{\tilde{\lambda}_i^j\}_{i=1}^{d_j}$.

From Lemma 16, for all i and j, $|\nu_i^j - \tilde{\lambda}_i^j| \leq \|\chi - \chi_N\|_\infty$. Therefore, when $j \geq J_1$,

$$\left| \mathrm{Tr}\big(\overline{P}_j(p(-\Delta) + [\chi] - p(\lambda_j))\overline{P}_j\big)^k - \mathrm{Tr}\big(\overline{P}_j^N(p(-\Delta) + [\chi_N] - p(\lambda_j))\overline{P}_j^N\big)^k \right|$$

$$= \sum_{i=1}^{d_j} \left|(\nu_i^j - p(\lambda_j))^k - (\tilde{\lambda}_i^j - p(\lambda_j))^k\right| \leq d_j k \|\chi\|_\infty^{k-1} \|\chi - \chi_N\|_\infty.$$

Then, with $j \geq J_1$,

$$\left| \langle \Psi_j, x^k \rangle - \int_X \chi^k d\mu \right| \leq k \|\chi\|_\infty^{k-1} \|\chi - \chi_N\|_\infty + \int_X |\chi_N^k - \chi^k| d\mu$$

$$+ d_j^{-1} \left| \mathrm{Tr}\big(\overline{P}_j^N(p(-\Delta) + [\chi_N] - p(\lambda_j))\overline{P}_j^N\big)^k - \int_X \chi_N^k d\mu \right|.$$

For sufficiently large N both $\|\chi - \chi_N\|_\infty$ and $\|\chi - \chi_N\|_1$ are less than ε and we can take $j > J_1$ so large that the last term is smaller than ε by Lemma 18. Applying the Stone-Weierstrass theorem completes the proof. $\qquad\square$

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