

# The Problem-Solving Cycle: A Model to Support the Development of Teachers’ Professional Knowledge

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This article focuses on the Problem-Solving Cycle (PSC), a model of professional development designed to assist teachers in supporting their students’ mathematical reasoning. Each PSC is a series of three interrelated workshops in which teachers share a common mathematical and pedagogical experience, organized around a rich mathematical task. Throughout the workshops, teachers delve deeply into issues involving mathematical content, pedagogy, and student thinking as they pertain to the selected task. We analyze this professional development model in relation to the ways it supports the development of content and pedagogical content knowledge. We highlight the ways in which specific knowledge strands are foregrounded during each of the three PSC workshops, while also demonstrating their interconnectedness.

The improvement of students’ opportunities to learn mathematics depends fundamentally on teachers’ skill and knowledge. No curriculum or framework is self-enacting, no students self-teaching. Moreover, teachers are often expected to teach mathematical topics and skills in ways substantially different from the ways in which they themselves learned that content.... Hence, if students’ learning is to improve, teachers’ professional learning opportunities are key. (Boaler & Humphreys, 2005)

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The professional development program featured in this article is one component of a larger research project entitled Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR). STAAR is supported by NSF Proposal No. 0115609 through the Interagency Educational Research Initiative (IERI). The views shared in this article are ours, and do not necessarily represent those of IERI.

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## MATHEMATICS EDUCATION REFORM AND MATHEMATICS TEACHING

The debate about what and how mathematics should be taught in American schools continues. However, most educational leaders and researchers agree that a balanced approach to mathematics instruction that focuses on both procedural and conceptual fluency is critical (National Research Council [NCR], 2005). Mathematics should not merely be taught as a set of procedural competencies; rather, teachers should help students gain sufficient conceptual knowledge along with a flexible understanding of procedures to become competent and efficient problem solvers (NRC, 2005). Getting this balance right in the classroom is a major challenge, and lies at the heart of current reforms (National Council of Teachers of Mathematics [NCTM], 2000).

Most educational practitioners and scholars ascribe to the core ideas of constructivist learning theories—that “learners actively construct their own understandings rather than passively absorb or copy the understanding of others” (Simon & Schifter, 1991, p. 310). From this perspective, the role of the mathematics learner is to engage in activities such as exploring, justifying, proving, critiquing, and generalizing the ideas, representations, and procedures of their solution strategies (Simon & Schifter, 1991; Fosnot, 1996; Lampert, 1990). Teaching can be understood as a dynamic process of inquiry into student reasoning rather than a process of transmitting a set of procedures (Zech, Gause-Vega, Bray, Secules, & Goldman, 2000). The role of the mathematics teacher is to build on students’ existing mathematical knowledge (both formal and informal), as opposed to providing them with new, disconnected pieces of information (Loucks-Horsley et al., 2003). As portrayed in the NCTM’s *Principles and Standards for School Mathematics* (NCTM, 2000), mathematics classrooms aligned with this vision for school mathematics:

The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding.... Students confidently engage in complex tasks carefully chosen by teachers.... Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures.... Alone or in groups and with access to technology, [students] work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. (p. 3)

This vision for school mathematics is highly ambitious. As noted in *Principles and Standards* (NCTM, 2000), achieving it requires attention to multiple sources of influence on classroom practice: “solid mathematics curricula, competent and knowledgeable teachers who can integrate instruction with assessment, educa-

tional policies that enhance and support learning, classrooms with ready access to technology, and a commitment to both equity and excellence” (p. 3). Of all these factors, arguably none is more important than the teacher. As indicated in the opening quote, enhancing students’ learning opportunities depends fundamentally on the knowledge and skills of teachers. Thus, the success of efforts to close the gap between reform visions of mathematics teaching and learning and the practices most common in mathematics classrooms today ultimately relies on teachers and their ability to make substantial changes in their classroom practices. Changes of this magnitude will require a great deal of learning on the part of teachers. This realization has led educational scholars and policymakers to focus their attention on the importance of professional development opportunities for teachers—opportunities that will help them to enhance their professional knowledge and develop new instructional practices.

This article focuses on the teacher and, more specifically, on a project that developed, enacted, and studied an approach to teacher professional development designed to help teachers meet the challenges of the mathematics reform agenda. The centerpiece of our contribution to the Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR) project is the “Problem-Solving Cycle” (PSC), a model of professional development that is situated in classroom practice and designed to help teachers deepen their knowledge of mathematics for teaching. This article describes the general focus and goals of the PSC workshops, details of their enactment, and professional learning opportunities that they provide. We begin by considering the conceptual framework for our work as part of the STAAR project, including key ideas about both the professional knowledge that teachers need to teach according to NCTM’s vision for school mathematics, and the processes of teacher learning.

## DEFINING THE PROFESSIONAL KNOWLEDGE MATHEMATICS TEACHERS NEED

*Principles and Standards* suggests “teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks” (NCTM, 2000, p. 17). There is substantial evidence that teachers do not typically hold this rich and connected knowledge of mathematics, nor do they teach in ways that are consistent with the NCTM *Standards* (Jacobs et al., 2006; Mewborn, 2003). Lloyd and Frykholm (2000) argued, “[Many] teachers possess weak knowledge and narrow views of mathematics and mathematics pedagogy that include conceptions of mathematics as a closed set of procedures, teaching as telling, and learning as the accumulation of information” (p. 576).

How can teachers shift into the roles suggested by the vision of school mathematics portrayed in the NCTM *Standards* (e.g., NCTM, 2000)? To address this

question, a number of researchers are engaged in specifying the professional knowledge that mathematics teachers need. The beginning of contemporary efforts to explore teachers' professional knowledge is typically attributed to Lee Shulman and his presidential address at the 1985 annual meeting of the American Educational Research Association, titled "Those who understand: Knowledge growth in teaching." In the published version of that address, Shulman (1986) suggested that "we distinguish among three categories of content knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge" (p. 9). In his own theoretical and empirical work, Shulman emphasized pedagogical content knowledge, which he characterized as "the dimension of subject matter knowledge *for teaching*...the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (p. 9), aspects such as understanding the conceptions, preconceptions, and misconceptions that students bring with them to the learning situation, and ways of representing ideas that make them comprehensible to students.

Educational scholars have built on and extended Shulman's seminal work on teachers' professional knowledge, particularly within the domains of subject matter and pedagogical content knowledge. There is increasingly widespread agreement that teachers' foundational knowledge draws on and connects knowledge about content, teaching, and students. In the field of mathematics education, Ball, Thames, and Phelps (2005) have attempted to identify and elucidate "knowledge of mathematics for teaching"—the mathematical knowledge that teachers must have in order to do the mathematical work of teaching effectively. This body of professional knowledge includes four components: (1) common knowledge of mathematics content, (2) specialized knowledge of mathematics content, (3) knowledge of mathematics and students, and (4) knowledge of mathematics and teaching (Ball, Thames, & Phelps, 2005).

Common and specialized knowledge of mathematics content are two aspects of *subject matter knowledge* that are necessary for teaching mathematics (Hill & Ball, 2004). Common content knowledge can be defined as a basic understanding of mathematical skills, procedures, and concepts acquired by any well-educated adult. This knowledge enables a teacher to solve mathematical problems, particularly those in their designated curriculum. Teachers also draw on their common content knowledge to recognize wrong answers and spot incorrect definitions in textbooks (Ball, Thames, & Phelps, 2005; Hill & Ball, 2004; Hill, Ball, & Schilling, 2004).

In addition to a strong base of common knowledge of mathematics content, mathematics teachers need *specialized* knowledge of mathematics (Hill & Ball, 2004; Hill, Ball, & Schilling, 2004). Specialized knowledge is the kind of mathematical knowledge that extends beyond the knowledge any well-educated adult might hold. It also differs from the professional knowledge used in other mathe-

matically intense occupations such as engineering, physics, accounting, and carpentry (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2005). Mathematics teachers draw on their specialized content knowledge to make connections among and between mathematical strands; to identify misconceptions and evaluate alternative ideas; to give mathematical explanations and use developmentally appropriate mathematical representations; and to be explicit about their mathematical language and practices.

Shulman (1987) described *pedagogical content knowledge* as “the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students” (p. 15). Like specialized content knowledge, pedagogical content knowledge is unique to teachers and is developed over time as teachers gain expertise in their field, with respect to both subject matter and effective pedagogical strategies (Wilson, Shulman, & Richert, 1987). For teachers to expand their pedagogical content knowledge, they need a sufficient content-knowledge base and classroom teaching experience.

Ball, Thames, and Phelps (2005) divided pedagogical content knowledge into two components: (1) knowledge of content and teaching, and (2) knowledge of content and students. The first component combines teachers’ knowledge of content with their knowledge of teaching. It includes, for example, the ability to recognize instructional affordances and constraints of different representations, and to sequence content to facilitate student learning (Ball, Thames, & Phelps, 2005). Knowledge of mathematical content and teaching is reflected in teachers’ instructional moves before, during, and after a lesson. Teachers draw on this knowledge when they plan for the use of pedagogical strategies and instructional materials in a lesson. They also make use of this knowledge during classroom lessons when they reconsider what tasks to pose, when to hold whole-class discussions or break into small groups, and when to use informal assessment techniques. Further, teachers invoke their knowledge of content and teaching when they critically reflect after a lesson, and consider how to improve their instructional practices the next time they implement a lesson with related mathematical content.

The second component of pedagogical content knowledge combines knowledge of content with knowledge of students. It includes, for example, the ability to predict how students are likely to approach specific mathematical tasks, anticipate student errors, and interpret incomplete student ideas. Teachers draw on this knowledge when they create lesson plans that take student thinking into account, and in particular, the thinking that a task is likely to evoke in their own student population. For example, teachers design lessons while keeping in mind the misconceptions their students are likely to bring to the classroom and the representations they are likely to use to gain access to a particular problem. Teachers also use this knowledge when they consider how to make pedagogical moves based on student

thinking about the content, such as how to respond to various correct or incorrect pathways students explore, misconceptions (anticipated or unanticipated), or incomplete thinking (Ball, Thames, & Phelps, 2005).

Although these domains of knowledge of mathematics for teaching can be separated for the purpose of analysis, they are inextricably intertwined in teachers' instructional practices. Teachers routinely make decisions that draw on all aspects of their knowledge as they engage in the numerous, complex activities of classroom instruction. Such activities include selecting, modifying, and using mathematical problems; selecting mathematical representations that are appropriate for a specific mathematical goal and group of students; explaining and justifying a solution strategy; understanding and building on student conceptions; and establishing and maintaining a discourse community that enhances students' mathematical understanding and their capacity to reason mathematically (Ball, Hill, & Bass, 2005; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005).

## A SITUATIVE PERSPECTIVE ON TEACHER LEARNING

Our research team developed a model of mathematics professional development—the PSC—to provide opportunities for teachers to enhance their professional knowledge and develop new instructional practices. Specifically, the PSC is designed to help teachers increase their knowledge of mathematics for teaching and improve their instructional practices through closely examining mathematics problems, pedagogical practices, and student thinking.

The PSC model is strongly influenced by both constructivist and situative theories of learning. Situative theorists define learning as changes in participation in socially organized activity. They consider the acquisition and use of knowledge as aspects of an individual's participation in social practices (Cobb, 1994; Greeno, 2003; Lave & Wenger, 1991). With respect to professional development, situative theorists focus on the importance of creating opportunities for teachers to work together on improving their practice, and locating these learning opportunities in the everyday practice of teaching (Ball & Cohen, 1999; Putnam & Borko, 2000).

We share with many teacher-educators the view that constructivist and situative theories can be seen as interrelated, and that learning involves both construction and enculturation (Cobb, 1994; Driver, Asoko, Leach, Mortimer, & Scott, 1994). Stemming from this framework, three design principals are central to our model—fostering active teacher participation in the learning process, using teachers' own classrooms as a powerful context for their learning, and enhancing teacher learning by creating a supportive professional community. In a previous article (Borko et al., 2005), we elaborated further on these theories and described how they formed the conceptual framework for our program of mathematics professional development and research.

## PROFESSIONAL DEVELOPMENT TO FOSTER KNOWLEDGE OF MATHEMATICS FOR TEACHING: THE PROBLEM-SOLVING CYCLE

The PSC model of professional development is designed to enhance teachers' knowledge of mathematics for teaching and improve their instructional practices. Each PSC is a series of three interconnected professional development workshops in which teachers share a common mathematical and pedagogical experience, organized around a rich mathematical task (see Figure 1). This common experience provides a framework upon which teachers can build a supportive community that encourages reflection on mathematical understandings and instructional practices. Throughout the workshops, teachers delve into issues involving mathematical content, pedagogy, and student thinking as they pertain to the selected task. All three workshops emphasize using artifacts of practice to situate teachers' learning opportunities in the context of their everyday work.

During the first workshop of the PSC, teachers collaboratively solve a rich mathematical problem ("the PSC problem") and develop plans for teaching it to their own students. The main goal of this workshop is to help teachers develop the content knowledge necessary for planning and implementing the PSC problem, and the majority of the time in Workshop 1 is spent by teachers doing the problem and debriefing their solution strategies (see Table 1). Teachers also discuss ideas about teaching the PSC problem and develop unique lesson plans that they will implement prior to Workshop 2. We call the framework for this workshop "doing for planning" to highlight the dual focus on mathematics and instructional planning.

After Workshop 1, each participant teaches the PSC problem in one of his or her classes, and the lesson is videotaped. Subsequent workshops focus on the teachers' experiences using the problem in their classrooms and rely heavily on video clips

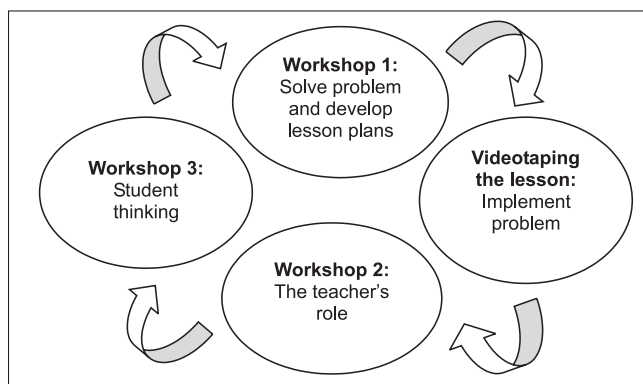


FIGURE 1 The Problem-Solving Cycle model of professional development.

TABLE 1  
Goals, Activities, and Knowledge Foregrounded in Each Workshop  
of the Problem-Solving Cycle (PSC)

	<i>Central goals</i>	<i>Key activities</i>	<i>Knowledge in the foreground</i>
Workshop 1	Develop the content knowledge necessary to teach the PSC problem effectively in the classroom.	1. Solve the PSC problem and debrief solution strategies. 2. Plan to teach the PSC problem to their students.	1. Specialized content knowledge. 2. Pedagogical content knowledge.
Workshop 2	Analyze the role played by the teacher when implementing the PSC problem in the classroom.	Analyze video clips using guiding questions that focus on the role of the teacher.	Pedagogical content knowledge, especially knowledge of content and teaching.
Workshop 3	Analyze student thinking in terms of the mathematics of the PSC problem.	Analyze video clips or student work using guiding questions that focus on the students' mathematical thinking.	Pedagogical content knowledge, especially knowledge of content and students.

and written student work from their lessons. The emphasis is on developing teachers' pedagogical content knowledge, as they consider more about the mathematical concepts and skills entailed in the problem, in conjunction with their role in teaching it and the student thinking that the problem generated. The major focus of Workshop 2 is the role played by the teacher in implementing the problem. Video clips serve as a springboard for exploring topics such as how the teachers introduced the task and managed the classroom discourse. Activities in Workshop 3 center on a critical examination of students' mathematical reasoning. In addition to watching video clips, teachers study their students' written work on the PSC problem and explore, for example, unexpected methods they used to solve the problem and the ways they explained and justified their ideas.

As they participate in successive iterations of the PSC—each focusing on a unique mathematical task and specific issues related to teaching and learning—teachers can gain new insights and continually add to their knowledge base. In particular, teachers have the opportunity to enhance their knowledge in each of the four domains identified by Ball, Thames, and Phelps (2005): (1) common content knowledge, (2) specialized content knowledge, (3) knowledge of content and teaching, and (4) knowledge of content and students.

Teachers use and expand their common content knowledge and specialized content knowledge in all three workshops as they work on the featured mathemati-



cal problem—sharing solution strategies, analyzing similarities and differences, and making connections between various strategies. In Workshop 1, participants draw on their common content knowledge as they consider the skills, procedures, and concepts entailed in the task. They draw on and expand their specialized content knowledge as they compare different solution strategies and discuss how different mathematical representations (e.g., pictures, tables, graphs, and equations) can support these different strategies. They continue to enhance their specialized content knowledge in Workshops 2 and 3 through examining the myriad mathematical strategies their students actually applied to the task, and gain a better understanding of the connections between strategies that are “naïve” and those that are more sophisticated.

Throughout the PSC, teachers tap into and enhance their pedagogical content knowledge—both knowledge of content and teaching, and knowledge of content and students—by planning, analyzing, and reflecting on lessons involving the PSC problem. Teachers use their knowledge of content and teaching during Workshop 1 when they designate mathematical learning goals for their lessons and consider appropriate formats for helping their students achieve those goals. During Workshop 2, participants confront and extend their knowledge of content and teaching as they focus on selected pedagogical moments and critically analyze the role of the teacher in shaping the lesson and fostering student learning. In Workshop 3, the teachers tap into this knowledge when they consider how they would modify their planning and implementation of the featured problem based on new insights gained throughout the PSC.

Participants use their knowledge of content and students during Workshop 1 as they consider their students’ background knowledge and the mathematical reasoning they are likely to apply to the problem, and as they predict what strategies their students might use in their solution attempts. In Workshop 2, the teachers use their knowledge of content and students when they examine how specific teacher moves support or constrain student thinking. In Workshop 3, teachers continue to enhance their knowledge in this domain by examining their students’ conversations and written work to critically analyze their thinking and generate ideas for appropriate pedagogical strategies.

Different knowledge strands are foregrounded in the three workshops of the PSC through the kinds of artifacts that are selected, the activities designed around the artifacts, and the questions posed to frame those activities. As shown in Table 1, Workshop 1 foregrounds teachers’ specialized knowledge of mathematics content and to a lesser extent, their pedagogical content knowledge; Workshop 2 foregrounds knowledge of content and teaching; and Workshop 3 foregrounds knowledge of content and students. However, as is true during moment-to-moment classroom teaching, participants in the PSC model inevitably tap into (and potentially extend) all aspects of their knowledge of mathematics for teaching as they engage in the workshop activities. In addition, because the model is designed to be flexible

and responsive to the needs and interests of participants, during any given iteration of the PSC, the knowledge strands that are foregrounded and backgrounded in a particular workshop may vary.

### IMPLEMENTING THE PROBLEM-SOLVING CYCLE: WITHIN THE STAAR PROFESSIONAL DEVELOPMENT PROGRAM

As part of the STAAR project we worked with middle school mathematics teachers for several years, developing, implementing, and refining our approach to professional development. During the 2003–2004 academic year, we held monthly, full-day professional development workshops with eight teachers. All were middle school mathematics teachers, with classroom experience ranging from 1 to 27 years. They represented six different schools in three school districts within the state. In 2004–2005, seven of the teachers continued working with us, and three additional middle school mathematics teachers joined the program. Each new teacher was a colleague of one of the current participants. During these two years, we conducted three iterations of the PSC. In keeping with the goals of the STAAR project, all three iterations centered on mathematical problems that would support the development of algebraic reasoning for both the participating teachers and their middle school students.

In this section of the article we elaborate the general focus and goals of each of the three workshops that comprise the PSC. We also present a detailed picture of the third iteration of the PSC. We chose the third PSC because it is the most refined version of the PSCs conducted as part of the STAAR program. As such, it provides the clearest illustration of the potential for the PSC model to develop and support teachers' professional knowledge. In our analyses, we point out opportunities for teacher learning and include examples of the growth in knowledge that was evident in some participants. These analyses draw on the variety of data that were collected throughout the STAAR program. We begin by briefly describing the data sources and methods of analysis.

#### Data Collection and Analysis

Our STAAR project team collected several types of data throughout the two years of research and development, in order to capture the processes involved in designing and carrying out the PSC model and its impact on participating teachers. We used multiple cameras to film each professional development workshop in its entirety, and we videotaped whole-group and small-group interactions. Members of the team attended the workshops and kept detailed field notes. We collected all the teachers' written work completed during the workshop (including structured writ-

ten reflections). In addition, we interviewed the facilitators after each workshop to document their goals, intentions, and reflections.

We use vignette analysis to mine particularly “rich pockets” of data that were representative of the third iteration of the PSC. Miles and Huberman (1994) defined a vignette as “a focused description of a series of events taken to be representative, typical, or emblematic” of the data (p. 81). Others have described vignettes as “normative depictions” or “realist tales”; that is, short descriptions intended to reconstruct and authentically represent the events, people, and activities under consideration (Erickson, 1986; LeCompte & Schensul, 1999; Van Maanen, 1988). As a researcher becomes familiar with the data, vignette analysis provides a way of analyzing by creating “characteristic stories.”

The vignette analysis for this study entailed an extensive examination of the entire data set including videotaped records of our third iteration of the PSC, artifacts from each of the workshops, and transcribed facilitator interviews. First, we studied the videotaped records, along with other relevant artifacts. We took detailed notes on teachers’ participation, with respect to the different types of knowledge, and on the ways in which discussions supported their developing knowledge of mathematics for teaching. We used these notes to write brief descriptive summaries of the three PSC workshops. Then we selected teachers to follow throughout each workshop to identify and understand the connections between their participation in the PSC and the development of their knowledge of mathematics for teaching.

Through this process, we identified themes related to the ways that knowledge was developing, and we discussed the themes within our research group. The prevalence of these themes was then verified by returning to the original data sources. This analytical process was helpful in choosing the aspects of each workshop to highlight through the vignettes, and for choosing teachers from the three workshops whose participation was characteristic of the larger group. The vignette analysis is written in present tense in order to help bring the reader into each workshop. Descriptive summaries are set off in italics. Interpretive commentary is interwoven using regular font.

### PROBLEM-SOLVING CYCLE THREE: “SKYSCRAPER WINDOWS”

Our third iteration of the PSC, conducted during spring 2005, centered on the “Skyscraper Windows” problem. This problem, adapted from *Fostering Algebraic Thinking: A Guide for Teachers, Grades 6-10* (Driscoll, 1999), is a relatively sophisticated algebraic problem intended for middle or high school students. It entails determining the cost of washing windows in a skyscraper where the cost per window increases for successive floors of the building (see Appendix A for a

common solution strategy with descriptive commentary). The Skyscraper Windows problem reads as follows:

A building is 12 stories high and is covered entirely by windows on all four sides. Each floor has 38 windows on it. Once a year, all the windows are washed. The cost for washing the windows is \$2.00 for each first-floor window, \$2.50 for each second-floor window, \$3.00 for each third-floor window, and so on. How much will it cost to wash the windows of this building? What if the building is 30 stories tall?  $n$  stories tall? (paraphrased from Driscoll, 1999, p. 70)

Our entire STAAR project team participated in planning the third PSC, which was then carried out by two members of the team. The workshop facilitators, Craig and Kim,<sup>1</sup> were doctoral students in mathematics education. Both had extensive experience as secondary mathematics teachers, as well as experience conducting previous iterations of the PSC and other professional development workshops.

### A Window into Workshop 1 of PSC Three

The vignette below depicts Workshop 1 from our third iteration of the PSC. We elected to focus the vignette largely on a small group of teachers—Laura, Penny, and Ken—as they actively engaged in solving the task. The vignette illustrates the three teachers’ evolving understanding of the mathematics content, including the background knowledge that they brought to the workshop and the knowledge that developed as they thought through and discussed the problem together, over the course of several hours. We also highlight the role that the facilitators played in supporting the development of teachers’ knowledge during the workshop. In addition, the importance of community is evident as the teachers worked together and supported one another throughout the process.

#### Vignette 1: Developing Specialized Content Knowledge

*The teachers gather in a university classroom and reconnect with their fellow colleagues. Their excitement is genuine and conversation centers on mathematics as well as personal issues. Craig begins by welcoming the teachers and setting the agenda for the day.*

*“In today’s workshop we are going to start our third Problem-Solving Cycle. We have split the day into phases. During Phase 1 we are going to integrate doing the mathematics with planning the lesson. So, when you start working*

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<sup>1</sup>Actual names are used for the STAAR project facilitators; pseudonyms are used for the teachers and students.

*on the problem with your small group today, we not only want you to solve the problem but we want you to also think about the way kids might solve the problem and how you might best generate a lesson plan to effectively implement this problem with your students.”*

*Laura, Penny, and Ken are seated around a small table and dive into the Skyscraper Windows problem. Laura and Penny immediately begin working together; while Ken elects to think about the problem independently at first. Penny and Laura explore the problem by making a table and writing a linear equation. Penny starts by explaining, “We have a \$1.50. That would be the zero floor or the y-intercept. And fifty cents is the slope. So the formula would be  $Y = .50x + 1.50$ .” Laura is listening intently and tries out the formula. She exclaims, “That is great! To go right to the equation is living in a perfect world.”*

Here we can see Penny and Laura utilizing their common knowledge of mathematics content to solve the problem. That is, they have brought to the workshop their prior knowledge that a linear equation can be derived from the y-intercept and slope. Once Penny abstracts what these two numbers would be from the story problem, she immediately substitutes them into the specified algorithm. Laura’s excitement over such an equation indicates that she too has common knowledge of how to derive a linear equation and of its potential usefulness in arriving at a solution. However, although the basic linear equation comes easily to Penny and Laura, they do not appear to realize that it provides only the cost of washing one window of a designated floor, rather than the cost of all the windows on each floor or the cost of all the windows in the building, which is the solution to the problem.

*Satisfied for the moment with their solution, Penny and Laura begin discussing issues currently unfolding in their classrooms, involving their students and their substitute teachers. When Ken joins in the conversation again, the three teachers start thinking about how to plan a lesson around the Skyscraper Windows problem based on how they imagine their students will attempt to solve the problem.*

*“Sixth-graders wouldn’t do what we did. What would they do?” wonders Laura.*

*Ken offers, “I think my kids would draw a picture or a table. They would know that the windows would be the same [cost] on each floor. They could put in the floor numbers and the cost and incrementally fill it in. What would eighth graders do?”*

*Laura responds, “My advanced seventh-graders will go to an equation like we did but my eighth-graders would not be able to... I think they may be able to make a table. They would know that the number of windows would be*

*the same each time. They would know the number of floors, and the cost each time. I think some of them would add fifty, then add fifty more, et cetera [identifying the recursive nature of the pattern in the table]. They would see that they are adding fifty cents each time....Gosh, this is going to be a perfect problem for where we are right now because we've been working on creating formulas from a context. I guess if we are going to plan I need to decide if the students could put this into a linear equation."*

This discussion of the mathematical strategies that middle school students would be likely to use to solve the problem engages the teachers' pedagogical content knowledge, especially their knowledge of content and students. They are able to predict how their own sixth-, seventh-, and eighth-graders would approach the problem and how they would move sequentially through a solution strategy (for example, starting with a table, filling in the number of windows on each floor, determining the cost of each floor, and noticing they always add 50 cents to the cost for one window each time they move up a floor).

As the teachers think through the reasoning their students are likely to use, they begin making plans for structuring their lessons. Interestingly, Ken's and Laura's plans correspond directly to the strategies *they* found helpful when solving the problem. For instance, Ken created a table to highlight patterns and to generate ideas as he worked through the problem, and he imagined that this was the most appropriate strategy for his sixth-graders. Although Laura also initially created a table, she largely focused on finding a formula. In thinking about planning, her main concern was about helping her eighth-graders similarly generate a formula.

It is not clear whether Ken and Laura *first* imagined the strategies that their students would be likely to use, and then adopted similar strategies themselves. Or, conversely, whether they predicted students' strategies *after* they solved the problem in the manner with which they were most comfortable. Either way, their initial thoughts about planning a lesson involving the Skyscraper Windows problem enables us to see how their knowledge of content and students could lead into an extended discussion related to pedagogy. However, gaps in these teachers' basic mathematical understanding of the problem limited that discussion, and led them instead to reconsider their solution.

*Kim comes over to Laura, Penny, and Ken and asks, "What did you guys come up with?"*

*Penny responds by giving an example using the 30<sup>th</sup> floor to show how her formula works. She explains, "So if this is the 30th floor, I would take 30 times 50 [cents], plus 1.50. So it should be 16.50 for thirty floors."*

*"So it is only sixteen bucks?" Kim asks in a surprised tone. The group carefully considers this question.*

*Then Laura responds, “So the question is about all the windows on a given floor. But we only found the cost of one window per floor, so we will have to multiply this by 38. So, it is the cost per window times 38.” Laura writes on her paper,  $y = (.50x + 1.50)38$ .*

Kim’s question helps raise important concerns about the accuracy of the group’s thinking and particularly about the linear equation that they derived. The teachers immediately recognize that there are gaps in their solution strategy, and they dig deeper into their content-knowledge base. Although their (corrected) equation produces the cost of a given floor, the problem requires that they determine the cost of the entire building.

*Ken and Penny nod along to suggest that they follow Laura’s thinking. Laura then suggests to Penny that they add a column to their table so that they can list the cost per floor, which will then lead them to the cost of the total building. Ken mentions that he has such a column on his table, and while he hasn’t completed filling it in for each floor, he did find that the cost of the first floor would be \$76.00.*

*“So let’s use a calculator to find the cost for the other floors using the new formula,” suggests Laura.*

*While Ken returns to using his table to calculate the total cost for each successive floor, Laura and Penny talk further about their formula. Laura walks through her derivation of the formula and the process she would use to find the floor costs: “So, since the first difference [between the floors] was constant, I just knew the slope was constant. That meant that the floor is  $x$  and the cost is  $y$  and the slope is 50 cents. We are at the zero floor, so  $y$  equals \$1.50 for this equation, which is the  $y$  intercept. But then we need to multiply the entire thing by 38. So now you wanted to figure out the cost of the 30<sup>th</sup> floor. Since we don’t know the answer to the 30<sup>th</sup> floor from our table, let’s test out a floor that we do have in the table. Let’s find the cost of the 10<sup>th</sup> floor windows to see if we are right.” Penny and Laura soon realize that their new formula gives them the total cost per floor rather than the total cost of a building. At the same time, they notice that they can add up the column of floor costs to arrive at the total building cost, and are inching closer to a final solution.*

*At this point, about an hour has passed since the teachers first began solving this problem and Craig and Kim decide to bring the full group back together to discuss their current progress. The facilitators guide a conversation in which the teachers share their (mostly incomplete) strategies and discuss the Gauss method of adding long sequences of numbers. (Gauss paired numbers in a sequence in such a way that all pairs summed to the same value. Then he multiplied that value by the number of pairs to arrive at*

*the sum of the entire sequence.) The facilitators connect the Gauss method to the mean value times the number of values equaling the sum.*

*After this full-group discussion, Laura, Ken, and Penny resume their work on the problem. Ken slowly uses the patterns in his table to find solutions for the cost of each floor. Penny and Laura are excited about the possibility of using a new strategy to extend their solution. They try to apply the Gauss method in order to generate a formula for the entire building rather than just each single floor. With additional probing and scaffolding from Kim, Laura and Penny recognize the pattern is quadratic and are able to develop a formula. Kim pulls Ken into the discussion to ensure that all three teachers understand the solution.*

*As the workshop draws to a close, Ken demonstrates his understanding by sharing this formula with the whole group. Ken, Laura, and Penny's formula is based on finding the average cost per floor in order to generate the total cost of washing the building. Other teachers share (more conventional) formulas based on finding the average cost per window. The teachers agree that both solution methods would work equally well to arrive at the correct answer.*

Throughout Workshop 1, Laura, Penny, and Ken work hard to engage and expand their content knowledge around this challenging problem. Although they do consider how students might reason through the Skyscraper Windows problem, these teachers are focused primarily on establishing a sufficient base of common content knowledge and specialized content knowledge. The development of specialized content knowledge is evident in the ways they compare, reason, and make connections between the various solution strategies.

### A Window into Workshop 2 of PSC Three

After all participating teachers taught the Skyscraper Windows problem to their own middle school students, they met in Workshop 2 to debrief their experiences and to watch selected video excerpts from each other's lessons. Members of the STAAR project team filmed the teachers' lessons and provided the teachers with DVD copies. The workshop facilitators, along with the entire project team, discussed all the lessons they filmed and considered which clips would be relevant for the teachers to watch and consider together.

The vignette below depicts an extended conversation in Workshop 2 regarding the teachers' implementation of the Skyscraper Windows problem. A large portion of this workshop involved analyzing a video excerpt from Peter's lesson, in which Peter queried a small group of students about their solution method. The STAAR team intended for this video clip to provide a foray into the central theme of the workshop: *teacher questioning*. Viewing and discussing the excerpt pushed the



teachers to consider more deeply how they could play an effective role in teaching the Skyscraper Windows problem, particularly with respect to asking questions that generate student reasoning about the specific mathematical content in the problem. In this vignette, we highlight the expansion of Peter's pedagogical content knowledge and note the extensive full-group analysis of key pedagogical actions and mathematical ideas that supported his learning.

## Vignette 2: Developing Pedagogical Content Knowledge (Content and Teaching)

*Once the teachers have gathered around a large table in a university classroom, Craig initiates a conversation about the implementation of the Skyscraper Windows problem by asking, "What things came up that you weren't expecting?" In response, teachers share different ways that their students approached the problem, including drawing graphs, making tables, and using ratios. Several report running into problems with graphing, especially selecting intervals.*

*Peter comments that he prepared his students for the problem by starting with a different task that required applying similar mathematical strategies in order to arrive at a solution. Peter's goal was for his students to approach the Skyscraper Windows problem with the understanding that they would be looking for patterns, sorting out various ideas, and summarizing. However, Peter notes that he was disappointed because he felt he pushed a particular idea onto his students in a misguided attempt to support their learning. Specifically, he encouraged his students to identify the cost of washing windows on a hypothetical "zero" floor, assuming that would help them find the y-intercept and lead to a simpler derivation of the direct formula. But he found that his good intentions backfired and actually made the problem more difficult for some students.*

*Using Peter's comment as a segue, Craig tells the group that they are going to watch a video clip from Peter's lesson in which he works with a small group of students involving exactly this issue. In the clip, Peter studies the solution method written on Kaitlin's paper, and he poses a number of questions to Kaitlin and her teammates. During the interaction, Peter is focused on the zero floor idea, whereas Kaitlin has another idea in mind. In preparation for watching the video, the facilitators distribute a handout with two questions to guide the teachers' viewing and discussion. Craig reads the discussion questions aloud: (1) How did Peter's questions help him understand how Kaitlin derived her expression? (2) What additional questions would you ask Kaitlin to further understand her mathematical thinking?*

The goal of this portion of the workshop is to investigate the relationship between teacher questioning and student reasoning. In particular, the spotlight is on considering appropriate teacher moves (specifically, teacher questioning) in relation to the mathematics of the Skyscraper Windows problem and students' current levels of thinking about the mathematics.

*The teachers watch a video clip that shows Peter examining an equation written on Kaitlin's paper:  $n \times 19 + 3 \times 19$ . Peter immediately asks if the students can rewrite the equation as beginning with  $19n$ , and the students nod. He then asks why they wrote 3 times 19, and lacking an audible response changes his question to, "What is 3 times 19?", to which they easily respond "57." Peter tries again to have Kaitlin explain how she got 3 times 19. This time she uses her pencil to point to numbers in her table. She shows Peter, "It was this number. So you could add 19 three times." After a few more prods and questions, Kaitlin rewrites her equation as  $19n + 57$ .*

From this clip, it is not entirely clear how Kaitlin arrived at her equation, and particularly, how she determined that it should include 19 times 3, other than using a guess-and-check approach. However, this lack of clarity regarding the student's mathematical reasoning is precisely the issue that the facilitators wanted the teachers to consider. Although Peter asks numerous questions of Kaitlin and her peers, ultimately, the questions do not help her to explain or clarify her thinking. Instead, she changes her equation to represent the solution process that Peter has in mind, specifically the idea that 57 can be thought of as a cost of the zero floor of the building or as the "base" price.

*The teachers watch this short clip several times and discuss it in small groups and then again as a full group. In their discussions, they unpack the mathematics that Kaitlin and her peers seem to have in mind and they also consider Peter's questioning strategies. The teachers carefully study Kaitlin's gestures, and even "freeze" the video on a shot of her finger pointing to numbers in her chart, which represent the costs for washing each individual floor. Eventually they began to understand Kaitlin's mathematical reasoning. Kaitlin knew that the cost per floor was increasing by 19 each time she went up a floor. She tried multiplying a selected floor number by 19 to see if that would generate the cost of that particular floor. She realized, however, that multiplying just the floor number by 19 yields the cost of the floor three floors below her selected floor number. Therefore, she needed to add three more 19s to the product to equal the cost of floor she intended to calculate.*

*One teacher, Kristen, comes to the conclusion that Kaitlin's strategy not only makes mathematical sense, but is an "elegant" way to approach the*

*problem: “I almost think it’s more elegant having the 3 times 19, instead of the 57. It has more meaning. She can actually see on the chart that the cost went up by 19, 3 times.” Other teachers nod in agreement with this statement. The group then expands on Kaitlin’s idea by suggesting that she could have graphed the points, using 19 as an interval, and then confirmed that the cost did increase by 19 each time. Graphing would also provide an additional representation to show why her equation includes 3 times 19.*

As the teachers work together to understand Kaitlin’s thinking, we can see the intersection between their specialized content knowledge, knowledge of content and teaching, and knowledge of content and students. Much of their discussion is focused on what they could do to help Kaitlin and her peers get a better grasp of the meaning behind the concept of “going up by 19, 3 times” in this context. Once the teachers understand why Kaitlin might have decided to write her formula with 3 times 19, they agree that this would have been a prime opportunity to incorporate graphing and related ideas into the lesson, such as how to choose meaningful intervals and how to conceptualize the  $y$ -intercept.

Toward the end of their conversation, Peter becomes increasingly reflective and relates powerful insights about some aspects of his teaching he feels could be improved. He suggests that his interaction with Kaitlin limited, rather than expanded, her thinking about the mathematics in the Skyscraper Windows problem. Peter sees that his preoccupation with conceptualizing 57 as the cost for the zero floor precluded him from understanding and encouraging Kaitlin’s own mathematical reasoning.

*Peter shares, “I would also say, I was trying to force her down toward that bottom thing, now that I have looked at it.” Craig clarifies Peter’s statement by pointing out the equation  $19^n + 57$  which was written on the bottom of Kaitlin’s paper.*

*Peter adds, “The more you guys have talked about it, the  $n$  times 19 plus 3 times 19. I kind of wish I had processed that a little bit more. Or, actually I just kind of wish I had stopped talking for about five seconds and looked at it.”*

*Kim notes, “I felt like you had the 57 in your mind...”*

*Peter responds, “I know I did.”*

*Kim continues, “...and you wanted her to go there...”*

*Peter agrees, “I know I did.”*

*Kim concludes, “...and she had this whole other idea.”*

The facilitators hoped to stretch the teachers’ pedagogical content knowledge in Workshop 2 by encouraging them to critically reflect on this “missed opportunity.” Their intention was to help the teachers become cognizant of the influence

their own ideas about how to solve a problem has on the moves they make in their classroom. In addition, they hoped that teachers would increasingly recognize the importance of supporting the development of students' reasoning by working from their current level of mathematical understanding. In the remainder of the conversation around Peter's video clip, the group continues to point out and unpack the many complexities and subtle nuances involved in this interaction.

*Several teachers in the group reassure Peter that they would have encountered the same difficulties that he did with respect to understanding Kaitlin's thinking in the heat of the moment. Kristen tells Peter, "But Peter, we were the group that kept asking them to replay it, replay it, replay it...[i.e., asking the facilitators to replay the video]." Peter nods and she continues, "So I would not have caught that myself. And it was finally seeing the motion of her pencil drawing that [Kristen gestures to the three groups of 19]. That was the key." Peter nodded and comments, "That's true."*

*Peter then suggests an alternative pedagogical approach that he now feels would have been preferable to the approach he took. He explains, "I was thinking maybe I could take this and kind of erase the  $19n + 57$ . I could almost make Kaitlin's approach into a lesson the next day, for the whole class. Like showing that little part and saying, 'Where did she come up with  $n \times 19 + 3 \times 19$ ? Go.'" Craig responds enthusiastically to this idea and comments, "That's a great strategy, in my opinion, to help kids. Just like we're doing in this workshop. We're trying to figure out what this person is thinking."*

This conversation clearly had a profound impact on Peter. In a reflection completed shortly after the group's discussion of his video he wrote, "The most valuable part of this year [was] watching the mistakes I made in forcing students toward a conclusion on the videotape. I have actively monitored myself since to not repeat this."

### A Window into Workshop 3 of PSC Three

In Workshop 3, the group continued to consider their implementation of the Skyscraper Windows problem, this time with a focus on student thinking. To prepare for the workshop, the facilitators, along with the research team, considered which clips would be relevant for the teachers to watch to foster their pedagogical content knowledge related to content and students. They selected a clip from Laura's lesson that shows a small group of students engaged in the problem. In the clip, one student explains his reasoning to three of his peers, using a solution method unanticipated by any of the teachers (or the facilitators). Our team selected this clip because it involved a creative use of the distributive property and demonstrated sophisticated reasoning about the problem. The facilitators prepared discussion questions that focused the group's attention primarily on the mathematical ideas evident in the video.

As teachers reflected on and discussed what these students seemed to be thinking, they gained insight into the complexities of both the mathematical concepts involved in the Skyscraper Windows task and student learning of those concepts. Although the students' solution method confounded the teachers at first, as they probed deeply into their mathematical ideas, they were able to understand and then extend this line of reasoning. Using the videotaped students' work as a springboard to re-examine the task, the teachers devised and shared multiple strategies that built off the students' numeric calculations and represented increasingly sophisticated and generalized mathematical thinking. They also reconsidered how they would teach the Skyscraper Windows task, given their stronger understanding of how students might approach the task.

### Vignette 3: Developing Pedagogical Content Knowledge (Content and Students)

*Teachers are seated around a table in a university classroom, reminiscing with the facilitators and one another about personal feats and accomplishments from when they were in middle school. Craig formally initiates the workshop by distributing handouts with the text of the Skyscraper Windows problem written out and asks with a smile, "Remember this one?" The teachers laugh as he continues, "Today we are going to watch a clip from Laura's classroom that focuses on a group of students solving the problem. They were trying to find a way to figure out the total cost for an eight-floor building." Craig notes that Laura modified the problem slightly for her students, using an 8-story building rather than a 12-story building. Laura explains, "I just thought 12 stories would scare them too much. They seem to get scared over those double digits."*

*Craig passes around another handout with three questions for the teachers to consider as they watch the video, and reads them aloud: (1) What are the students doing mathematically? (2) What mathematical background do the students appear to draw on? (3) What new ideas does this clip give you about teaching this problem? As the video begins to play, the teachers become quiet and concentrate on the clip. After it is over, they move into three small groups, eager to talk about the interactions they just watched. Craig encourages the teachers to work out the mathematics for themselves, acknowledging, "That's certainly what I needed to do." The small groups all have laptop computers so they can view the clip, or selected portions, as many times as they want. Laura distributes copies of the written work she collected from the videotaped students.*

*One small group is composed of Celia, Nancy, and Peter. Celia initiates the conversation by noting her lack of understanding of the student thinking: "I don't understand what the boy is talking about."*

*"I don't understand either," Nancy agrees.*

*Peter offers, "Well I now think they were thinking about the order of operations."*

*"They also talked about finding patterns," Celia adds.*

*"Yes, they first did computations and they recognized some patterns," says Peter.*

*"But where did the 30 come from?" wonders Nancy.*

The conversation begins with the teachers thinking aloud and making concrete observations of the students work, and provides them with a starting place to explore student thinking. This interaction highlights their struggle to apply their mathematical content knowledge to the classroom context, and their need to obtain a more sophisticated mathematical understanding of the problem.

*Kim joins the conversation and poses several questions in an attempt to help extend the teachers' discussion, "So, the kids took 30 times 38. Where did they get those numbers? Did their strategy work? Does it give them the right answer?"*

Kim's questions help to steer the group in the intended direction. Kim picks up on and reframes Nancy's question about how the student came up with the number 30, and encourages a focused look at some of the specific mathematical components of the students' solution method. Her goal is to help the teachers make sense of students' computations that differ from their own solution processes. The group then engages in some of the same calculations they imagine the students must have done, and they discuss if and why these computations are accurate.

*After a short time Peter exclaims, "It does work!" referring to the numeric computation of 30 times 38. He then wonders, "Why does it work?" After a short pause while the teachers think about his question, Peter generates a hypothesis about the origin of the students' method (see Figure 2). "I think this works because 38 times 30 is the same as 38 times [2.00 + 2.50 + 3.00 + 3.50 + 4.00 + 4.50 + 5.00 + 5.50]. Does that make sense?"*

*Nancy agrees, "Yes, because if you took 38 times 2.00 you would get the first floor and if you took 38 times [2.00 + 2.50] then you would get the price of first and second floor. This is what the kids were doing. They were adding the string of numbers [2.00 + 2.50 + ... 5.50], which equaled thirty. And then they multiplied it by 38 to get the total for an eight-story building."*

*Nancy's comments lead Peter to abstract the strategy further by identifying the underlying mathematics, "It is a form of the distributive property. Let's try it out for a six-story building and see if it was just a coincidence."*

FIGURE 2 Student's idea for the total cost of washing an 8-story building.

$$\begin{array}{r}
 5.50 \text{ floor 8} \\
 5.00 \\
 4.50 \text{ } (\$30.00) (38 \text{ windows}) \\
 4.00 \text{ } = \$1,140 \\
 3.50 \\
 3.00 \\
 2.50 \\
 \underline{2.00 \text{ floor 1}} \\
 \$30.00
 \end{array}$$

*As the teachers work on calculating the price for a six-story building they begin to notice patterns. Peter, for instance, relates this strategy to an idea that the teachers discussed in Workshop 1. "Remember when we used the Gaussian method to find the average of large strings of numbers? I wonder if there is a way to come up with a formula that is simpler?"*

*"I think it is interesting that instead of coming up with a price per window like we did, they skipped that step and just found the total cost of the building," Celia adds.*

This conversation indicates that Celia, Nancy, and Peter have made substantial progress in their understanding of these students' thinking, and are beginning to explore a novel way to solve the problem by connecting to their prior understanding of the problem and attempting to extend the students' numeric strategy to a symbolic direct formula.

*The facilitators continue to push the small groups of teachers to work through the students' mathematical ideas and then to extend the ideas in order to find the cost of washing a building with "n" number of floors. After the teachers talk a while longer in small groups, the facilitators decide to hold a whole-group discussion and help the teachers come to firmer conclusions about the mathematics.*

*Ken begins by describing some of the ideas his small group generated for a six-story building, building on the videotaped students' thinking. Specifically, they started with the idea of adding  $[2.0 + 2.5 + 3.0 + 3.5 + 4 + 4.5]$ . Then Ken's group noticed that they could pair these six numbers together to form three groups of 6.5. Next they saw that they could multiply 3 by 6.5 to get 19.50, the cost of washing one window per floor (See Figure 3). Craig follows up on this idea by applying the same procedure to determine the cost of washing one window per floor for an eight-story building. Afterwards, Laura makes a suggestion that pushes the group's thinking even further.*

$$\begin{aligned}
 &2.00 + 2.50 + 3.00 + 3.50 + 4.00 + 4.50 \\
 &= 3 \cdot (6.50) \\
 &= 19.50
 \end{aligned}$$

FIGURE 3 Ken's method of addition by grouping.

Laura suggests, "Let's find the cost of all the windows on a five-story building."

Craig questions, "Why a five-story building?"

Laura explains, "Because it is an odd number of floors and all the others have been even, with even pairs."

Peter agrees to take on this challenge and comes to the front of the room to share his groups' ideas. He shows how his group first generated a simplified method for calculating a sum for buildings with an even number of floors because, as Laura alluded, they were easy to pair up. Peter explains, while drawing a table on the board, "We made a table. We started with the even numbers and then we were able to see the patterns to figure out what the odd numbers would look like."

Craig and Kim help to clarify and connect Peter and Ken's ideas. Craig notes, "If you look at a five-story building, you would add  $[2 + 2.5 + 3 + 3.5 + 4]$ . Using Ken's method, you can see how 6.00 is the cost per pair. And in this case there are 2.5 pairs."

Kim adds, "You can see that if you play around with pairing the numbers you can find what to multiply by. For odd numbers, you multiply by half of a pair. Peter's group discovered the same thing by identifying a pattern."

Ken interjects, "But in order to find the solution we still are adding down the column in order to figure out the total cost."

Kristen responds, "No, we have found a formula. If you take the cost per pair times the number of pairs, then you will get the total cost of one window on each floor. Then you multiply by 38."

Building on Kristen's explanation, the teachers agree on an expression to find the total cost of washing one window per floor: (the cost of a pair)  $\times$  (the number of pairs). They note that this product would have to be multiplied by 38 to find the cost of washing all the windows. At this point, the teachers realize they need mathematical expressions for "the cost of a pair" and "the number of pairs." After discussions among the teachers they agree that "the cost of a pair" is  $[2 + (.50n + 1.50)]$ . This represents the cost of one window on the first floor (\$2.00) and the cost of one window on the  $n$ th floor  $(.50n + 1.50)$ . Recall that the teachers developed the linear expression  $(.50n + 1.50)$  during Workshop 1. The teachers also agree that the "number of pairs" for any building is  $n/2$ , where  $n$  is the number of floors. Thus the generalized ex-



pression for the cost of washing the windows of an  $n$  story building can be written as:  $(38) \bullet [2 + (.50n + 1.50)] \bullet (n/2)$ .

At this point in the workshop, the teachers have spent several hours re-considering their conceptual understanding of the Skyscraper Windows problem in light of the student reasoning they saw taking place in Laura's lesson. The facilitators, along with the teachers, remark on the depth of their continued exploration into the algebraic concepts in one problem, even after two previous workshops focused on the same problem. To link their expanded content knowledge with pedagogical issues, Kim and Craig encourage the teachers to discuss how they would now teach a lesson involving this problem. Several teachers comment that these detailed investigations into the mathematical content of the problem, particularly through the lens of student thinking, motivated them to modify their instructional plans and approaches to the problem.

Kristen leads off the discussion by saying, "I am thinking instructionally now. The week before I do this lesson again, I could teach the students how to add a string of numbers using the Gaussian approach or the pairing method. Then, they would have a strategy to use for problems like this one. I think it is important to give them the tools they need to understand problems."

Kim capitalizes on this segue and suggests, "So what you are doing, Kristen, is rethinking your instructional strategy within your curriculum, right?"

Craig adds, "You just said 'give them the tools.' But you, the teacher, have to have the tools. So how did you get them?"

Kristen responds, "Well, you gave them to us. During Workshop 1 you told us about Gauss."

Kim challenges this response by noting, "We tried to push your mathematical thinking in Workshop 1, and strategies such as averaging and the Gauss method were mentioned. But for most of you, you were not going there. In this workshop you studied the students' ideas in the video. You used their idea of adding  $[2.00 + 2.50 + 3.00 + \dots + 6.50]$ . Initially when we work out a math problem, we might see the mathematics involved only to a certain degree. Then we give the problem to a bunch of kids and we see a lot more strategies we haven't even thought about."

Kristen shares what seems to be a common feeling among the group, "Their thinking, obviously, was just as good as ours."

Kim continues, "Now that we realize that it is students' thinking that brought us here, what do we do? Can we rethink our instructional strategies for using this problem?"

*Ken offers, "I will be able to help kids more because I now understand these two ways to solve the problem. I understand how one connects to the other. So I would probably ask better questions."*

*Laura adds, "Today was the first day that really solidified my understanding of this problem. We've looked at it for three days [i.e., three workshops] and I'm finally like, 'Oh gosh. Now I get it.' I think having that knowledge will help me to understand where the kids are going next time I teach it, and to understand their thinking a little bit better."*

As this conversation suggests, the teachers found the process of exploring student thinking extremely helpful. Rather than tiring of the continued examination of the Skyscraper Windows problem, they gained new understandings that they planned to take back to their classrooms, particularly when they might teach this problem or related lessons again. There was general agreement among the group that these new understandings would assist them in future decision-making processes, including how to prepare their students for the problem and how to capitalize on their students' developing ideas.

## CONCLUSION

The knowledge needed for teaching mathematics in accordance with the vision of classrooms portrayed in *Principles and Standards* (NCTM, 2000) is multifaceted and complex. The PSC professional development model is designed to provide teachers with the opportunity to expand on their existing knowledge base through the exploration and teaching of specific mathematical tasks in ways that are intended to inform their classroom practices and impact student achievement. More specifically, the PSC model is intended to help teachers increase all strands of their knowledge of mathematics for teaching: (1) common content knowledge, (2) specialized content knowledge, (3) knowledge of content and teaching, and (4) knowledge of content and students. The series of PSC workshops reflects the interconnected nature of these four strands of knowledge, as well as the unique aspects of each. Our central goal in this article was to highlight the ways in which specific strands are foregrounded during each of the three PSC workshops, while also demonstrating their interconnectedness.

In Workshop 1, specialized content knowledge is foregrounded, as teachers solve the selected mathematical task and prepare to teach the problem in their classroom. In Workshop 1 of the third PSC conducted as part of the STAAR project, the teachers tapped into their knowledge of linear equations as they worked on the Skyscraper Windows problem and considered multiple ways to represent the cost of washing the windows on each floor. They built on their pedagogical content

knowledge as they thought through the ways in which middle school students might solve the problem, and how to best structure the presentation of the problem.

In Workshop 2, pedagogical content knowledge is foregrounded, especially teachers' knowledge of content and teaching. In the third PSC conducted as part of the STAAR project, the teachers used this knowledge to examine the questions Peter posed to Kaitlin as she solved the problem. A detailed analysis of Peter's questioning supported the teachers' developing knowledge about how to plan, enact, and critically reflect on lessons, taking into account what instructional moves were likely to be most effective within a specific mathematical domain and for specific groups of students.

Another aspect of pedagogical content knowledge—knowledge of content and students—is foregrounded in Workshop 3. Specifically, in the third PSC conducted as part of the STAAR project, the teachers critically reflected on the way a group of students in Laura's class approached the Skyscraper Windows problem. As they embarked on an extended conversation about the nature of these students' thinking, the teachers gained an increasingly sophisticated understanding of the mathematical constructs embedded in the problem. They also considered how they might teach the problem differently to better support the variety of approaches students might use when solving it.

Building teachers' professional knowledge lies at the heart of the success of the mathematics reform movement. When teachers effectively engage and draw from multiple knowledge domains in the planning, implementation, and reflection stages of their classroom teaching, they are likely to make more-informed instructional decisions and produce more-capable students. It is this complex knowledge base that professional development programs must address. At the same time, it is important to bear in mind that the development of knowledge is a gradual process—one that requires a delicate balance between supporting teachers' current knowledge and challenging them to gain new understandings that will serve as motivation to make incremental changes to their classroom practices. The PSC holds promise as one way to support teacher learning in all domains, challenge their mathematical understandings, and support gradual change by situating learning in participants' classrooms through the use of artifacts of practice and sharing knowledge within a professional community. While we cannot provide evidence that all participants in the STAAR program engaged all aspects of their knowledge in all of the workshops, our analyses—as presented in the three vignettes—suggest that they had the opportunity to do so.

We consider the PSC model to be complementary to other professional development efforts with similar goals of helping teachers match the vision for school mathematics presented in *Principles and Standards* (NCTM, 2000). In fact, participation in multiple programs with different emphases offers teachers a variety of routes to gain professional knowledge. Derry, Wilsman, and Hackbarth described one such program in their article (this issue). These two professional development

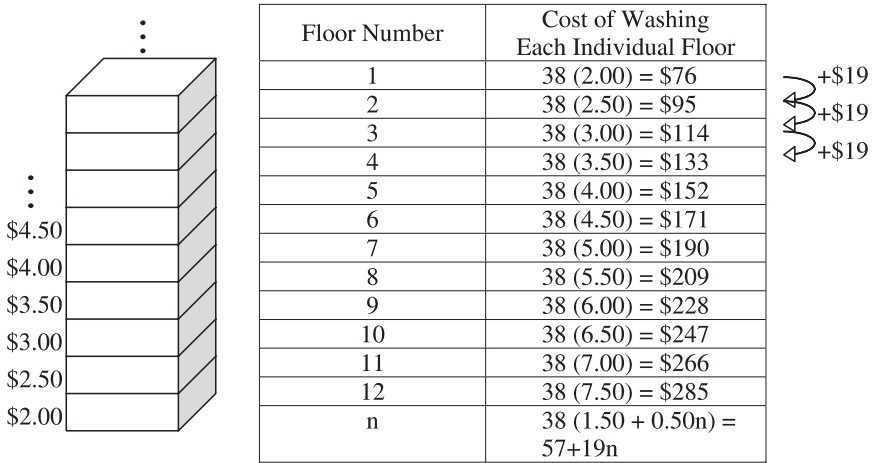
programs, like most others for mathematics teachers, are still in their infancy with respect to research data on their enactment and impact. We are deeply grateful to the teachers who are participating in such programs, and view them as partners in the dual effort to help us implement and improve these professional development models while also striving to further their own knowledge base and improve their practices.

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APPENDIX A  
A Solution Strategy for the “Skyscraper Windows” Problem



To find the total cost for washing a 12-story building, one strategy is to use straightforward arithmetic. Multiply 38 windows by the sum of the cost of washing an individual window on floors 1 through 12.

$$\begin{aligned}
 &= 38 (2.00 + 2.50 + 3.00 + 3.50 + 4.00 + 4.50 + 5.00 + 5.50 + 6.00 + 6.50 + 7.00 \\
 &\quad + 7.50) \\
 &= \$2166
 \end{aligned}$$

Reorganizing the 12 window costs by pairing the 1st floor with the 12th floor, the 2nd floor with the 11th, etc.) allows for the beginning of a more generalizable approach.

$$\begin{aligned}
 &= 38 [(2.00 + 7.50) + (2.50 + 7.00) + (3.00 + 6.50) + (3.50 + 6.00) + (4.00 + \\
 &\quad 5.50) + (4.50 + 5.00)] \\
 &= 38 [9.50 + 9.50 + 9.50 + 9.50 + 9.50 + 9.50] \\
 &= 38 [9.50] \bullet 6 \\
 &= \$2166
 \end{aligned}$$

We can use a straightforward arithmetic approach to find the total cost for washing an  $n$ -story building. However, we need to include the cost of washing an individual window on the last floor (i.e., floor  $n$ ). The cost of floor  $n$  can be represented as  $(1.50+0.50n)$ , where 1.50 is the cost of a hypothetical “floor 0” and 0.50 is the cost increase per floor.

$$= 38 [2.00 + 2.50 + 3.00 + \dots + (1.50 + 0.50n)]$$

We can also use a pairing approach: 38 windows multiplied by the sum of a matched pair (e.g., floor 1 and floor  $n$ ), which is then multiplied by the number of pairs in an  $n$ -story building.

$$= 38 [2.00 + (1.50 + 0.50n)] \bullet \frac{n}{2}$$

This simplifies to a quadratic expression, representing the total cost of washing the windows of an  $n$ -story building.

$$\begin{aligned} &= 19n [3.50 + 0.50n] \\ &= 66.50n + 9.5n^2 \end{aligned}$$

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