2005-2006 年度第二学期拓扑学评分细则（试卷三）

第 1 题 State the definition of a topology. .... (10 points)

Proof of (ii). By the assumption, for every $x \in K$, there is an open set U_x such that $x \in U_x \subset K$. This implies that $K = \bigcup_{x\in K} U_x$, that is, K is the union of open sets $\{U_x : x \in K\}$ and hence is open. .... (5 points)

第 2 题 If W is open in Z, since $h : Y \to Z$ is continuous, it follows that $h^{-1}(W)$ is open in Y. (10 points)

Since $f : X \to Y$ is continuous, it follows that $f^{-1}(h^{-1}(W))$ is open in X. Noting that $(h \circ f)^{-1}(W) = f^{-1}(h^{-1}(W))$, ... (4 points)

we obtain that $(h \circ f)^{-1}(W)$ is open in X, so $h \circ f$ is continuous. .... (1 point)

第 3 题 Set $T := \bigcup_{\alpha\in\Lambda} K_\alpha$. If T is not connected, then there is a separation $T = C \cup D$, ($\emptyset \neq C \subset T$, $\emptyset \neq D \subset T$, C and D are open in T, and $C \cap D = \emptyset$) .... (5 points)

Since K_β is connected, K_β is contained in either C or D. .... (5 points)

Without loss of generality, suppose that $K_\beta \subset C$. For every $\alpha \in \Lambda$, since K_α is connected and $K_\alpha \cap K_\beta \neq \emptyset$, it follows that $K_\alpha \subset C$. .... (5 points)

This implies that $T = \bigcup_{\alpha\in\Lambda} K_\alpha \subset C$, contradicting the fact that D should be nonempty. . (5 points)

第 4 题 For every $y \in Y$, $y \neq x$, so there are neighborhoods U_y of x and V_y of y such that $U_y \cap V_y = \emptyset$. (5 points)

Since $Y \subset \bigcup_{y\in Y} V_y$ and Y is compact, there is a finite set $\{y_1, y_2, \ldots, y_n\}$ such that $Y \subset \bigcup_{i=1}^{n} V_{y_i}$. .... (10 points)

Take $U := \bigcap_{i=1}^{n} U_{y_i}$. Then U is a neighborhood of x and $U \cap V = \emptyset$. .... (5 points)

第 5 题 (i) By the definition of metric, $x \in B(x, 1)$ for every $x \in X$. If $x \in B(y, r) \cap B(z, t)$ where $y, z \in X$ and $r, t > 0$, then $d(x, y) < r$ and $d(x, z) < t$. Take $\delta := \min\{r - d(x, y), t - d(x, z)\}$. Then $\delta > 0$ and $x \in B(x, \delta) \subset B(y, r) \cap B(z, t)$. This verifies (i). ....(10 points)

(ii) If $x_n \to x$, then for any $\varepsilon > 0$, since $B(x, \varepsilon)$ is a neighborhood of x, there is N such that when $n > N$, $x_n \in B(x, \varepsilon)$, that is, $d(x_n, x) < \varepsilon$. (5 points)

Conversely, let U be a neighborhood of x, that is, U is open and $x \in U$. So $B(x, r) \subset U$ for some $r > 0$. By the assumption, there is N such that when $n > N$, $d(x_n, x) < r$, so $x_n \in B(x, r) \subset U$. This shows that $x_n \to x$. .... (5 points)

(iii) If $x \in \bar{K}$, then for any n, there is $x_n \in B(x, 1/n) \cap K \neq \emptyset$. For every neighborhood U of x, there is $r > 0$ such that $B(x, r) \subset U$. Since $1/n \to 0$, there is N such that when $n > N$, $x_n \in B(x, 1/n) \subset B(x, r) \subset U$. Therefore, $x_n \to x$. .... (5 points)

Conversely, if there is a sequence $\{x_n\} \subset K$ such that $x_n \to x$, then for every neighborhood U of x, there is N such that $x_N \in U$ and hence $x_N \in K \cap U \neq \emptyset$. This implies that $x \in \bar{K}$.(5 points)