2005-2006 年度第二学期拓扑学评分细则（试卷一）

第 1 题 This set is not open, because all open intervals form a basis for the standard topology on R and this set contains no open interval. For example, there is no open interval (a, b) contained in the set such that $1/2 \in (a, b)$ .... (7 points)

This set is not closed, because a set is closed if and only if it contains all its limit points. Here 0 is a limit point of the set but 0 itself does not belong to the set. .... (8 points)

第 2 题 The topology K inherits as a subspace of Y is the collection $\tau_1 := \{V \cap K : V \text{ is open in } Y\}$. (8 points)

Since V is open in Y if and only if $V = U \cap Y$ for some open subset U of X, $\tau_1 = \{U \cap Y \cap K : U \text{ is open in } X\} = \{U \cap K : U \text{ is open in } X\}$. .... (7 points)

第 3 题 Suppose that $f(X)$ is not connected. Then there is a separation $f(X) = C \cup D$ ($\emptyset \neq C \subset f(X)$, $\emptyset \neq D \subset f(X)$, C and D are open in $f(X)$, and $C \cap D = \emptyset$). So $X = f^{-1}(C) \cup f^{-1}(D)$. . (5 points)

Since C is open in $f(X)$, there is an open set U of Y such that $C = U \cap f(X)$, so $f^{-1}(C)$ equals $f^{-1}(U)$ which is an open set as f is continuous. Similarly, $f^{-1}(D)$ is open in X. (10 points)

Since $f^{-1}(C) \cap f^{-1}(D)$ equals $f^{-1}(C \cap D)$ which is empty, it follows that $f^{-1}(C)$ and $f^{-1}(D)$ constitute a separation of X, contradicting to the assumption that X is connected. ......(5 points)

第 4 题 (i) By the definition of metric, $x \in B(x, 1)$ for every $x \in X$. If $x \in B(y, r) \cap B(z, t)$ where $y, z \in X$ and $r, t > 0$, then $d(x, y) < r$ and $d(x, z) < t$. Take $\delta := \min\{r - d(x, y), t - d(x, z)\}$. Then $\delta > 0$ and $x \in B(x, \delta) \subset B(y, r) \cap B(z, t)$. This verifies (i). ....(10 points)

(ii) If $x_n \to x$, then for any $\varepsilon > 0$, since $B(x, \varepsilon)$ is a neighborhood of x, there is N such that when $n > N$, $x_n \in B(x, \varepsilon)$, that is, $d(x_n, x) < \varepsilon$. (5 points)

Conversely, let U be a neighborhood of x, that is, U is open and $x \in U$. So $B(x, r) \subset U$ for some $r > 0$. By the assumption, there is N such that when $n > N$, $d(x_n, x) < r$, so $x_n \in B(x, r) \subset U$. This shows that $x_n \to x$. .... (5 points)

(iii) If $x \in \bar{K}$, then for any n, there is $x_n \in B(x, 1/n) \cap K \neq \emptyset$. For every neighborhood U of x, there is $r > 0$ such that $B(x, r) \subset U$. Since $1/n \to 0$, there is N such that when $n > N$, $x_n \in B(x, 1/n) \subset B(x, r) \subset U$. Therefore, $x_n \to x$. .... (5 points)

Conversely, if there is a sequence $\{x_n\} \subset K$ such that $x_n \to x$, then for every neighborhood U of x, there is N such that $x_N \in U$ and hence $x_N \in K \cap U \neq \emptyset$. This implies that $x \in \bar{K}$.(5 points)

第 5 题 For every $y \in Y$, $y \neq x$, so there are neighborhoods U_y of x and V_y of y such that $U_y \cap V_y = \emptyset$. (5 points)

Since $Y \subset \bigcup_{y \in Y} V_y$ and Y is compact, there is a finite set $\{y_1, y_2, \ldots, y_n\}$ such that $Y \subset \bigcup_{i=1}^n V_{y_i}$. .... (10 points)

Take $U := \bigcap_{i=1}^n U_{y_i}$. Then U is a neighborhood of x and $U \cap V = \emptyset$. .... (5 points)