

Topology

by Munkres James

何诣然

7-405, Sichuan Normal University

2nd semester, 2010-2011

- 2 Lecture 2
 - Topological Spaces and Continuous Functions
 - Basis for a Topology
 - Metric and topology
 - Subbasis
 - Compare different topologies

- 3 Lecture 3
 - Prelude
 - The Subspace Topology

- 4 Lecture 4
 - Closed Sets
 - Closure

- 5 Lecture 5
 - Closure (continued)
 - Limit point
 - Interior

- 6 Lecture 6
 - Boundary
 - Convergence
 - Hausdorff Space

- 7 Lecture 7

- 8 Lecture 8
 - Continuous function: other equivalent statements
 - Continuous function: rules
 - Continuous \leftrightarrow subspace

- 9 Lecture 9
 - Continuous function (continued)
 - Continuous \leftrightarrow product
 - Homeomorphism
 - Metric space
 - Metric space \leftrightarrow basis, open set, topology
 - Metric space \leftrightarrow product, subspace

- Metric space \leftrightarrow convergence, closure

10 Lecture 10: metric space (continued)

- Continuity
- Hausdorff
- New things

11 Lecture 11

- Separation
- Separation \leftrightarrow subspace
- Connected space
- Properties of connected space
- Connected \leftrightarrow subspace
- Connected \leftrightarrow continuous
- Connected \leftrightarrow product
- Connected \leftrightarrow closure, boundary
- Intermediate value theorem
- Path connected

- Connected component

12 Lecture 12

- Compact
- Compact \leftrightarrow subspace
- Compact \leftrightarrow Hausdorff
- Compact \leftrightarrow continuous
- Compact \leftrightarrow product
- Compact \leftrightarrow connected
- Finite intersection property
- Compact sets in \mathbb{R}^n
- Extreme value theorem
- Compact \leftrightarrow metric space
 - Lebesgue number lemma
 - Equivalent characterization of compactness

13 Lecture 13

- Countability axioms
- Separation axioms

- Regular space

Definition 13.26

Let Y be a topological space and $A \subset Y$. $y \in A$ is said to be an **isolated point** of A , if there is $V \in \mathcal{N}_Y(y)$, $V \cap A = \{y\}$.

Proposition 13.27

Let Y be a topological space. y is an isolated point of $Y \iff \{y\}$ is open in Y .

Theorem 13.28

Let X be a compact Hausdorff space. If X has no isolated points, then it is uncountable.

Example 13.29

(1) $X := \{a, b, c\}$. $\tau := \{\emptyset, X, \{1, 2\}\}$. (2) \mathbb{R} .

Definition 13.26

Let Y be a topological space and $A \subset Y$. $y \in A$ is said to be an **isolated point** of A , if there is $V \in \mathcal{N}_Y(y)$, $V \cap A = \{y\}$.

Proposition 13.27

Let Y be a topological space. y is an isolated point of $Y \iff \{y\}$ is open in Y .

Theorem 13.28

Let X be a compact Hausdorff space. If X has no isolated points, then it is uncountable.

Example 13.29

(1) $X := \{a, b, c\}$. $\tau := \{\emptyset, X, \{1, 2\}\}$. (2) \mathbb{R} .

Definition 13.26

Let Y be a topological space and $A \subset Y$. $y \in A$ is said to be an **isolated point** of A , if there is $V \in \mathcal{N}_Y(y)$, $V \cap A = \{y\}$.

Proposition 13.27

Let Y be a topological space. y is an isolated point of $Y \iff \{y\}$ is open in Y .

Theorem 13.28

Let X be a compact Hausdorff space. If X has no isolated points, then it is uncountable.

Example 13.29

(1) $X := \{a, b, c\}$, $\tau := \{\emptyset, X, \{1, 2\}\}$. (2) \mathbb{R} .

Definition 13.26

Let Y be a topological space and $A \subset Y$. $y \in A$ is said to be an **isolated point** of A , if there is $V \in \mathcal{N}_Y(y)$, $V \cap A = \{y\}$.

Proposition 13.27

Let Y be a topological space. y is an isolated point of $Y \iff \{y\}$ is open in Y .

Theorem 13.28

Let X be a compact Hausdorff space. If X has no isolated points, then it is uncountable.

Example 13.29

(1) $X := \{a, b, c\}$. $\tau := \{\emptyset, X, \{1, 2\}\}$. (2) \mathbb{R} .

Theorem 13.30

Let X be a metric space. Then the following are equivalent:

- (1) X is compact.
- (2) For every infinite subset $A \subset X$, $A' \neq \emptyset$. (limit point compact)
- (3) Every sequence $\{x_n\} \subset X$ has a convergent subsequence. (sequentially compact)

思路.

(1) \Rightarrow (2). If $A' = \emptyset$, then A is closed, and hence compact in X . For every $x \in A$, $\exists U \in \mathcal{N}(x)$ such that $U \cap A = \{x\}$. Compactness enters here...

(2) \Rightarrow (3).

(3) \Rightarrow (1). Omitted. □

Theorem 13.30

Let X be a metric space. Then the following are equivalent:

- (1) X is compact.
- (2) For every infinite subset $A \subset X$, $A' \neq \emptyset$. (*limit point compact*)
- (3) Every sequence $\{x_n\} \subset X$ has a convergent subsequence. (*sequentially compact*)

思路.

(1) \Rightarrow (2). If $A' = \emptyset$, then A is closed, and hence compact in X . For every $x \in A$, $\exists U \in \mathcal{N}(x)$ such that $U \cap A = \{x\}$. Compactness enters here...

(2) \Rightarrow (3).

(3) \Rightarrow (1). Omitted. □

Theorem 13.30

Let X be a metric space. Then the following are equivalent:

- (1) X is compact.
- (2) For every infinite subset $A \subset X$, $A' \neq \emptyset$. (limit point compact)
- (3) Every sequence $\{x_n\} \subset X$ has a convergent subsequence. (sequentially compact)

思路.

(1) \Rightarrow (2). If $A' = \emptyset$, then A is closed, and hence compact in X . For every $x \in A$, $\exists U \in \mathcal{N}(x)$ such that $U \cap A = \{x\}$. Compactness enters here...

(2) \Rightarrow (3).

(3) \Rightarrow (1). Omitted. □

Outline

- 12 Lecture 13
 - Countability axioms
 - Separation axioms
 - Regular space

Definition 14.1

A topological space X is said to have a **countable basis at x** if there is a countable collection $\mathcal{B} \subset \mathcal{N}(x)$ such that

$$\forall V \in \mathcal{N}(x), \exists U \in \mathcal{B} \text{ such that } x \in U \subset V.$$

A space that has a countable basis at each of its points is said to satisfy the **first countable axiom**, or to be **first-countable**.

Theorem 14.2 (see Theorem 8.15)

Let X be a topological space. If there exists a sequence $\{x_n\} \subset A$ such that $x_n \rightarrow x$, then $x \in \overline{A}$. The converse holds if X is *first-countable*.

Corollary 14.3

Let A be a subset in a *first-countable* space. A is closed \iff For any sequence $\{x_n\} \subset A$, if $x_n \rightarrow x$, then $x \in A$.

Theorem 14.4

Let X and Y be topology spaces. If f is continuous at x_0 , then $x_n \rightarrow x_0$ implies $f(x_n) \rightarrow f(x_0)$. The converse holds if X is *first-countable*.

Definition 14.5

If a topological space X has a **countable basis** for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

Definition 14.6

$A \subset X$ is said to be **dense** in X if $\bar{A} = X$.

Theorem 14.7

If X is second-countable, then there exist a countable subset of X that is dense in X .

The latter is just the definition of a **separable space**.
second-countable \Rightarrow separable.

Definition 14.5

If a topological space X has a **countable basis** for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

Definition 14.6

$A \subset X$ is said to be **dense** in X if $\bar{A} = X$.

Theorem 14.7

If X is second-countable, then there exist a countable subset of X that is dense in X .

The latter is just the definition of a **separable space**.
second-countable \Rightarrow separable.

Definition 14.5

If a topological space X has a **countable basis** for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

Definition 14.6

$A \subset X$ is said to be **dense** in X if $\bar{A} = X$.

Theorem 14.7

If X is second-countable, then there exist a countable subset of X that is dense in X .

The latter is just the definition of a **separable space**.
second-countable \Rightarrow **separable**.

Example 14.8

\mathbb{R}_ℓ is separable, but not second-countable.

Proof.

Let \mathcal{B} be a basis for \mathbb{R}_ℓ . For every $x \in \mathbb{R}_\ell$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subset [x, x+1)$. Since $x = \inf B_x$, if $x \neq y$, then $B_x \neq B_y$. Therefore, \mathcal{B} is uncountable. \square

Proposition 14.9

Separable metric space is second-countable.

Corollary 14.10

\mathbb{R}_ℓ is not metrizable.

(千呼万唤始出来, 没有半遮面)

Example 14.8

\mathbb{R}_ℓ is separable, but not second-countable.

Proof.

Let \mathcal{B} be a basis for \mathbb{R}_ℓ . For every $x \in \mathbb{R}_\ell$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subset [x, x + 1)$. Since $x = \inf B_x$, if $x \neq y$, then $B_x \neq B_y$. Therefore, \mathcal{B} is uncountable. \square

Proposition 14.9

Separable metric space is second-countable.

Corollary 14.10

\mathbb{R}_ℓ is not metrizable.

(千呼万唤始出来, 没有半遮面)

Example 14.8

\mathbb{R}_ℓ is separable, but not second-countable.

Proof.

Let \mathcal{B} be a basis for \mathbb{R}_ℓ . For every $x \in \mathbb{R}_\ell$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subset [x, x + 1)$. Since $x = \inf B_x$, if $x \neq y$, then $B_x \neq B_y$. Therefore, \mathcal{B} is uncountable. \square

Proposition 14.9

Separable metric space is second-countable.

Corollary 14.10

\mathbb{R}_ℓ is not metrizable.

(千呼万唤始出来, 没有半遮面)

Example 14.8

\mathbb{R}_ℓ is separable, but not second-countable.

Proof.

Let \mathcal{B} be a basis for \mathbb{R}_ℓ . For every $x \in \mathbb{R}_\ell$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subset [x, x + 1)$. Since $x = \inf B_x$, if $x \neq y$, then $B_x \neq B_y$. Therefore, \mathcal{B} is uncountable. \square

Proposition 14.9

Separable metric space is second-countable.

Corollary 14.10

\mathbb{R}_ℓ is not metrizable.

(千呼万唤始出来, 没有半遮面)

Example 14.8

\mathbb{R}_ℓ is separable, but not second-countable.

Proof.

Let \mathcal{B} be a basis for \mathbb{R}_ℓ . For every $x \in \mathbb{R}_\ell$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subset [x, x + 1)$. Since $x = \inf B_x$, if $x \neq y$, then $B_x \neq B_y$. Therefore, \mathcal{B} is uncountable. \square

Proposition 14.9

Separable metric space is second-countable.

Corollary 14.10

\mathbb{R}_ℓ is not metrizable.

(千呼万唤始出来, 没有半遮面)

Outline

12 Lecture 13

- Countability axioms
- Separation axioms
 - Regular space

Definition 14.11

A topological space X is said to be **regular** if for each point x and each closed set K satisfying $x \notin K$, there exist open sets U and V such that

$$x \in U, K \subset V, \text{ and } U \cap V = \emptyset.$$

Definition 14.12

A topological space X is said to be **normal** if for each pair A, B of disjoint closed sets of X , there exist open sets U and V such that

$$A \subset U, B \subset V, \text{ and } U \cap V = \emptyset.$$