Topology

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- Regular space

Definition 13.26

Let Y be a topological space and $A \subset Y$. $y \in A$ is said to be an isolated point of A, if there is $V \in \mathcal{N}_Y(y)$, $V \cap A = \{y\}$.

Proposition 13.27

Let Y be a topological space. y is an isolated point of $Y \iff \{y\}$ is open in Y.

Theorem 13.28

Let X be a compact Hausdorff space. If X has no isolated points, then it is uncountable.

Example 13.29

(1) $X := \{a, b, c\}$. $\tau := \{\emptyset, X, \{1, 2\}\}$. (2) $\mathbb{R}$.

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Theorem 13.30

Let X be a metric space. Then the following are equivalent:

(1) *X is compact.*

(2) *For every infinite subset $A \subset X$, $A' \neq \emptyset$.* (limit point compact)

(3) *Every sequence $\{x_n\} \subset X$ has a convergent subsequence.* (sequentially compact)

思路.

(1)⇒(2). If $A' = \emptyset$, then A is closed, and hence compact in X. For every $x \in A$, $\exists U \in \mathcal{N}(x)$ such that $U \cap A = \{x\}$. Compactness enters here...

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Outline

Definition 14.1

A topological space X is said to have a countable basis at x if there is a countable collection $\mathscr{B} \subset \mathscr{N}(x)$ such that

$$\forall V \in \mathscr{N}(x), \exists U \in \mathscr{B} \text{ such that } x \in U \subset V.$$

A space that has a countable basis at each of its points is said to satisfy the first countable axiom, or to be first-countable.

Theorem 14.2 (see Theorem 8.15)

Let X be a topological space. If there exists a sequence $\{x_n\} \subset A$ such that $x_n \to x$, then $x \in \overline{A}$. The converse holds if X is first-countable.

Corollary 14.3

Let A be a subset in a first-countable space. A is closed $\iff$ For any sequence $\{x_n\} \subset A$, if $x_n \to x$, then $x \in A$.

Theorem 14.4

Let X and Y be topology spaces. If f is continuous at x_0, then $x_n \to x_0$ implies $f(x_n) \to f(x_0)$. The converse holds if X is first-countable.

Definition 14.5

If a topological space X has a countable basis for its topology, then X is said to satisfy the second countability axiom, or to be second-countable.

Definition 14.6

$A \subset X$ is said to be dense in X if $\overline{A} = X$.

Theorem 14.7

If X is second-countable, then there exist a countable subset of X that is dense in X.

The latter is just the definition of a separable space.

second-countable $\Rightarrow$separable.

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second-countable $\Rightarrow$separable.

Example 14.8

$\mathbb{R}_\ell$ is separable, but not second-countable.

Proof.

Let $\mathscr{B}$ be a basis for $\mathbb{R}_\ell$. For every $x \in \mathbb{R}_\ell$, there is $B_x \in \mathscr{B}$ such that $x \in B_x \subset [x, x+1)$. Since $x = \inf B_x$, if $x \neq y$, then $B_x \neq B_y$. Therefore, $\mathscr{B}$ is uncountable. □

Proposition 14.9

Separable metric space is second-countable.

Corollary 14.10

$\mathbb{R}_\ell$ *is not metrizable.*

(千呼万唤始出来，没有半遮面)

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Outline

12 Lecture 13
- Countability axioms
- Separation axioms
  - Regular space

Definition 14.11

A topological space X is said to be regular if for each point x and each closed set K satisfying $x \notin K$, there exist open sets U and V such that

$$x \in U, K \subset V, \text{ and } U \cap V = \emptyset.$$

Definition 14.12

A topological space X is said to be normal if for each pair A, B of disjoint closed sets of X, there exist open sets U and V such that

$$A \subset U, B \subset V, \text{ and } U \cap V = \emptyset.$$