Topology

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2 Lecture 12

- Compact
- Compact↔subspace
- Compact↔Hausdorff
- Compact↔continuous
- Compact↔product
- Compact↔connected
- Finite intersection property
- Compact sets in $\mathbb{R}^n$
- Extreme value theorem
- Compact↔metric space
  - Lebesgue number lemma

Outline

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     - Lebesgue number lemma

Throughout this section, we assume that X is a metric space with metric d, unless otherwise specified.

Definition 13.20

Let C be a nonempty subset of X. For $x \in X$,

$$\mathrm{d}(x, C) := \inf_{y \in C} \mathrm{d}(x, y)$$

is the distance from x to C.

Proposition 13.21 (Theorem ??)

Let C be a nonempty subset of X. Then $\mathrm{d}(x, C) = 0 \iff x \in \overline{C}$.

Corollary 13.22

If C is closed in X, then $x \notin C \iff \mathrm{d}(x, C) > 0$.

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Proposition 13.23

1. $|\,\mathrm{d}(x,C) - \mathrm{d}(y,C)| \leq \mathrm{d}(x,y)$, $\forall\, x, y \in X$.
2. $x \mapsto \mathrm{d}(x,C)$ is a continuous function.

Lemma 13.24 (Lebesgue number lemma)

Let $\{A_i : 1 \leq i \leq n\}$ be an open covering of a compact metric space X. Then there is $\delta > 0$ such that for any $K \subset X$ satisfying $\mathrm{diam}(K) < \delta$,

$$\exists i, \ni: K \subset A_i.$$

We may assume that K is a ball of radius δ, say, $\mathbb{B}(x_0, \delta)$. The problem reduces to $\mathbb{B}(x_0, \delta) \cap C_i = \emptyset$ for some i, where $C_i = X \setminus A_i$. $\mathrm{d}(x_0, C_i) > \delta$?

- Finite open covering → arbitrary open covering.
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Theorem 13.25

Let X and Y be metric spaces and $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.