Topology

by Munkres James

何诣然

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2. Lecture 12
   - Compact
   - Compact↔subspace
   - Compact↔Hausdorff
   - Compact↔continuous
   - Compact↔product
   - Compact↔connected
   - Finite intersection property
   - Compact sets in $\mathbb{R}^n$
   - Extreme value theorem
   - Compact↔metric space
     - Lebesgue number lemma

Outline

Exercise 11 on Page 171.

Question

What about intersection of compact sets; union of compact sets?

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Definition 13.16

Let $\{C_\alpha : \alpha \in \Lambda\} \subset 2^X$ be a collection. We say that $\{C_\alpha : \alpha \in \Lambda\}$ has finite intersection property if for every finite index set $\mathscr{F} \subset \Lambda$, $\bigcap_{\alpha \in \mathscr{F}} C_\alpha \neq \emptyset$.

Theorem 13.17 (有限交蕴含任意交)

X is compact $\iff$ For every collection of closed sets $\{C_\alpha : \alpha \in \Lambda\} \subset 2^X$, if it has finite intersection property, then $\bigcap_{\alpha \in \Lambda} C_\alpha \neq \emptyset$.

Application

$\{C_n : n \in \mathbb{N}\} \subset 2^X$ and $C_1 \supset C_2 \supset \cdots \supset C_n \supset \ldots$. If X is compact and each C_n is closed, then $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.

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A known fact: Closed interval $[a, b]$ is compact in $\mathbb{R}$.

Theorem 13.18

In $\mathbb{R}^n$, Y is compact $\Longleftrightarrow$ Y is bounded and closed.

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Topologist's sine curve.

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Theorem 13.19

Let $f : X \to \mathbb{R}$ be continuous. If X is compact, then $\exists c, d \in X$ such that

$$f(c) \leq f(x) \leq f(d), \quad \forall\, x \in X.$$

思路.

Let $A := f(X)$. Then A is compact, and hence bounded and closed. Aim to prove $\sup A$ is attained. There is a sequence $\{a_n\} \subset A$ such that $a_n \to a$. Since A is closed, $a \in A$. □

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