Topology

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2. Lecture 11
    - Separation
    - Separation↔subspace
    - Connected space
    - Properties of connected space
    - Connected↔subspace
    - Connected↔continuous
    - Connected↔product
    - Connected↔closure, boundary
    - Intermediate value theorem
    - Path connected
    - Connected component

数形结合只是帮助理解, 不能当作结论. For example, $[a, b)$ is connected in $\mathbb{R}$, but not connected in $\mathbb{R}_\ell$. Topologist's sine curve is connected in $\mathbb{R}^2$.

Proposition 12.20 (Exercise 6 on Page 152)

Let Y be a connected subspace of X and $C \subset X$. If $Y \cap C \neq \emptyset$ and $Y \cap (X \setminus C) \neq \emptyset$, then $Y \cap \mathrm{bd}(C) \neq \emptyset$.

Outline

Theorem 12.21 (Intermediate value theorem)

Let X be a connected space and $f : X \to \mathbb{R}$ be continuous. If $r \in \mathbb{R}$ satisfies $f(a) < r < f(b)$ for two points $a, b \in X$, then there exists $c \in X$ such that $f(c) = r$.

Corollary 12.22

Let X be a connected subspace of $\mathbb{R}$. Then $[a, b] \subset X$ for any $a, b \in X$ with $a \neq b$. In particular, X is an interval.

Outline

Definition 12.23

A topological space is said to be path connected if for any $x, y \in X$, there exist a closed interval $[a, b] \subset \mathbb{R}$ and a continuous function $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$.

Path connected $\Rightarrow$connected.

Let X be path connected. If it is not connected, assume that $\{A, B\}$ is its separation. Let $x \in A$ and $y \in B$. Then there exist a closed interval $[a, b] \subset \mathbb{R}$ and a continuous function $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. By Theorem 12.15, $f([a, b])$ is connected in X. By Corollary 12.11, we assume $f([a, b]) \subset A$. Then $y = f(b) \in A$, contradicting $y \in B$. □

But connected $\not\Rightarrow$ path connected: topologist's sine curve.

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Let X be a topological space. Define a relation on X by

$$x \sim y \iff \exists \text{ a connected subspace } Y \text{ of } X \text{ such that } x, y \in Y.$$

Then $\sim$ is an equivalent relation on X. Each of its equivalent class is called a connected component.

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