# Topology
## by Munkres James

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2. Lecture 11
   - Separation
   - Separation↔subspace
   - Connected space
   - Properties of connected space
   - Connected↔subspace
   - Connected↔continuous
   - Connected↔product
   - Connected↔closure, boundary
   - Intermediate value theorem
   - Path connected
   - Connected component

Supplement

- Singleton is connected.

**Question**

Let $\tau_1$ and $\tau_2$ be two topologies on $X$. If $\tau_1 \subset \tau_2$, is there any relationship of connectedness in the two topologies?

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# Outline

### Proposition 12.10

Let $\{C, D\}$ be a separation of $X$, $Y$ be a subspace of $X$. If $C \cap Y \neq \emptyset$ and $D \cap Y \neq \emptyset$, then $\{C \cap Y, D \cap Y\}$ is a separation of $Y$.

### Corollary 12.11

*Let $\{C, D\}$ be a separation of $X$. If $Y$ is a connected subspace of $X$, then either $Y \subset C$ or $Y \subset D$.*

社会语言：要守住连通的底线，就一定要站队，不能骑墙.

### Example 12.12

Let $X = [-3, 0) \cup (0, 3]$ with the subspace topology inherited from $\mathbb{R}$. Let $Y = [1, 2]$. Then $\{[-3, 0), (0, 3]\}$ forms a separation of $X$.

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### Theorem 12.13

*Let $\{A_\alpha\}$ be a collection of connected subspaces of $X$. If $\bigcap_\alpha A_\alpha \neq \emptyset$, then $\bigcup_\alpha A_\alpha$ is connected.*

交不空, 并连通.

### Theorem 12.14 (Exercise 3 on Page 152)

*Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of connected subspaces of a topological space $X$. If $\exists \beta \in \Lambda$ such that $A_\alpha \cap A_\beta \neq \emptyset, \forall \alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected in $X$.*

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Is there any relationship between Theorem 12.13 and Theorem 12.14?

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See Exercise 2 on Page 152.

**2.** Let $\{A_n\}$ be a sequence of connected subspaces of $X$, such that $A_n \cap A_{n+1} \neq \varnothing$ for all $n$. Show that $\bigcup A_n$ is connected.

**Question**

If $A$ and $B$ are connected in $X$ and $A \cap B \neq \emptyset$, is $A \cap B$ connected?

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# Outline

## Theorem 12.15

*Let $f : X \to Y$ be continuous. If $A$ is connected in $X$, then $f(A)$ is connected in $Y$.*

## Application

- $(a, b)$ is connected in $\mathbb{R}$.
- Topologist's sine curve is connected in $\mathbb{R}^2$.
- Connected↔product (come soon...)

## Application (Exercise 1 on Page 157)

As subspaces of $\mathbb{R}$, $(0, 1)$ and $(0, 1]$ are not homeomorphic(同胚).

Theorem 12.15既符合理论本身发展的需要，又能够处理以前不能处理或者不容易处理的问题. 这就是数学发展的动力源泉和魅力. 我们从中也看到连续映射的妙用.

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Let $f : X \to Y$ be continuous. If $B$ is connected in $Y$ and $f^{-1}(B) \neq \emptyset$, is $f^{-1}(B)$ connected in $X$?

No! $f : \mathbb{R}_\ell \to \mathbb{R}$ defined by $f(x) = x$.

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## Theorem 12.16

*$X$ and $Y$ are connected spaces $\iff$ $X \times Y$ is connected.*

## Proof.

- $\Rightarrow$ Theorem 23.6: consequence of Theorem 12.15 and Theorem 12.13 (▸ Figure).
- $\Leftarrow$ Exercise 17 on Page 124 in the Chinese version. □

## Application

$\mathbb{R}^n$是连通的，$\mathbb{R}_l^n$不是连通的.

## Proposition 12.17 (Exercise 9 on Page 152)

Let $A \subsetneqq X$ and $B \subsetneqq Y$. If $X \times Y$ is connected, then so is $(X \times Y) \setminus (A \times B)$.

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**Theorem 12.18**

*Let $Y$ be a connected subspace of $X$ and $B \subset X$. If $Y \subset B \subset \overline{Y}$, then $B$ is connected.*

**Corollary 12.19**

*If $Y$ is a connected subspace of $X$, then so is $\overline{Y}$.*

The converse does not hold: $\overline{Y}$ is connected $\nRightarrow$ $Y$ is connected. (say, $Y = \mathbb{Q}$)

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$\mathbb{R}$上所有区间都是连通的.

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