Topology

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7-405, Sichuan Normal University

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2 Lecture 11

- Separation
- Separation↔subspace
- Connected space
- Properties of connected space

Aim: extend the intermediate value theorem from $\mathbb{R}$ to topological space.

Outline

1. Lecture 11
   - Separation
   - Separation↔subspace
   - Connected space
   - Properties of connected space

Definition 12.1

Let X be a topological space and $C, D \subset X$. $\{C, D\}$ is said to be a separation of X if the following hold

1. $C \cap D = \emptyset$, $C \cup D = X$; (不重复、无遗漏)
2. $C \neq \emptyset, D \neq \emptyset$;
3. C and D are open.

Remark 12.2

$\{C, D\}$ is a separation of X $\iff$ The following hold

1. $C \cap D = \emptyset$, $C \cup D = X$;
2. $C \neq \emptyset, D \neq \emptyset$;
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Remark 12.3

$\{C, D\}$ is a separation of X $\iff$ The following hold

1. $C \cap D = \emptyset$, $C \cup D = X$;
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Question

In the following cases, is $\{C, D\}$ a separation of X?

1. $X = \mathbb{R}$, $C = (-\infty, 1)$, $D = (1, +\infty)$.
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3. $X = \mathbb{R}_\ell$, $C = (-\infty, 1)$, $D = [1, +\infty)$.
4. $X = (0, 1)$ (the subspace topology inherited from $\mathbb{R}$), $C = (0, 1/2)$, $D = [1/2, 1)$.
5. $X = \mathbb{Q}$, $C = (-\infty, \alpha) \cap \mathbb{Q}$, $D = (\alpha, +\infty) \cap \mathbb{Q}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. How about $\alpha \in \mathbb{Q}$?

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Proposition 12.4

Let Y be a subspace of X and $C, D \subset Y$. Then $\{C, D\}$ is a separation of $Y \Leftrightarrow$

$$\left\{\begin{array}{l} C \cap D = \emptyset, C \cup D = \not{X}Y; \\ C \neq \emptyset, D \neq \emptyset; \\ C \text{ and } D \text{ are open in } Y. \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} C \cap D = \emptyset, C \cup D = \not{X}Y; \\ C \neq \emptyset, D \neq \emptyset; \\ C \cap \overline{D} = \emptyset \text{ and } \overline{C} \cap D = \emptyset. \end{array}\right.$$

Revisit Item 4 of Question 3.

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Proposition 12.5

X has a separation $\iff$ X has a nonempty proper subset which is both open and closed in X.

Revisit Question 3.

Other concern.

separation↔product topology

separation↔continuous function

We will discuss these topics in the framework of connected space.

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Definition 12.6

A topological space X is said to be connected if there does NOT exist a separation of X.

Example 12.7

Is the following topological space connected?

1. $X = \{1, 2, 3\}$ with $\tau = \{\emptyset, X, \{1, 2\}, \{3\}\}$.
2. $\mathbb{R}_\ell$.
3. $\mathbb{R}$.

$\mathbb{R}$ is connected.

If not, there is a separation $\{C, D\}$ of $\mathbb{R}$. Let $a \in C$ and $b \in D$. Set $A := C \cap [a, b]$ and $B := D \cap [a, b]$. Let $x := \sup A$. Then $x \in A$ and $x < b$. Moreover, $(x, b] \cap A = \emptyset$, equivalently, $(x, b] \subset B$. So $[x, b] \subset B$ as B is closed. It follows that $x \in A \cap B \subset C \cap D = \emptyset$, impossible! □

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The above proof essentially shows that every closed interval $[a, b]$ is connected in $\mathbb{R}$.

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Yes! But how to prove?

1. 改编"$[a, b]$ is connected"的证明(初学者的基本想法)
2. 探索新的证明方法. (基本靠积累和感觉)

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X is not connected $\Longleftrightarrow$ There exists a nonempty proper subset of X that are both open and closed.

Corollary 12.9

X is connected $\Longleftrightarrow$ The only subsets of X that are both open and closed are $\emptyset$ and X.

一般来说，给出不连通空间的证据，比给连通空间的证据更容易.

Question (Topologist's sine curve)

Let

$$X := \left\{(x,y) \in \mathbb{R}^2 : x > 0, y = \sin\frac{1}{x}\right\} \cup (\{0\} \times [-1,1]).$$

Is X connected? (方向在哪里?)

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