Topology

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2 Lecture 6

- Boundary
- Convergence
- Hausdorff Space

Outline

Let A be a subset of a topological space X.

$$\mathrm{bd}(A) := \overline{A} \setminus \mathrm{int}(A)$$

is called the boundary of A.

Theorem 1.1

$x \in \mathrm{bd}(A) \iff$ *For every* $V \in \mathcal{N}(x)$,

$$V \cap A \neq \emptyset \text{ and } V \cap (X \setminus A) \neq \emptyset.$$

Other properties:

1. $\mathrm{int}(A) \cap \mathrm{bd}(A) = \emptyset$, $\overline{A} = \mathrm{int}(A) \cup \mathrm{bd}(A)$.
2. $\mathrm{bd}(A) = \overline{A} \cap \overline{X \setminus A}$.
3. A is open $\iff$ $\mathrm{bd}(A) = \overline{A} \setminus A$.

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Boundary $\leftrightsquigarrow$ Subspace

Question

$\mathrm{bd}_Y(A) = \mathrm{bd}(A) \cap Y$?

No! But $\mathrm{bd}_Y(A) \subset \mathrm{bd}(A) \cap Y$.

Boundary $\leftrightsquigarrow$ Product topology

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$\mathrm{bd}(A \times B) = \mathrm{bd}(A) \times \mathrm{bd}(B)$?

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Let X be a topological space.

Definition 1.2 (Convergent sequence)

We say that a sequence $\{x_n\} \subset X$ converges to $x \in X$, if for every $U \in \mathscr{N}(x)$, there exists N such that

$$x_n \in U, \quad \forall\, n > N.$$

Example 1.3

$\mathbb{R}$, metric space.

Example 1.4

$X = \{a,\ b,\ c\}$, $\tau = \{\emptyset,\ \{a\},\ \{a,\ b\},\ X\}$, $x_n = a$. We have $x_n \to a$, $x_n \to b$.

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To exclude the nonuniqueness of limit, we need to restrict the space.

Definition 1.5

A topological space X is called a Hausdorff space, if $\forall\ x,\ y \in X$ with $x \neq y$, there exist $U \in \mathscr{N}(x)$ and $V \in \mathscr{N}(y)$ such that $U \cap V = \emptyset$.

It is actually a separation property. A micro definition.
Macro? Exercise 13 on Page 101, too artificial!
Difference: space, not set.

Theorem 1.6

If X is a Hausdorff space, then every sequence of points of X converges to at most one point of X.

Theorem 1.7

Let X be a Hausdorff space. Then for every $x \in X$, $\{x\}$ is closed in X.

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Let Y be a subspace of a Hausdorff space X. Then Y is a Hausdorff space.

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