Topology

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2 Lecture 5

- Closure (continued)
- Limit point
- Interior

Outline

1. Lecture 5
   - Closure (continued)
   - Limit point
   - Interior

Micro: pointwise

Theorem 1.1

Let X be a topological space and $A \subset X$.

1. $x \in \overline{A} \iff$ *For any open set U in X, if $x \in U$, then $U \cap A \neq \emptyset$.*
2. $x \in \overline{A} \iff \forall B \in \mathscr{B}$, *if $x \in B$, then $B \cap A \neq \emptyset$.*

Each open set containing x is called a neighborhood of x. We use $\mathscr{N}(x)$ to denote all neighborhoods of x.

Theorem 1.2

Let X be a topological space whose basis is $\mathscr{B}$ and $A \subset X$.

- $x \in \overline{A} \iff$ *For any $V \in \mathscr{N}(x)$, $U \cap A \neq \emptyset$.*

Exercise: The closure of $\{1/n : n \in \mathbb{N}\}$?

Closure ⇜⇝ Subspace

Let Y be a subspace of a topological space X. Let $A \subset Y$ be a nonempty set. Then the closure of A in Y is the set

$$\overline{A}^{Y} := \bigcap \{B : \; A \subset B \text{ and } B \text{ is closed in } Y\}$$

Theorem 1.3

$$\overline{A}^{Y} = \overline{A} \cap Y.$$

Closure ⇜⇝ Product topology

Theorem 1.4 (See Exercise 9 on Page 101)

Let $A \subset X$ and $B \subset Y$. Then

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

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Revisit an example

The closure of $\{1/n : n \in \mathbb{N}\}$?

Outline

Let A be a subset of a topological space X.

Definition 1.5

$x \in X$ is said to be a limit point of A, if for every $V \in \mathscr{N}(x)$,

$$V \cap (A \setminus \{x\}) \neq \emptyset.$$

We use A' to denote the set of all limit points of A.

Remark 1.6

1. a limit point of A is not necessarily in A.
2. a point in A is not necessarily a limit point of A.

$\{1/n : n \in \mathbb{N}\}$.

Theorem 1.7

$\overline{A} = A \cup A'$

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Definition 1.8

Let A be a subset of a topological space X.

$$\mathrm{int}(A) := \bigcup \{B : \ B \subset A \text{ and } B \text{ is open in } X\}$$

is called the interior of A.

Remark 1.9

1. $\mathrm{int}(A)$ is open in X.
2. $\mathrm{int}(A) \subset A$.
3. A is open $\iff \mathrm{int}(A) = A$.

Theorem 1.10

$x \in \mathrm{int}(A) \iff$ *There exists $V \in \mathcal{N}(x)$ such that $V \subset A$.*

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Interior $\leftrightsquigarrow$ Subspace

Question

Let Y be a subspace of a topological space X and let $A \subset Y$. $\mathrm{int}_Y(A) = \mathrm{int}(A) \cap Y$?

No! Take $A = Y$. Dramatically different from Closure. Why does this happen?

Interior $\leftrightsquigarrow$ Product topology

Theorem 1.11

$\mathrm{int}(A \times B) = \mathrm{int}(A) \times \mathrm{int}(B)$.

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