Topology

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2nd semester, 2010-2011

2 Lecture 3
- Prelude
- The Subspace Topology

Outline

1 Lecture 3
- Prelude
- The Subspace Topology

Exercise 1

$d(x, y) := \min\{\rho(x, y), 1\}$ is a metric, if so is ρ.

$d(x, y) \leq d(x, z) + d(z, y)$.

Proof.

(i) If either $\rho(x, z) \geq 1$ or $\rho(z, y) \geq 1$, no problem!
(ii) If both $\rho(x, z) < 1$ and $\rho(z, y) < 1$,

$$d(x, y) \leq \rho(x, y) \leq \rho(x, z) + \rho(z, y) = d(x, z) + d(z, y).$$

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作业中出现的:

1. 应该对$\rho(x,y),\rho(x,z),\rho(y,z)$与1的大小仔细分类，有的同学只分$\rho(x,y)>1$和$\rho(x,y)\le 1$两种情况.
   分成3×2种情况，是最初等的方式，也是最完美的方式.
2. 因为$d(x,y)=\min\{\rho(x,y),1\}$和$d(y,z)=\min\{\rho(y,z),1\}$, 所以
$$d(x,y)+d(y,z)=\min\{\rho(x,y)+\rho(y,z),1\}.$$

要注意逻辑推理的严密性.

Let X be a topological space with the topology τ. The following are the same.

1. U is an open set in the topological space X.
2. U is open in X in the topology τ.
3. U is open in X.
4. $U \in \tau$.

Several typical examples of topological spaces:

1. $\mathbb{R}$, $\mathbb{R}^2$, $\ldots$.
2. metric space
3. $\mathbb{R}_\ell$
4. Set of finite elements. It's too artificial!

Example 1.1 ($\tau_X \times \tau_Y$ is not a topology)

Let $X := \{1, 2, 3\}$, $\tau_X := \{\emptyset, X, \{1\}, \{1, 2\}\}$; $Y := \{a, b, c\}$, $\tau_Y := \{\emptyset, Y, \{a\}, \{a, b\}\}$. Then both $\{1\} \times \{a, b\}$ and $\{1, 2\} \times \{a\}$ are in $\tau_X \times \tau_Y$, but NOT their union. Note that

$$(\{1\} \times \{a, b\}) \cup (\{1, 2\} \times \{a\}) = \{(1, a), (1, b), (2, a)\}.$$

If there exist $A \in \tau_X$ and $B \in \tau_Y$ such that

$$\{(1, a), (1, b), (2, a)\} = A \times B, \tag{1}$$

then A should be $\{1, 2\}$ and B should be $\{a, b\}$. However, $(2, b)$ is in $A \times B$, but not in $\{(1, a), (1, b), (2, a)\}$. So

$$\{(1, a), (1, b), (2, a)\} \subsetneq A \times B.$$

That is, (1) is impossible to hold.

Outline

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Proposition 1.2

Let X be a topological space with topology τ. If Y is a subset of X, then the collection

$$\tau_Y := \{Y \cap U : \; U \in \tau\}$$

is a topology on Y, which is called the subspace topology on Y

When we say that Y is a subspace of X in Topology, it means:

1. $Y \subset X$.
2. The topology on Y is the subspace topology.

V is open in Y means: $V \subset Y$ and $V \in \tau_Y$.

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Remark 1.3

Open set in the subspace Y is possibly not open in the space X.

Let $X := \mathbb{R}$ and $Y = [0, 2)$. Then $[0, 1)$ is open in Y, but not in X.

Proposition 1.4

1. If $U \subset Y$ is open in Y and if Y is open in X, then U is open in X.
2. If $U \subset Y$ is open in X, then U is open in Y.

Basis of the subspace topology

Proposition 1.5

If $\mathscr{B}$ is a basis for the topology of X, then the collection

$$\mathscr{B}_Y := \{Y \cap B : B \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.

Let X and Y be two topological spaces, $A \subset X$, $B \subset Y$. Let τ_A and τ_B denote the subspace topology on A and B, respectively.

Proposition 1.6

The subspace topology of $A \times B$ (as a subspace of $X \times Y$) is the same as the product topology of $A \times B$ (as a product of $(A, \tau_A) \times (B, \tau_B)$).

Hint.

Let $\mathscr{B}_1 := \{(U \times V) \cap (A \times B) : \ U \in \tau_X, V \in \tau_Y\}$. Then $\mathscr{B}_1 = \{(U \cap A) \times (V \cap B) : \ u \in \tau_X, v \in \tau_Y\} = \tau_A \times \tau_B$. □