# Topology

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2 Lecture 2

- Topological Spaces and Continuous Functions
- Basis for a Topology
- Metric and topology
- Subbasis
- Compare different topologies

# Outline

**Definition 1.1**

Let $X$ be a set, a topology on $X$ is a collection $\tau$ of subsets of $X$ satisfying the following conditions:
(1) $\emptyset \in \tau, X \in \tau$.
(2) The union of arbitrary elements of $\tau$ is an element of $\tau$.
(3) The intersection of finite elements of $\tau$ is an element of $\tau$.

Every element of $\tau$ is called an open set. $(X, \tau)$ is called a topological space.

# Outline

1. Lecture 2
   - Topological Spaces and Continuous Functions
   - Basis for a Topology
   - Metric and topology
   - Subbasis
   - Compare different topologies

拓扑，作为一个集合族，包括的元素太没有特点，不容易把握，找不到感觉.

Topology, as a collection, whose elements are too vague to be caught.

# Basis

### Definition 1.2

Let $X$ be a set. A collection $\mathscr{B} \subset P(X)$ is a basis for a topology if

1. $\forall\, x \in X,\ \exists\, B \in \mathscr{B}$ such that $x \in B$.
   (Equivalently, $X = \bigcup_{B \in \mathscr{B}} B$)
2. If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there is an element $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Each element of $\mathscr{B}$ is called a basis element.

### Example 1.3 (Examples of basis)

1. In the real line $\mathbb{R}$, all open intervals.
2. In the real line $\mathbb{R}$, all intervals of the form $[a, b)$.
3. Every topology itself.

## Basis

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**Proposition 1.4 (Basis$\rightarrowtail\rightarrow$Topology: Micro)**

Let $X$ be a set and $\mathscr{B}$ be a basis. Define $\tau$ to be the collection of subsets $U \subset X$ satisfying the following property:

$$U \in \tau \Leftrightarrow \forall\, x \in U,\ \exists B \in \mathscr{B}, \ni: x \in B \subset U. \qquad (*)$$

Then $\tau$ is a topology.

We say that $\tau$ is the topology generated by the basis $\mathscr{B}$.

**Proposition 1.5 (Basis$\rightarrowtail\rightarrow$Topology: Macro)**

Let $X$ be a set, $\mathscr{B}$ a basis, and $\tau$ the topology generated by $\mathscr{B}$. Then $\tau$ is equal to the collection of union of arbitrary elements in $\mathscr{B}$.

# Outline

**Definition 1.6 (Metric Space)**

Let $X$ be a set. If a function $\rho : X \times X \to \mathbb{R}_+$, satisfies:

1. $\rho(x,\ y) \geq 0$; $\rho(x,\ y) = 0 \Longleftrightarrow x = y$.
2. $\rho(x,\ y) = \rho(y,\ x)$.
3. $\rho(x,\ z) \leq \rho(x,\ y) + \rho(y,\ z)$.

Then $\rho$ is said to be a metric on $X$, $(X,\ \rho)$ is called a metric space.

For $x \in X$, $\varepsilon > 0$, let $B_\rho(x, \varepsilon) := \{y \in X : \rho(x, y) < \varepsilon\}$.

### Proposition 1.7 (Metric $\to$ topology)

If $\rho$ is a metric on $X$, then the collection $B := \{B_\rho(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a basis for a topology on $X$.

We call such topology as the metric topology.

### Proposition 1.8

Let $(X, \ \rho)$ be a metric space and $U \subset X$. Then $U$ is open in the metric topology $\Longleftrightarrow \forall x \in U, \ \exists \delta > 0, \ \ni: \ B_\rho(x, \ \delta) \subset U$.

**Topology** $\to$ **metric**: metrizable topology. A topology $\tau$ on $X$ is said to be metrizable if there is a metric $\rho$ on $X$ such that the metric topology is the same as $\tau$.

# Outline

### Definition 1.9

$\mathscr{A} \subset P(X)$ is said to be a subbasis of $X$ if for each $x \in X$ there exist $S \in \mathscr{A}$ such that $x \in S$.

### Proposition 1.10

Let $\mathscr{B}$ denote the collection of intersection of finite elements in $\mathscr{A}$. Then $\mathscr{B}$ is a basis on $X$.

# Outline

## Definition 1.11

Let $\tau_1$ and $\tau_2$ be two topologies on a set $X$. Then $\tau_1$ is finer(更细致) than $\tau_2$ if $\tau_2 \subset \tau_1$. If $\tau_2 \subsetneq \tau_1$, then $\tau_1$ ia said to be strictly finer than $\tau_2$. We also say that $\tau_2$ is coarser than $\tau_1$ if $\tau_1$ is finer than $\tau_2$.

## Remark 1.12

If either $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$, then we say that $\tau_1$ and $\tau_2$ are comparable.

## Example 1.13

Let $X = \{1, 2, 3\}$, then

$$\{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\} \supset \{\emptyset, \{1\}, \{1, 2\}, X\} \supset \{\emptyset, \{1, 2\}, X\}.$$

But $\{\emptyset, \{1, 2\}, X\}$ and $\{\emptyset, \{2, 3\}, X\}$ can't compare with each other.

Compare topologies: the role of basis.

### Proposition 1.14 (Compare topologies by their bases)

Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\tau$ and $\tau'$ on $X$, respectively. Then the following are equivalent.

1. $\tau \subset \tau'$.
2. $\forall\ B \in \mathscr{B}$ and $\forall\ x \in B,\ \exists\ B' \in \mathscr{B}',\ \ni: x \in B' \subset B$.

### Example 1.15

Consider the set $\mathbb{R}$.

1. $\mathscr{B} := \{(a,b) :\ a, b \in \mathbb{R}\}$. Then $\mathscr{B}$ is a basis on $\mathbb{R}$. (standard topology, still write as $\mathbb{R}$)
2. $\mathscr{B}' := \{[a,b) :\ a, b \in \mathbb{R}\}$. Then $\mathscr{B}'$ is a basis on $\mathbb{R}$. (lower limit topology $\mathbb{R}_\ell$)

Compare the two topologies.

### Exercise 1

In a metric space $(X, \rho)$, $\{B_\rho(x,\ \varepsilon):\ x \in X,\ 0 < \varepsilon < 1\}$ is also a basis, which generates the same topology as the usual basis.