GENERALIZED FRONTS FOR ONE-DIMENSIONAL REACTION-DIFFUSION EQUATIONS

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Abstract. For a class of one-dimensional reaction-diffusion equations, we establish the existence of generalized fronts, as recently defined by Berestycki and Hamel. We also prove uniform nondegeneracy estimates, such as a lower bound on the time derivative on some level sets, as well as a lower bound on the spatial derivative.

1. Introduction

Consider the following reaction-diffusion equation:

$$u_t - \partial_{xx} u = g(x) f(u) \quad \text{in } \mathbb{R} \times \mathbb{R}, \tag{1}$$

where f is a smooth function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} f(u) \geq 0 \text{ for all } u \in \mathbb{R} \quad \text{and} \quad \operatorname{supp} f = [\theta, 1] \\ f'(1) < 0. \end{cases} \tag{2}$$

The function $g(x)$, satisfies

$$0 < g_{\min} \leq g(x) \leq g_{\max} \qquad \text{for all } x \in \mathbb{R} \tag{3}$$

for some positive constants $g_{\min}$ and $g_{\max}$. Equation (1) represent the simplest way to model the propagation of a flame in a heterogeneous medium. The scalar theta (usually called the ignition temperature) represents a threshold below which no reaction may occur.

Traveling fronts are particular solutions of (1) that are important in the study of transition phenomena between different states (typically $u = 0$ and $u = 1$). When $g(x)$ is independent of x, such transition fronts are described by traveling wave solutions, of the form

$$u(t, x) = \phi(x + ct) \tag{4}$$

The existence and uniqueness (c, ϕ) have been extensively studied (see e.g. [3] for the one-dimensional case, [5, 4] in the multi-D case). The unique

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speed c is positive and $s \mapsto \phi(s)$ is a smooth increasing function satisfying $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$.

When $g(x)$ is not constant, it is clear that solutions of the form (4) no longer exist. In some cases however, it is still possible to study front propagation. A particular framework which has received a lot of attention is that of periodic media, i.e. when $g(x + L) = g(x)$ for all $x \in \mathbb{R}$ and for some $L \in \mathbb{R}$. Xin [9, 10], then Berestycki-Hamel [1] establish the existence of pulsating fronts which are global solutions of (1) connecting 0 and 1, and satisfying

$$u(t, x + L) = u\left(t + \frac{L}{c}, x\right) \qquad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

Such solutions are proved to be monotone increasing with respect to t.

It is natural to try to generalize this to situation in which g only satisfies (3) and no other structural assumptions such as periodicity or quasi-periodicity. A definition of generalized front was first given by Matano and later formalized by Shen in [8]. More recently, an alternative notion of generalized traveling fronts was proposed by Berestycki-Hamel in [2]. Their definition is quite general and we give here a one-dimensional version of it:

Definition 1.1. *A generalized front for equation (1) is a global in time solution u such that there exists a function $x : \mathbb{R} \to \mathbb{R}$ with*

(5)
$$\begin{cases} u(t, x) \longrightarrow 1 \text{ uniformly in } t \in \mathbb{R} \text{ and } x \geq x(t) \text{ as } |x - x(t)| \longrightarrow +\infty \\ u(t, x) \longrightarrow 0 \text{ uniformly in } t \in \mathbb{R} \text{ and } x \leq x(t) \text{ as } |x - x(t)| \longrightarrow +\infty. \end{cases}$$

We stress out the crucial role of the uniformity in the above definition. Naturally there is more than one possible choice of $x(t)$ for a given generalized front, though it is easy to show that $x(t)$ should be close to a level set of the function u. In the proof we will take $x(t)$ such that $u(t, x(t)) = \theta$ (we will see that this defines $x(t)$ uniquely).

Our goal here is to prove the existence of such a generalized front for the ignition temperature model (1), in the Berestycki-Hamel sense. For that purpose, we will need the following additional assumption on f:

$$uf'(u) \geq f(u) \quad \text{ for } u \leq \theta + \lambda_0 \tag{6}$$

for some $\lambda_0 > 0$. Such an assumption is satisfied, for instance if

$$f(u) \sim (u - \theta)^p \quad \text{ when } u \to \theta^+$$

for some $p \geq 1$.

Our main result is the following:

Theorem 1.2. *Assume that f satisfies (2) and (6) and that g satisfies (3). Then there exists a generalized front $u(t, x)$ solution of (1) connecting the states* 0 *and* 1*. Furthermore, u satisfies:*

(i) *$t \mapsto u(t, x)$ is monotone increasing.*

(ii) *There exists* $\gamma > 0$ *such that*

$$u_x(t, x(t)) \geq \gamma.$$

The proof of Theorem 1.2 relies on a simple idea: We solve a Cauchy problem starting at some time $t = -n$ and translate to ensure that $u^n(0,0) = \theta$. We then take the limit $n \to +\infty$. The main point is to show that this limit is non trivial (not identically equal to some constant) and that it converges uniformly to 0 and 1 to the left and right of the interface $u = \theta$. This is achieved using Lemma 3.1 which is the keystone of the proof (it is also the only place where we use assumption (6)).

The plan of the paper is as follows: in the next section, we set up the problem and prove some basic facts. In Section 3, we prove the key uniform estimates. In Section 4, we prove Theorem 1.2. Section 5 is devoted to some concluding remarks.

As this paper was being written, we learned about a similar work by J. Nolen and L. Ryzhik [6], in which they prove the existence of generalized fronts (under slightly different assumptions on f and via a completely different method), as well the existence of random travelling waves as introduced by Matano-Shen. The uniqueness and stability of the generalized fronts given by Theorem 1.2 is the object of a recent paper by Mellet-Nolen-Roquejoffre-Ryzhik [7].

2. Setting of the problem

When g is constant, the existence of traveling waves is well known (see for instance [4]). In particular, there exists a unique pair $(\overline{\phi}, \overline{c})$ solution of

$$\begin{cases} -\overline{\phi}'' + \overline{c}\overline{\phi}' = g_{\min} f(\overline{\phi}) & \text{for all } x \in \mathbb{R} \\ \lim_{x \to -\infty} \overline{\phi}(x) = 0 \\ \lim_{x \to +\infty} \overline{\phi}(x) = 1 \\ \overline{\phi}(0) = \theta. \end{cases} \tag{7}$$

Furthermore, $x \mapsto \overline{\phi}(x)$ is monotone increasing and $\overline{c} > 0$. The function $\overline{u}(t,x) = \overline{\phi}(x + \overline{c}t)$ is then a traveling wave solution of

$$u_t - u_{xx} = g_{\min} f(u).$$

In particular, it is a sub-solution of (1).

Finally, we note that $\overline{\phi}(x) \leq \theta$ for $x \leq 0$ and thus $-\overline{\phi}'' + \overline{c}\overline{\phi} = 0$ for $x \leq 0$. This implies:

$$\overline{\phi}(x) = \theta e^{\overline{c}x} \quad \text{for all } x \leq 0. \tag{8}$$

The goal of this preliminary section is to establish some key properties of the function $u(t,x)$ solution of the following Cauchy problem

$$\begin{cases} u_t - u_{xx} = g(x)f(u) & \text{for all } t > s \text{ and } x \in \mathbb{R} \\ u(s,x) = \overline{\phi}(x) \end{cases} \tag{9}$$

where s is any given real number.

We immediately note that $\overline{\phi}(x + \overline{c}(t-s))$ is a sub-solution for (9) and so

$$u(t,x) \geq \overline{\phi}(x + \overline{c}(t-s)) \text{ for all } t \geq s \text{ and } x \in \mathbb{R}.$$

The choice of a sub-solution as an initial data also implies that $u(t,x)$ is increasing in time:

Lemma 2.1. *Let u be a solution of (9). Then $u_t(t,x) > 0$ for all $(t,x) \in (s,\infty) \times \mathbb{R}$.*

Proof. Differentiating (9) with respect to t, we check that the function $v(t,x) = u_t(t,x)$ solves:

$$v_t - v_{xx} = g(x)f'(u)v. \tag{10}$$

Since $\overline{\phi}'(x) > 0$, we have

$$\overline{\phi}'' + g(x)f(\overline{\phi}) \geq \overline{\phi}'' + g_{\min}f(\overline{\phi}) = \overline{c}\overline{\phi}' > 0$$

and so $v(s,x) = \overline{\phi}''(x) + g(x)f(\overline{\phi}(x)) > 0$ for all $x \in \mathbb{R}$. The strong maximum principle applied to (10) thus gives $v(t,\cdot) = u_t(t,\cdot) > 0$ for all $t > s$. □

Next, we want to show that $u(t,x)$ has some uniform behavior away from its level set $\{u = \theta\}$. For that purpose, we first prove that the level set can be parametrized by t:

Lemma 2.2. *For all $t \geq s$, there exists a unique $x(t)$ such that $u(t,x(t)) = \theta$ (note that $x(s) = 0$).*

Proof. Assume that there exists $x_1 < x_2$ such that $u(t_0,x_1) = u(t_0,x_2) = \theta$.

If $\sup\{u(t_0,x)\,,\, x \leq x_2)\} = \theta$, then $x \mapsto u(t_0,x)$ has a local maximum at x_1. We then have $u_{xx}(t_0,x_1) \leq 0$ and $f(u(t_0,x_1)) = 0$ which contradicts $u_t > 0$.

If $\sup\{u(t_0,x)\,,\, x \leq x_2)\} > \theta$, then we define $t_1 < t_0$ to be the first time such that $\sup\{u(t_1,x)\,,\, x \leq x_2\} = \theta$. Since $u_t > 0$, we have $u(t_1,x_2) < \theta$, and so $x \mapsto u(t_1,x)$ has a local maximum for some $x_3 < x_2$ with $u(t_1,x_3) = \theta$. We get a contradiction again, and the result is proved. □

Note that this argument works if we replace θ by any number $\gamma \in (0,\theta]$. In particular, it implies:

$$u_x(t,x) \geq 0 \qquad \text{for all } (t,x) \text{ such that } x \leq x(t).$$

Next, we need to strengthen the result of Lemma 2.1:

Lemma 2.3. *For any $\lambda \in (0, 1-\theta)$, there exists a constant $q(\lambda) > 0$ such that*

$$\textit{if} \quad u(t,x) = \theta + \lambda \quad \textit{then} \quad u_t(t,x) \geq q(\lambda)$$

Proof. This lemma follows from a compactness argument: assume that there exist sequences s_n, u^n, (t_n, x_n) such that u^n is a solution of (9) with $s = s_n$, and (t_n, x_n) is such that

$$u^n(t_n, x_n) = \theta + \lambda \quad \text{and} \quad \lim_{n\to+\infty} u^n_t(t_n, x_n) = 0.$$

We set $v^n(t,x) = u^n(t+t_n, x+x_n)$ and $g^n(x) = g(x+x^n)$. Classical parabolic estimates show that up to a subsequence we have:

$$v^n(t,x) \longrightarrow v^\infty(t,x) \text{ and } g^n(x) \longrightarrow g^\infty(x)$$

uniformly on every compact in $\mathbb{R}\times\mathbb{R}$. And since $\overline{\phi}_t(x_0) > 0$ we have $t_n > s_n + M$ for all n.

Furthermore, $w^n = v^n_t \geq 0$ is a solution of

$$w^n_t - w^n_{xx} = g(x+x_n) f'(v^n) w^n$$

with $\lim_{n\to+\infty} w^n(0,0) = 0$. Parabolic Harnack's inequality thus yields:

$$v^n_t \to 0 \quad \text{uniformly w.r.t. } t \text{ and } x \text{ on every compact set in } [-M, 0)\times\mathbb{R}.$$

We deduce

$$\begin{cases} -v^\infty_{xx} = g^\infty f(v^\infty) & \text{for all } (t,x) \in (-M,0)\times\mathbb{R} \\ v^\infty(0,0) = \theta+\lambda & \\ |v^\infty(x,t)| \leq 1 & \text{for all } (t,x) \in (-M,0)\times\mathbb{R} \end{cases}$$

which is impossible. As a matter of fact, the first equality implies that $x \mapsto v^\infty_x(t,x)$ is monotone decreasing and since v^∞ is bounded, we must have v^∞_x identically zero. This, in turn, implies that $-v^\infty_{xx} = g^\infty f(v^\infty) = 0$ which contradicts the fact that $g^\infty f(v^\infty(0,0)) = g^\infty f(\theta+\lambda) > 0$. □

3. Uniform nondegeneracy estimates

The following lemma can be viewed as the keystone of our proof, and will later be used to prove the non-degeneracy of u:

Lemma 3.1. *There exists $\lambda \in (0, 1-\theta)$ and $q > 0$ such that if $u \leq \theta + \lambda$, then*

$$u_t(t,x) \geq q u(t,x). \tag{11}$$

Proof. We denote

$$\Omega = \{(x,t) \in \mathbb{R}\times\mathbb{R}\,;\, u(t,x) \leq \theta + \lambda\}$$

and we introduce the function $v(t,x) = u_t(t,x) - q u(t,x)$.

This function satisfies

$$v_t - v_{xx} = g(x)f'(u)v + qg(x)\left[f'(u)u - f(u)\right]$$

and if $\lambda < \lambda_0$, (6) yields:

$$v_t - v_{xx} \geq g(x)f'(u)\,v \qquad \text{in } \Omega.$$

Next, we note that

$$\begin{aligned} v(s,x) &= u_{xx}(s,x) + g(x)f(u(s,x)) - qu(s,x) \\ &\geq \overline{\phi}''(x) + g_{\min}f(\overline{\phi}(x)) - q\overline{\phi}(x) \\ &\geq \overline{c}\overline{\phi}'(x) - q\overline{\phi}(x). \end{aligned}$$

When $\overline{\phi}(x) \leq \theta$ (i.e. for $x \leq 0$) a simple computation using (8) yields

$$v(s,x) \geq (\overline{c}^2 - q)\theta e^{\overline{c}x}$$

and by continuity we deduce that for q sufficiently small, we have

$$v(s,x) \geq 0 \qquad \text{in } \{u(s,\cdot) \leq \theta + \lambda\} = \Omega \cap \{t = s\}.$$

Finally, Lemma (2.3) gives $v \geq 0$ on $\{u = \theta + \lambda\}$.

We finally recall that $u_t > 0$ on $\partial\{u = \theta+\lambda\} \cap \{t > 0\}$, this set is therefore a smooth subgraph in the (x,t)-plane, and Ω is a smooth subset of the (x,t)-plane. Consequently, we may apply the standard maximum principle for v in Ω, and deduce

$$v(t,x) = u_t(t,x) - qu(t,x) \geq 0 \qquad \text{in } \Omega$$

which concludes the proof. □

We now deduce:

Corollary 3.2. *For all $\varepsilon \in (0,\theta)$, there exists K_ε such that if $t - s \geq K_\varepsilon$, then*

$$u(t,x) \leq \varepsilon + (\theta - \varepsilon)e^{\alpha(x - x(t))} \qquad \text{for all } x \leq x(t). \tag{12}$$

In particular, if $t - s$ is large enough, there exists a constant $\gamma > 0$ such that

$$u_x(t, x(t)) \geq \gamma. \tag{13}$$

Proof. We fix $t_0 > s$. Then Lemma 3.1 (integrating (11) from t to t_0) yields

$$u(t, x(t_0)) \leq \theta e^{-q(t_0 - t)}$$

for all $t \in (s, t_0)$. In particular, for any $\varepsilon > 0$, there exists a constant K_ε such that if $t_0 - s \geq K_\varepsilon$, then there exists $\sigma \in (s, t_0)$ such that

$$u(\sigma, x(t_0)) \leq \varepsilon.$$

Since $u_x(\sigma, x) \geq 0$ for $x \leq x(\sigma)$, we deduce

$$u(\sigma, x) \leq \varepsilon \qquad \text{for all } x \leq x(t_0).$$

We now define the function

$$w(t,x) = \varepsilon + (\theta - \varepsilon)e^{\alpha(x - x(t_0) + \alpha(t - t_0))}.$$

It satisfies $w_t - w_{xx} = 0$ and $w(t,x) \geq \varepsilon$ and therefore

$$w(\sigma, x) \geq u(\sigma, x) \quad \text{for all } x \leq x(t_0).$$

Next, we note that if α is small enough, then

$$w(t, x(t_0)) \geq \varepsilon + (\theta - \varepsilon)e^{-\alpha^2(t_0 - t)} \geq \theta e^{-q(t_0-t)} \geq u(t, x(t_0)) \quad \text{for all } t \in (\sigma, t_0).$$

The maximum principle thus yields

$$w(t,x) \geq u(t,x) \quad \text{for all } t \in (\sigma, t_0) \text{ and } x \leq x(t_0).$$

which implies (12). Furthermore, since $u(t_0, x(t_0)) = w(t_0, x(t_0)) = \theta$, we deduce

$$u_x(t_0, x(t_0)) \geq w_x(t_0, x(t_0)) = (\theta - \varepsilon)\alpha$$

which gives (13). □

This corollary will be the key in getting the uniform convergence of the generalized front to 0 as $x - x(t)$ goes to $-\infty$. We now need to control the behavior of u as $x - x(t) \to +\infty$. We start with the following lemma:

Lemma 3.3. *There exists $\lambda_0 > 0$ and C such that if $\lambda \leq \lambda_0$ and*

$$y(t) = \inf\{x\,;\, u(t,x) = \theta + \lambda\},$$

then $y(t) - x(t) \leq C$ and $u_x(t, y(t)) \geq \gamma/2$.

Proof. This follows easily from the fact that $u_x(t, x(t)) \geq \gamma > 0$ and classical $\mathcal{C}^2$ estimate for u. □

We can then prove:

Corollary 3.4.

(i) *$u(t,x) \geq \theta + \lambda$ for all $x \geq y(t)$.*

(ii) *There exists $\eta > 0$ such that*

$$u(t,x) \geq 1 - (1 - \theta - \lambda)\left(e^{-\eta(x - y(t))} + e^{-g_{\min}\gamma(t-s)}\right)$$

for all $x \geq y(t)$.

Proof. (i) By definition of $y(t)$, we have $u(t, y(t)) = \theta + \lambda$. Differentiating this equality with respect to t, we get:

$$y'(t) = -\frac{u_t(t, y(t))}{u_x(t, y(t))}.$$

Since $u_t(t, y(t)) \geq q$ and $|u_x| \leq C$, there exists a positive constant q_2 such that $y'(t) \leq -q_2$. Furthermore, we have $u_x(t, y(t)) \geq \gamma/2$ and $|u_t| \leq C$, so there exists q_3 such that

$$q_2 \leq -y'(t) \leq q_3.$$

We now introduce $w(t,x) = u(t, x + y(t))$, satisfying

$$\begin{cases} w_t - y'(t)w_x - w_{xx} = g(x + y(t))f(w) & x \geq 0 \quad t \geq s \\ w(t,0) = \theta + \lambda & t \geq s \\ w(s,x) \geq \theta + \lambda & x \geq 0 \end{cases}$$

If there exists (t_0, x_0) such that $w(t_0, x_0) = \theta + \lambda$ with $x_0 > 0$, $t_0 > s$, then we have (assuming that t_0 is the first time for which this happens):

$$w_t(t_0, x_0) \leq 0, \quad w_x(t_0, x_0) = 0 \quad \text{and } w_{xx}(t_0, x_0) \geq 0$$

which leads to a contradiction since $f(\theta + \lambda) > 0$.

(ii) Let γ be such that

$$f(u) \geq \gamma(1 - u) \qquad \forall u \in [\theta + \lambda, 1]$$

(such a γ exists thanks to (6)) and let $\bar{w}(x)$ be the solution of

$$\begin{cases} -\bar{w}'' + q_3 \bar{w}' = g_{\min}\gamma(1 - \bar{w}) & x \geq 0 \\ \bar{w}(0) = \theta + \lambda \\ \bar{w}(+\infty) = 1 \end{cases}$$

One can actually compute $\bar{w}$ explicitly:

$$\bar{w}(x) = 1 - (1 - \theta - \lambda)e^{-\eta x}$$

with $\eta = (-q_3 + \sqrt{q_3^2 + 4g_{\min}\gamma})/2 > 0$. In particular, we have $\bar{w}' \geq 0$ and so the function $\varphi(t,x) = w(t,x) - \bar{w}(x)$ satisfies

$$\begin{cases} \varphi_t - y'(t)\varphi_x - \varphi_{xx} \geq -g_{\min}\gamma\varphi & x \geq 0 \quad t \geq s \\ \varphi(t,0) = 0 & t \geq s \\ \varphi(t,+\infty) = 0 & t \geq s \\ \varphi(s,x) \geq \theta + \lambda - 1 & x \geq 0. \end{cases}$$

We easily deduce that $\varphi(t,x) \geq -(1 - \theta - \lambda)e^{-g_{\min}\gamma(t-s)}$ and so

$$w(t,x) \geq 1 - (1 - \theta - \lambda)\left(e^{-\eta x} + e^{-g_{\min}\gamma(t-s)}\right) \qquad \text{for all } x \geq 0, \quad t \geq s.$$

Since $u(t,x) = w(t, x - y(t))$, we deduce the corollary. □

4. Construction of the generalized front

We are now ready to construct the generalized traveling front and complete the proof of Theorem 1.2. We take $s = -n$ and we denote by $u^n(t,x)$ the corresponding solution of (9). By Lemma 2.2, there exists a unique x_n such that $u^n(0, x_n) = \theta$. We then define $v^n(t,x) = u^n(t, x + x_n)$. Up to a subsequence, v^n converges to $v(t,x)$ locally uniformly in $\mathcal{C}^2(\mathbb{R}^2)$. Theorem 1.2 now follows from the following proposition:

Proposition 4.1. *Let $x(t) = \inf\{x\,;\, v(t,x) = \theta\}$, then there exists α, η and C positive constants such that*

(i) $v(t,x) \leq \theta e^{\alpha(x-x(t))}$ *for all* $x \leq x(t)$ *and for all* $t \in \mathbb{R}$. *In particular*

$$v(t,x) \longrightarrow 0 \text{ uniformly as } x - x(t) \to -\infty.$$

(ii) $v(t,x) \geq 1 - Ce^{-\eta(x-x(t))}$ *for all* $x \geq x(t)$ *and for all* $t \in \mathbb{R}$. *In particular*

$$v(t,x) \longrightarrow 1 \text{ uniformly as } x - x(t) \to +\infty.$$

Proof. First, we have to check that $x(t)$ is well defined. For all n, by Lemma 2.2, there exists a function $x^n(t)$ such that

$$v^n(t, x^n(t)) = \theta.$$

Furthermore, using (13), we get

$$0 \geq x^{n\prime}(t) = -\frac{v_t^n(t, x^n(t))}{v_x^n(t, x^n(t))} \geq -C$$

for some constant C. By Ascoli's theorem, we can thus assume (extracting another subsequence) that

$$x^n(t) \longrightarrow x(t) \quad \text{uniformly in any compact set.}$$

Finally the uniform convergence of v^n yields:

$$v(t, x(t)) = \theta.$$

Now, for every $\varepsilon > 0$, let K_ε be as in Corollary 3.2. For a given t, and for all n such that $n \geq K_\varepsilon - t$, Corollary 3.2 implies that

$$v^n(t,x) \leq \varepsilon + (\theta - \varepsilon) e^{\alpha(x - x^n(t))} \qquad \text{for all } x \leq x^n(t).$$

Taking the limit $n \to \infty$, we deduce

$$v(t,x) \leq \varepsilon + (\theta - \varepsilon) e^{\alpha(x - x(t))} \qquad \text{for all } x \leq x(t), \quad t \in \mathbb{R}.$$

and since the result holds for every $\varepsilon > 0$, (i) follows.

Similarly Corollary 3.4 yields

$$v^n(t,x) \geq 1 - (1 - \theta - \lambda)\left(e^{-\eta(x - y^n(t))} + e^{-g_{\min}\gamma(t+n)}\right) \qquad \text{for all } x \geq y^n(t)$$

and Lemma 3.3 gives $|y^n(t) - x^n(t)| \leq C$. We deduce

$$v(t,x) \geq 1 - (1 - \theta - \lambda) e^{-\eta(x - x(t) - C)} \qquad \text{for all } x \geq x(t) + C \tag{14}$$

which implies (ii). □

5. Concluding remarks, possible generalizations

In this last part, we discuss easy generalisations of our result, as well as cases that the ideas developed here cannot treat.

5.1. **More general terms.** Let us first comment on assumption (6). Although it does not handle all the f having an ignition temperature, it encompasses the most reasonable nonlinearities. Moreover, one should remember that, in the context of flame propagation theory, the ignition temperature assumption is a technical assumption made to avoid the so-called 'cold boundary difficulty' - i.e. a nonzero reaction rate in the fresh gases. There is therefore no physical assumption dictating a particular form of f, and we may as well choose the one that will avoid unnecessary technical difficulties.

One may now want to consider more general reaction term by replacing $g(x)f(u)$ by a nonlinearity of the form $f(x,u)$ satisfying:

(i) For all $x \in \mathbb{R}$, there is $\theta(x) \in (0,1)$ such that

$$f(x,u) \equiv 0 \text{ on } [0,\theta(x)] \quad \text{and} \quad f(x,u) > 0 \text{ on}_(\theta(x),1)$$

with $\theta(x) \in [\theta_{min}, \theta_{max}]$ and $0 < \theta_{min} \leq \theta_{max} < 1$.

(ii) There exists $\lambda_0 > 0$ small enough so that

$$u f_u(.,u) \geq f(.,u) \quad \text{for } u \leq \theta + \lambda_0 \tag{15}$$

and a constant $\mu_0 > 0$ small enough so that

$$f(.,u) \geq \gamma_0(1-u) \quad \text{for } u \geq \theta(x) + \lambda_0 \tag{16}$$

We claim that our existence theorem holds for such nonlinearity. Indeed, one simply has to replace the initial subsolution $(\bar{c}, \overline{\phi})$ by a solution $(\bar{c}, \overline{\phi})$ of $-\overline{\phi}'' + \bar{c}\overline{\phi}' = \underline{f}(\overline{\phi})$, where $\underline{f}(u) = \gamma_0(1-u)$ if $u \geq \theta_{max} + \lambda_0$, 0 everywhere else. The main estimate Lemma 3.1 is then true without any modification: one simply has to work in the domain $\{u(t,x) \leq \theta(x) + \lambda\}$ which, by the strict monotonicity of u, is a smooth sub-graph in the (x,t)-plane. This implies the uniform nondegeneracy of u at $\{u(t,x) = \theta_{min}\}$; the uniform bound on the front width is implied by assumption (16).

A further generalization consists in replacing the constant diffusion ∂_{xx} by a nonhomogeneous diffusion $\partial_x(a(x)\partial_x)$. This can also be treated similarly. The only modification is again in the choice initial sub-solution, which now has to solve solving $-(a(x)\overline{\phi}')' + \bar{c}\overline{\phi}' = \underline{f}(\overline{\phi})$ (this can be solved by a standard shooting method).

Note that adding an advection term would be a more subtle problem, which we shall not discuss here.

5.2. **Other types of nonlinearities.** The first other type of nonlinearity one may think about is the bistable nonlinearity, i.e. a reaction term of the form $f(x,u)$ with $f(x,.) < 0$ in a (uniform in x) neighbourhood of 0, $f(x,.) > 0$ in a (uniform in x) neighbourhood of 1 and

$$\sup_x f_u(x,0) < 0, \quad \sup_x f_u(x,1) < 0.$$

The existence of generalised fronts in this setting is far from clear, even the case where f is periodic in x is not, to our knowledge, settled in full generality. They require a new set of ideas, and we hope to treat them in a forthcoming work.

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