Existence and regularity of extremal solutions for a mean-curvature equation

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Abstract

We study a class of mean curvature equations $-\mathcal{M}u = H + \lambda u^p$ where $\mathcal{M}$ denotes the mean curvature operator and for $p \geq 1$. We show that there exists an extremal parameter λ^* such that this equation admits a minimal weak solutions for all $\lambda \in [0, \lambda^*]$, while no weak solutions exists for $\lambda > \lambda^*$ (weak solutions will be defined as critical points of a suitable functional). In the radially symmetric case, we then show that minimal weak solutions are classical solutions for all $\lambda \in [0, \lambda^*]$ and that another branch of classical solutions exists in a neighborhood $(\lambda_* - \eta, \lambda^*)$ of λ^*.

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The aim of this paper is to study the existence and regularity of non-negative solutions for the following mean-curvature problem:

$$\begin{cases} -\mathrm{div}(Tu) = H + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where

$$Tu := \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}$$

and

$$f(u) = |u|^{p-1}u, \qquad p \geq 1.$$

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Formally, Equation (P_λ) is the Euler-Lagrange equation for the minimization of the functional

$$\mathscr{F}_\lambda(u) := \int_\Omega \sqrt{1+|\nabla u|^2} - \int_\Omega Hu + \lambda F(u)\, dx + \int_{\partial\Omega} |u|\, d\mathscr{H}^{n-1}(x) \qquad (1)$$

with $F(u) = \frac{1}{p+1}|u|^{p+1}$ (convex function).

When $\lambda = 0$, Problem (P_λ) reduces to a prescribed mean-curvature equation, which has been extensively studied (see for instance Bernstein [Ber10], Finn [Fin65], Giaquinta [Gia74], Massari [Mas74] or Giusti [Giu76, Giu78]). In particular, it is well known that a necessary condition for the existence of a classical solution of (P_λ) when $\lambda = 0$ (or the existence of a minimizer of $\mathscr{F}_{\lambda=0}$) is

$$\left|\int_A H\, dx\right| < P(A), \text{ for all proper subset } A \text{ of } \Omega, \qquad (2)$$

where $P(A)$ is the perimeter of A (see (5) for the definition of the perimeter). It is also known that the following is a sufficient condition (see Giaquinta [Gia74]):

$$\left|\int_A H\, dx\right| \leq (1-\varepsilon_0)P(A), \text{ for all measurable set } A \subset \Omega, \qquad (3)$$

for some $\varepsilon_0 > 0$.

Equation (P_λ) has also been studied for $\lambda < 0$ and $p = 1$ ($f(u) = u$), in particular in the framework of capillary surfaces (in that case, the Dirichlet boundary condition is often replaced by a contact angle condition. We refer the reader to the excellent book of Finn [Fin86] for more details on this topic). The existence of minimizers of (1) when $\lambda < 0$ is proved, for instance, by Giusti [Giu76] and Miranda [Mir64].

In this paper, we are interested in the case $\lambda > 0$. In that case, the functional $\mathscr{F}_\lambda$ is not convex, and the existence and regularity results that hold when $\lambda \leq 0$ no longer apply. The particular case $p = 1$ corresponds to the classical pendent drop problem (with the gravity pointing upward in our model). The pendent drop in a capillary tube (Equation (P_λ) in a fixed domain but with contact angle condition rather than Dirichlet condition) has been studied in particular by Huisken [Hui83]-[Hui84], while the corresponding free boundary problem, which describes a pendent drop hanging from a flat surface has been studied by Gonzalez, Massari and Tamanini [GMT80] and Giusti [Giu80]. In [Hui84], Huisken also studies the Dirichlet boundary problem (P_λ) when $p = 1$ (with possibly non-homogeneous boundary condition). This problem models a pendent drops hanging from a fixed boundary, such as the end of a pipette. Establishing suitable gradient estimates, Huisken proves the existence of a solution for small λ (see also Stone [Sto94] for a proof by convergence of a suitable evolution process). In [CF78], Concus and Finn characterize the profile of the radially symmetric pendent drops, thus finding explicit solutions for this mean curvature problem. Finally, in the case $H = 0$, other power like functions $f(u)$

have been considered, in particular by Pucci and Serrin [PS86] and Bidaut-Véron [BV93]. In that case, non-existence results can be obtained for $f(u) = u^p$ if $p \geq \frac{N+2}{N-2}$. Note however that in our paper, we will always assume that $H > 0$ (see condition (16)), and we will in particular show that a solution exists for all values of p, at least for small $\lambda > 0$.

1.1 Branches of minimal and non-minimal weak solutions

Through most of the paper, we will study *weak* solutions of (P_λ), which we will define as critical points of a suitable functional in $\mathrm{BV}(\Omega) \cap L^{p+1}(\Omega)$ (see Definition 2.2). In the radially symmetric case, we will see that those weak solutions are actually *classical* solutions (see Section 2.2) in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$ of (P_λ).

As noted above, a first difficulty when $\lambda > 0$ is that the functional $\mathscr{F}_\lambda$ is not convex and not bounded below. So global minimizers clearly do not exist. However, under certain assumptions on H (which guarantee the existence of a solution for $\lambda = 0$), it is still possible to show that solutions of (P_λ) exist for small values of λ (this is proved in particular by Huisken [Hui84] in the case $p = 1$). The goal of this paper is to show, under appropriate assumptions on H and for $p \geq 1$ that

1. there exists an extremal parameter $\lambda^* > 0$ such that (P_λ) admits a minimal non-negative weak solutions u_λ for all $\lambda \in [0, \lambda^*]$, while no weak solutions exists for $\lambda > \lambda^*$ (weak solutions will be defined as critical points of the energy functional that satisfy the boundary condition (see Definition 2.2), and by minimal solution, we mean the smallest non-negative solution),

2. minimal weak solutions are uniformly bounded in L^∞ by a constant depending only on Ω and the dimension.

We then investigate the regularity of the minimal weak solutions, and prove that

3. in the radially symmetric case, the set $\{u_\lambda\,;\, 0 \leq \lambda \leq \lambda^*\}$ is a branch of classical solutions (see Section 2.2 for a precise definition of classical solution). In particular, we will show that the extremal solution u_{λ^*}, which is the increasing limit of u_λ as $\lambda \to \lambda^*$, is itself a classical solution.

4. It follows that in the radially symmetric case, there exists another branch of (non-minimal) solutions for λ in a neighborhood $[\lambda^* - \eta, \lambda^*]$ of λ^*.

Those results will be stated more precisely in Section 2.3, after we introduce some notations in Sections 2.1 and 2.2. The rest of the paper will be devoted to the proofs of those results.

1.2 Semi-linear elliptic equations

These results and our analysis of Problem (P_λ) are guided by the study of the following classical problem:

$$\begin{cases} -\Delta u = g_\lambda(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

It is well known that if $g_\lambda(u) = \lambda f(u)$, with f superlinear and $f(0) > 0$, then there exists a critical value $\lambda^* \in (0, \infty)$ for the parameter λ such that one (or more) solution exists for $\lambda < \lambda^*$, a unique weak solution u^* exists for $\lambda = \lambda^*$ and there is no solution for $\lambda > \lambda^*$ (see [CR75]). And one of the key issue in the study of (4) is whether the extremal solution u^* is a classical solution or u_λ blows up when $\lambda \to \lambda^*$ (see [KK74, BCMR96, MR96, Mar97]).

Classical examples that have been extensively studied include power growth $g_\lambda(u) = \lambda(1+u)^p$ $(p > 1)$ and the celebrated Gelfand problem $g_\lambda(u) = \lambda e^u$ (see [JL73, MP80, BV97]). For such non-linearities, the minimal solutions, including the extremal solution u^* can be proved to be classical, at least in low dimension. In particular, for $g_\lambda(u) = \lambda(1+u)^p$, u^* is a classical solution if

$$n - 2 < F(p) := \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$$

(see Mignot-Puel [MP80]) while when $\Omega = B_1$ and $n - 2 \geq F(p)$, it can be proved that $u^* \sim Cr^{-2}$ (see Brezis-Vázquez [BV97]). For very general non-linearities of the form $g_\lambda(u) = \lambda f(u)$ with f superlinear, Nedev [Ned00] proves the regularity of u^* in low dimension while Cabré [Cab06] and Cabré-Capella [CC06, CC07] obtain optimal regularity results for u^* in the radially symmetric case.

Other examples of non-linearity have been studied, such as $g_\lambda(x, u) = f_0(x, u) + \lambda\varphi(x) + f_1(x)$ (see Berestycki-Lions [BL81]) or $g_\lambda(x, u) = \lambda f(x)/(1-u)^2$ (see Ghoussoub et al. [GG07, EGG07, GG08]).

Our goal is to study similar behavior for the mean-curvature operator. In the present paper, we only consider functions $g_\lambda(u) = H + \lambda u^p$, but the techniques introduced here can and will be extended to more general non-linearities in a forthcoming paper.

2 Definitions and main theorems

2.1 Weak solutions

We recall that $\mathrm{BV}(\Omega)$ denotes the set of functions in $L^1(\Omega)$ with bounded total variation over Ω, that is:

$$\int_\Omega |Du| := \sup\left\{ \int_\Omega u(x)\mathrm{div}(g)(x)\, dx; g \in \mathcal{C}^1_c(\Omega)^n, |g(x)| \leq 1 \right\} < +\infty.$$

The space $\mathrm{BV}(\Omega)$ is equipped with the norm

$$\|u\|_{\mathrm{BV}(\Omega)} = \|u\|_{L^1(\Omega)} + \int_\Omega |Du|.$$

If A is a Lebesgue subset of $\mathbb{R}^n$, its perimeter $P(A)$ is defined as the total variation of its characteristic function φ_A:

$$P(A) := \int_{\mathbb{R}^n} |D\varphi_A|, \quad \varphi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For $u \in \mathrm{BV}(\Omega)$, we define the "area" of the graph of u by

$$\mathscr{A}(u) := \int_\Omega \sqrt{1+|\nabla u|^2} = \sup\left\{\int_\Omega g_0(x) + u(x)\mathrm{div}(g)(x)\,dx\right\}, \tag{6}$$

where the supremum is taken over all functions $g_0 \in \mathcal{C}^1_c(\Omega), g \in \mathcal{C}^1_c(\Omega)^n$ such that $|g_0| + |g| \leq 1$ in Ω. An alternative definition is $\mathscr{A}(u) = \int_{\Omega\times\mathbb{R}} |D\varphi_U|$ where U is the subgraph of u. We have, in particular

$$\int_\Omega |Du| \leq \int_\Omega \sqrt{1+|\nabla u|^2} \leq |\Omega| + \int_\Omega |Du|. \tag{7}$$

A major difficulty, when developing a variational approach to (P_λ), is to deal with the boundary condition. It is well known that even when $\lambda = 0$, minimizers of $\mathscr{F}_\lambda$ may not satisfy the homogeneous Dirichlet condition (we need an additional condition on H and the curvature of $\partial\Omega$, see below condition (13)). Furthermore, the usual techniques used to handle this issue, which work when $\lambda \leq 0$ do not seem to generalize easily to the case $\lambda > 0$. For this reason, we will not use the functional $\mathscr{F}_\lambda$ in our analysis. Instead, we will define the solutions of (P_λ) as the "critical points" (the definition is made precise below, see Definition 2.2 and Remark 2.3) of the functional

$$\mathscr{I}_\lambda(u) := \int_\Omega \sqrt{1+|\nabla u|^2} - \int_\Omega H(x)u + \lambda F(u)\,dx \tag{8}$$

which satisfy the boundary condition $u = 0$ on $\partial\Omega$.

Proposition 2.1 (Directional derivative of the area functional)**.** *For any $u, \varphi \in \mathrm{BV}(\Omega)$ the limit*

$$\mathcal{L}(u)(\varphi) := \lim_{t\downarrow 0} \frac{1}{t}\left(\mathscr{A}(u+t\varphi) - \mathscr{A}(u)\right) \tag{9}$$

exists and, for all $u, v \in \mathrm{BV}(\Omega)$

$$\mathscr{A}(u) + \mathcal{L}(u)(v-u) \leq \mathscr{A}(v). \tag{10}$$

Proof. The existence of the limit in (9) follows from the convexity of the application $t \mapsto \mathscr{A}(u + t\varphi)$. By convexity also, we have

$$\mathscr{A}(u + t(v - u)) \leq (1 - t)\mathscr{A}(u) + t\mathscr{A}(v), \quad 0 \leq t \leq 1,$$

whence

$$\mathscr{A}(u) + \frac{1}{t}\left(\mathscr{A}(u + t(v - u)) - \mathscr{A}(u)\right) \leq \mathscr{A}(v), \quad 0 < t \leq 1,$$

which gives (10) at the limit $t \to 0$. $\square$

We stress out the fact $\mathcal{L}(u)$ is not linear, since we might not have

$$\mathcal{L}(u)(-\varphi) = -\mathcal{L}(u)(\varphi)$$

for all φ (for instance if φ is the characteristic function of a set A).

With the definition of $\mathcal{L}(u)$ given by Proposition 2.1, it is readily seen that local minimizers of $\mathscr{J}_0 \colon u \mapsto \mathscr{A}(u) - \int_\Omega Hu\,dx$ in $\mathrm{BV}(\Omega)$ satisfy

$$\mathcal{L}(u)(\varphi) \geq \int_\Omega H\varphi \qquad \text{for all } \varphi \in \mathrm{BV}(\Omega). \tag{11}$$

There is equality in (11) if u and φ are smooth enough, but strict inequality if, for instance, $\varphi = \varphi_A$ and $\mathscr{J}_0(u) < \mathscr{J}_0(u + t\varphi_A)$ for a $t > 0$ since

$$\frac{1}{t}\left(\mathscr{A}(u + t\varphi_A) - \mathscr{A}(u)\right) = P(A), \forall t > 0.$$

We thus consider the following definition:

Definition 2.2 (Weak solution)**.** *A function $u \in L^{p+1} \cap \mathrm{BV}(\Omega)$ is said to be a weak solution of (P_λ) if it satisfies*

$$\begin{cases} \mathcal{L}(u)(\varphi) \geq \displaystyle\int_\Omega [H + \lambda f(u)]\varphi\,dx, \quad \forall \varphi \in L^{p+1} \cap \mathrm{BV}(\Omega) \textit{ with } \varphi = 0 \textit{ on } \partial\Omega, \\ u \geq 0 \quad \textit{in } \Omega, \\ u = 0 \quad \textit{on } \partial\Omega. \end{cases} \tag{12}$$

Furthermore, a weak solution will be said to be minimal *if it is the smallest among all non-negative weak solutions.*

Remark 2.3 (Local minimizer and weak solution)**.** *With this definition, it is readily seen that a local minimizer u of $\mathscr{J}_\lambda$ in $L^{p+1} \cap \mathrm{BV}(\Omega)$ satisfying $u = 0$ on $\partial\Omega$ and $u \geq 0$ in Ω is a weak solution of (P_λ).*

Note that the boundary condition in Definition 2.2 makes sense because functions in $\mathrm{BV}(\Omega)$ have a unique trace in $L^1(\partial\Omega)$ if $\partial\Omega$ is Lipschitz (see [Giu84]).

2.2 Classical solutions

A classical solution of (P_λ) is a function $u \in \mathcal{C}^2(\overline{\Omega})$ which satisfies equation (P_λ) pointwise.

In the case of the semi-linear equation (4), it is well known that it is enough to show that a weak solution u is in $L^\infty(\Omega)$, to deduce that it is a classical solution of (4) (using, for instance, Calderon-Zygmund inequality and a bootstrap argument).

Because of the degenerate nature of the mean curvature operator, an L^∞ bound on u is not enough to show that it is a classical solution of (P_λ). When $H + \lambda f(u)$ is bounded in L^∞, classical results of the calculus of variation (see [Mas74] for instance), imply that for $n \leq 6$, the surface $(x, u(x))$, the graph of u, is $\mathcal{C}^\infty$ (analytic if H is analytic) and that u is continuous almost everywhere in Ω. However, to get further regularity on u itself, we need to show that u is Lipschitz continuous on Ω, as shown by the following proposition. In the rest of our paper we will thus focus in particular on the Lipschitz regularity of weak solutions.

Proposition 2.4. *Assume that H satisfies the conditions of Theorem 2.7, and let $u \in L^{p+1} \cap \mathrm{BV}(\Omega)$ be a weak solution of (P_λ) for some $\lambda > 0$. If $u \in \mathrm{Lip}(\Omega)$, then u is a classical solution of (P_λ). In particular, $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$ and u satisfies $-\mathrm{div}(Tu) = H + \lambda f(u)$ in Ω, $u = 0$ on $\partial\Omega$.*

Proof. This result follows from fairly classical arguments of the theory of prescribed mean curvature surfaces and elliptic equations (see for instance [GT01]). Anticipating a little bit, we can also notice that (modulo the regularity *up to* the boundary) it will be a consequence of Theorem 2.5 (ii) below (with $\overline{H} = H + \lambda f(u)$ instead of H), using the characterization of weak solutions given in Lemma 3.1 (ii). □

2.3 Main results

Before we state our main results, we recall the following theorem concerning the case $\lambda = 0$, which plays an important role in the sequel:

Theorem 2.5 (Giaquinta [Gia74])**.**

(i) Let Ω be a bounded domain with Lipschitz boundary and assume that $H(x)$ is a measurable function such that (3) holds for some $\varepsilon_0 > 0$. Then the functional

$$\mathscr{F}_0(u) := \mathscr{A}(u) - \int_\Omega H(x)u(x)\,dx + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1}$$

has a minimizer u in $\mathrm{BV}(\Omega)$.

(ii) Furthermore, if $\partial\Omega$ is $\mathcal{C}^1$, $H(x) \in \mathrm{Lip}(\Omega)$ and

$$|H(y)| \leq (n-1)\Gamma(y) \quad \textit{for all } y \in \partial\Omega \tag{13}$$

where $\Gamma(y)$ denotes the mean curvature of $\partial\Omega$ (with respect to the inner normal), then the unique minimizer of $\mathscr{F}_0$ belongs to $\mathcal{C}^{2,\alpha}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ for all $\alpha \in [0,1)$ and is solution to

$$\begin{cases} -\mathrm{div}(Tu) = H & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{14}$$

(iii) Finally, if $\partial\Omega$ is $\mathcal{C}^3$ and the hypotheses of (ii) hold, then $u \in \mathrm{Lip}(\Omega)$.

The key in the proof of *(i)* is the fact that (3) and the coarea formula for BV functions yield

$$\varepsilon_0 \int_\Omega |Du| \leq \int_\Omega |Du| - \int_\Omega H(x)u(x)\,dx$$

for all $u \in \mathrm{BV}(\Omega)$. This is enough to guarantee the existence of a minimizer. The condition (13) is a sufficient condition for the minimizer to satisfy $u = 0$ on $\partial\Omega$. In the sequel, we assume that Ω is such that (3) holds, as well as the following strong version of (13):

$$|H(y)| \leq (1-\varepsilon_0)(n-1)\Gamma(y) \quad \text{for all } y \in \partial\Omega. \tag{15}$$

Remark 2.6. *When $H(x) = H_0$ is constant, Serrin proves in [Ser69] that (13) is necessary for the equation $-\mathrm{div}(Tu) = H$ to have a solution for* any *smooth boundary data. However, it is easy to see that (13) is not always necessary for (14) to have a solution: when $\Omega = B_R$ and $H = \frac{n}{R}$, (14) has an obvious solution given by an upper half sphere, even though (13) does not hold since $(n-1)\Lambda = (n-1)/R < H = n/R$.*

Several results in this paper only require Equation (14) to have a solution with $(1+\delta)H$ in the right-hand side instead of H. In particular, this is enough to guarantee the existence of a minimal branch of solutions and the existence of an extremal solution. When $\Omega = B_R$, we can thus replace (15) with

$$|H(y)| \leq (1-\varepsilon_0)n\Gamma(y) \quad \textit{for all } y \in \partial B_R.$$

However, the regularity theory for the extremal solution will require the stronger assumption (15).

Finally, we assume that there exists a constant $H_0 > 0$ such that:

$$H \in \mathrm{Lip}(\Omega) \text{ and } H(x) \geq H_0 > 0 \text{ for all } x \in \Omega. \tag{16}$$

This last condition will be crucial in the proof of Lemma 4.2 to prove the existence of a non-negative solution for small values of λ. Note that Pucci and Serrin [PS86] proved, using a generalization of Pohozaev's Identity, that if $H = 0$ and $p \geq (n+2)/(n-2)$, then (P_λ) has no non-trivial solutions for any values of $\lambda > 0$ when Ω is star-shaped (see also Bidaut-Véron [BV93]).

Our main theorem is the following:

Theorem 2.7. *Let Ω be a bounded subset of $\mathbb{R}^n$ such that $\partial\Omega$ is $\mathcal{C}^3$. Assume that $H(x)$ satisfies conditions (3), (15) and (16). Then, there exists $\lambda^* > 0$ such that:*

(i) For all $\lambda \in [0, \lambda^*]$, (P_λ) has one minimal weak solution u_λ.

(ii) For $\lambda > \lambda^*$, (P_λ) has no weak solution.

(iii) The application $\lambda \mapsto u_\lambda$ is non-decreasing.

The proof of Theorem 2.7 is done in two steps: First we show that the set of λ for which a weak solution exists is a non-empty bounded interval (see Section 4). Then we prove the existence of the extremal solution for $\lambda = \lambda^*$ (see Section 6). The key result in this second step is the following uniform L^∞ estimate:

Proposition 2.8. *There exists a constant C depending only on Ω and H, such that the minimal weak solution u_λ of (P_λ) satisfies*

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } \lambda \in [0, \lambda^*].$$

Next we investigate the regularity of minimal weak solutions: We want to show that minimal weak solutions are classical solutions of (P_λ) (in view of Proposition 2.4, we need to obtain a Lipschitz estimate). This, it seems, is a much more challenging problem and we obtain some results only in the radially symmetric case. More precisely, we show the following:

Theorem 2.9. *Assume that $\Omega = B_R \subset \mathbb{R}^n$ ($n \geq 1$), $H = H(r)$, and that the conditions of Theorem 2.7 hold. Then the minimal weak solution of (P_λ) is radially symmetric and lies in $\mathrm{Lip}(\Omega)$. In particular there exists a constant C such that*

$$|\nabla u_\lambda(x)| \leq \frac{C}{\lambda^* - \lambda} \qquad \text{a.e. in } \Omega, \quad \forall \lambda \in [0, \lambda^*). \tag{17}$$

In particular u_λ is a classical solution of (P_λ), and if $H(x)$ is analytic in Ω, then u_λ is analytic in Ω for all $\lambda < \lambda^*$.

Note that the Lipschitz constant in (17) blows up as $\lambda \to \lambda^*$. However, we are then able to show the following:

Theorem 2.10. *Assume that the conditions of Theorem 2.9 hold. Then there exists a constant C such that for any $\lambda \in [0, \lambda^*]$, the minimal weak solution $u_\lambda \in \mathrm{Lip}(\Omega)$ and satisfies*

$$|\nabla u_\lambda(x)| \leq C \qquad \text{a.e. in } \Omega.$$

In particular the extremal solution u^* is a classical solution of (P_λ).

The classical tools of continuation theory developed for example in [KK74, CR75] can be modified in our context (non-linear leading order differential operator, radial case) to show that there exists a second branch of solution in the neighborhood of λ^*:

Theorem 2.11. *Assume that the conditions of Theorem 2.9 hold. Then there exists $\delta > 0$ such that for $\lambda^* - \delta < \lambda < \lambda^*$ there are at least two classical solutions to (P_λ).*

To prove this result, we will need to consider the linearized operator

$$L_\lambda(v) = -\partial_i(a^{ij}(\nabla u_\lambda)\partial_j v) - \lambda f'(u_\lambda)v$$

where

$$a^{ij}(\mathbf{p}) = \frac{1}{(1+|\mathbf{p}|^2)^{1/2}}\left(\delta_{ij} - \frac{\mathbf{p}_i\mathbf{p}_j}{1+|\mathbf{p}|^2}\right), \qquad \mathbf{p} \in \mathbb{R}^n.$$

If we denote by $\mu_1(\lambda)$ the first eigenvalue of L_λ, we will prove in particular:

Lemma 2.12. *Assume that the conditions of Theorem 2.9 hold. Then the linearized operator L_λ has positive first eigenvalue $\mu_1(\lambda) > 0$ for all $\lambda \in [0, \lambda^*)$. Furthermore, the linearized operator L_{λ^*} corresponding to the extremal solution has zero first eigenvalue $\mu_1(\lambda^*) = 0$, and λ^* corresponds to a turning point for the (λ, u_λ) diagram.*

A turning point means that there exists a parametrized family of classical solutions

$$s \mapsto (\lambda(s), u(s)), \qquad s \in (-\varepsilon, \varepsilon)$$

with $\lambda(0) = \lambda^*$ and $\lambda(s) < \lambda^*$ both for $s < 0$ and $s > 0$. In particular we will prove that $\lambda'(0) = 0$ and $\lambda''(0) < 0$.

In the radially symmetric case, we can thus summarize our results in the following corollary:

Corollary 2.13. *Assume that $\Omega = B_R \subset \mathbb{R}^n$ $(n \geq 1)$, $H = H(r)$, and that the conditions of Theorem 2.7 hold. Then there exists $\lambda^* > 0$, $\delta > 0$ such that*

1. *if $\lambda > \lambda^*$, there is no weak solution of (P_λ),*
2. *if $\lambda \leq \lambda^*$, there is a minimal classical solution of (P_λ).*
3. *if $\lambda^* - \delta < \lambda < \lambda^*$, there are at least two classical solutions of (P_λ).*

Finally, we point out that numerical computation suggest that for some values of n and H, a third branch of solutions may arise (and possibly more).

The paper is organized as follows: In Section 3, we give some a priori properties of weak solutions. In Section 4 we show the existence of a branch of minimal weak solutions for $\lambda \in [0, \lambda^*)$. We then establish, in Section 5, a uniform L^∞ bound for these minimal weak solutions (Proposition 2.8), which we use, in Section 6, to show the existence of an extremal solution as $\lambda \to \lambda^*$ (thus completing the proof of Theorem 2.7). In the last Section 7 we prove the regularity of the minimal weak solutions, including that of u_{λ^*}, in the radial case (Theorems 2.9 and 2.10) and we give the proof of Theorem 2.11. In appendix, we prove a comparison lemma that is used several times in the paper.

Remark 2.14. *One might want to generalize those results to other non-linearity $f(u)$: In fact, all the results presented here holds (with the same proofs) if f is a $\mathcal{C}^2$ function satisfying:*

(H1) $f(0) = 0$, $f'(u) \geq 0$ *for all* $u \geq 0$.

(H2) There exists C *and* $\alpha > 0$ *such that* $f'(u) \geq \alpha$ *for all* $u \geq C$.

(H3) If $u \in L^q(\Omega)$ *for all* $q \in [0, \infty)$ *then* $f(u) \in L^n(\Omega)$.

The last condition, which is used to prove the L^∞ bound (and the Lipschitz regularity near $r = 0$) of the extremal solution u_{λ^*} is the most restrictive. It excludes in particular non-linearities of the form $f(u) = e^u - 1$. However, similar results hold also for such non-linearities, though the proof of Proposition 2.8 has to be modified in that case. This will be developed in a forthcoming paper.

We can also consider right-hand sides of the form $\lambda(1+u)^p$ (or λe^u). In that case, Theorem 2.7, Proposition 2.8 and Theorem 2.9 are still valid (but require different proofs), but Theorem 2.10 is not. Indeed, our proof of the boundary regularity of the extremal solution u_{λ^*} (Lemma 7.3) relies heavily on condition (15), which should be replaced here by the condition

$$\lambda^* < (n-1)\Gamma(y) \qquad \text{for all } y \in \partial\Omega. \tag{18}$$

However, it is not clear that λ^* should satisfy (18).

3 Properties of weak solutions

3.1 Weak solutions as global minimizers

Non-negative minimizers of $\mathscr{J}_\lambda$ that satisfy $u = 0$ on $\partial\Omega$ are in particular critical points of $\mathscr{J}_\lambda$, and thus weak solutions of (P_λ). But not all critical points are minimizers. However, the convexity of the perimeter yields the following result:

Lemma 3.1. *Assume that $\partial\Omega$ is $\mathcal{C}^1$ and let u be a non-negative function in $L^{p+1} \cap \mathrm{BV}(\Omega)$. The following propositions are equivalent:*

(i) u is a weak solution of (P_λ),

(ii) $u = 0$ on $\partial\Omega$ and for every $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$, we have

$$\mathscr{A}(u) - \int_\Omega (H + \lambda f(u))\, u\, dx \leq \mathscr{A}(v) - \int_\Omega (H + \lambda f(u))\, v\, dx + \int_{\partial\Omega} |v|\, d\mathscr{H}^{N-1}, \tag{19}$$

(iii) $u = 0$ on $\partial\Omega$ and for every $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$, we have

$$\mathscr{J}_\lambda(u) \leq \mathscr{J}_\lambda(v) + \int_\Omega \lambda G(u, v)\, dx + \int_{\partial\Omega} |v|\, d\mathscr{H}^{N-1} \tag{20}$$

where

$$G(u, v) = F(v) - F(u) - f(u)(v - u) \geq 0.$$

In particular, *(ii)* implies that any weak solution u of (P_λ) is a global minimizer in $L^{p+1}\cap \mathrm{BV}(\Omega)$ of the functional (which depends on u)

$$\mathscr{F}_\lambda^{[u]}(v) := \mathscr{A}(v) - \int_\Omega (H+\lambda f(u))\, v\, dx + \int_{\partial\Omega} |v|\, d\mathscr{H}^{N-1}.$$

Furthermore, since $G(u,u)=0$, *(iii)* implies that any weak solution u of (P_λ) is also a global minimizer in $L^{p+1}\cap \mathrm{BV}(\Omega)$ of the functional

$$\begin{aligned}\mathscr{G}_\lambda^{[u]}(v) &:= \mathscr{A}(v) - \int_\Omega Hv + \lambda F(v)\, dx + \int_{\partial\Omega} |v|\, d\mathscr{H}^{N-1} + \int_\Omega \lambda G(u,v)\, dx \\ &= \mathscr{J}_\lambda(v) + \int_{\partial\Omega} |v|\, d\mathscr{H}^{N-1} + \int_\Omega \lambda G(u,v)\, dx.\end{aligned}$$

Proof of Lemma 3.1. The last two statements *(ii)* and *(iii)* are clearly equivalent (this follows from a simple computation using the definition of G).

Next, we notice that if *(ii)* holds, then taking $v=u+t\varphi$ in (19), where $\varphi\in L^{p+1}\cap\mathrm{BV}(\Omega)$ with $\varphi=0$ on $\partial\Omega$, we get

$$\frac{1}{t}(\mathscr{A}(u+t\varphi)-\mathscr{A}(u)) \geq \int (H+\lambda f(u))\varphi\, dx.$$

Passing to the limit $t\to 0$, we deduce $\mathcal{L}(u)(\varphi)\geq \int_\Omega (H+f(u))\varphi dx$, i.e. u is a solution of (12). In view of Definition 2.2, we thus have *(ii)⇒(i)*.

So it only remains to prove that *(i)* implies *(ii)*, that is

$$\mathscr{F}_\lambda^{[u]}(u) = \min_{v\in L^{p+1}\cap \mathrm{BV}(\Omega)} \mathscr{F}_\lambda^{[u]}(v).$$

By definition of weak solutions, we have

$$\mathcal{L}(u)(\varphi) \geq \int_\Omega (H+\lambda f(u))\varphi\, dx$$

for all $\varphi\in L^{p+1}\cap\mathrm{BV}(\Omega)$ with $\varphi=0$ on $\partial\Omega$. Furthermore, by (10), we have

$$\mathscr{A}(u)+\mathcal{L}(u)(v-u)\leq \mathscr{A}(v),$$

for every $v\in L^{p+1}\cap\mathrm{BV}(\Omega)$ with $v=0$ on $\partial\Omega$. We deduce (taking $\varphi=v-u$):

$$\mathscr{A}(u)+\int_\Omega (H+\lambda f(u))(v-u)\, dx \leq \mathscr{A}(v),$$

which implies

$$\mathscr{F}_\lambda^{[u]}(u) \leq \mathscr{F}_\lambda^{[u]}(v) \tag{21}$$

for all $v\in L^{p+1}\cap\mathrm{BV}(\Omega)$ satisfying $v=0$ on $\partial\Omega$.

It thus only remains to show that (21) holds even when $v\neq 0$ on $\partial\Omega$. For that, the idea is to apply (21) to the function $v-w^\varepsilon$ where (w^ε) is a sequence of functions in $L^{p+1} \cap \mathrm{BV}(\Omega)$ converging to 0 in $L^{p+1}(\Omega)$ such that $w^\varepsilon = v$ on $\partial\Omega$. Heuristically the mass of w^ε concentrates on the boundary $\partial\Omega$ as ε goes to zero, and so $\mathscr{A}(v - w^\varepsilon)$ converges to $\mathscr{A}(v) + \int_{\partial\Omega} |v| d\mathscr{H}^{N-1}$. This type of argument is fairly classical, but we give a detailed proof below, in particular to show how one can pass to the limit in the non-linear term.

First, we consider $v \in L^\infty \cap \mathrm{BV}(\Omega)$. Then, for every $\varepsilon > 0$, there exists $w^\varepsilon \in L^\infty \cap \mathrm{BV}(\Omega)$ such that $w^\varepsilon = v$ on $\partial\Omega$ satisfying the estimates:

$$\|w^\varepsilon\|_{L^1(\Omega)} \leq \varepsilon \int_{\partial\Omega} |v| d\mathscr{H}^{N-1}, \quad \int_\Omega |Dw^\varepsilon| \leq (1+\varepsilon) \int_{\partial\Omega} |v| d\mathscr{H}^{N-1} \tag{22}$$

and $\|w^\varepsilon\|_{L^\infty(\Omega)} \leq 2\|v\|_{L^\infty(\Omega)}$ (see Theorem 2.16 in [Giu84]). In particular we note that

$$\|w^\varepsilon\|^{p+1}_{L^{p+1}(\Omega)} \leq 2^p \|v\|^p_{L^\infty(\Omega)} \|w^\varepsilon\|_{L^1(\Omega)} \to 0 \tag{23}$$

when $\varepsilon \to 0$. Using (21), (22) and the fact that $\mathscr{A}(v - w^\varepsilon) \leq \mathscr{A}(v) + \int_\Omega |Dw^\varepsilon|$, we deduce:

$$\begin{aligned}
\mathscr{F}^{[u]}_\lambda(u) &\leq \mathscr{F}^{[u]}_\lambda(v - w^\varepsilon) \\
&\leq \mathscr{A}(v) - \int_\Omega (H + \lambda f(u))\, v\, dx + \int_\Omega |Dw^\varepsilon| + \int_\Omega (H + f(u)) w^\varepsilon\, dx \\
&\leq \mathscr{A}(v) - \int_\Omega (H + \lambda f(u))\, v\, dx + (1+\varepsilon) \int_{\partial\Omega} |v| d\mathscr{H}^{N-1} \\
&\qquad + \|w^\varepsilon\|_{L^{p+1}} \|H + f(u)\|_{L^{\frac{p+1}{p}}} \\
&= \mathscr{F}^{[u]}_\lambda(v) + \varepsilon \int_{\partial\Omega} |v| d\mathscr{H}^{N-1} + \|w^\varepsilon\|_{L^{p+1}} \|H + f(u)\|_{L^{\frac{p+1}{p}}}.
\end{aligned} \tag{24}$$

(Note that $f(u) \in L^{\frac{p+1}{p}}(\Omega)$ since $u \in L^{p+1}(\Omega)$). Using (23) and taking the limit $\varepsilon \to 0$ in (24), we obtain (21) for any $v \in L^\infty \cap \mathrm{BV}(\Omega)$.

We now take $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$. Then, the computation above shows that for every $M > 0$ we have:

$$\mathscr{F}^{[u]}_\lambda(u) \leq \mathscr{F}^{[u]}_\lambda(T_M(v)),$$

where T_M is the truncation operator $T_M(s) := \min(M, \max(s, -M))$. Clearly, we have $T_M(v) \to v$ in $L^{p+1}(\Omega)$ as $M \to \infty$. Furthermore, one can show that $\mathscr{A}(T_M(v)) \to \mathscr{A}(v)$. As a matter of fact, the lower semi-continuity of the perimeter gives $\mathscr{A}(v) \leq \liminf_{M\to+\infty} \mathscr{A}(T_M(v))$, and the coarea formula implies:

$$\begin{aligned}
\mathscr{A}(T_M(v)) &\leq \mathscr{A}(v) + \int_\Omega |D(v - T_M(v))| \\
&= \mathscr{A}(v) + \int_0^{+\infty} P(\{v - T_M(v) > t\}) dt \\
&= \mathscr{A}(v) + \int_M^{+\infty} P(\{v > t\}) dt \\
&\longrightarrow \mathscr{A}(v) \text{ when } M \to +\infty.
\end{aligned}$$

We deduce that $\mathscr{F}_\lambda^{[u]}(T_M(v)) \longrightarrow \mathscr{F}_\lambda^{[u]}(v)$, and the proof is complete. $\square$

3.2 A priori bounds

Next, we want to derive some a priori bounds satisfied by any weak solution u of (P_λ).

First, we have the following lemma:

Lemma 3.2. *Let u be a weak solution of (P_λ), then*

$$\int_A H + \lambda f(u)\, dx \leq P(A)$$

for all measurable sets $A \subset \Omega$.

Proof. When u is smooth, this lemma can be proved by integrating (P_λ) over the set A and noticing that $|\frac{\nabla u \cdot \nu}{\sqrt{1+|\nabla u|^2}}| \leq 1$ on ∂A. If u is not smooth, we use Lemma 3.1 (ii): for all $A \subset \Omega$, we get (with $v = \varphi_A$):

$$\mathscr{A}(u) - \int_\Omega [H + \lambda f(u)]u \leq \mathscr{A}(u+\varphi_A) - \int_\Omega [H+\lambda f(u)](u+\varphi_A) + \mathscr{H}^{n-1}(\partial\Omega \cap A).$$

We deduce

$$0 \leq \int_\Omega |D\varphi_A| + \mathscr{H}^{n-1}(\partial\Omega \cap A) - \int_A H + \lambda f(u)\, dx.$$

and so

$$0 \leq P(A) - \int_A H + \lambda f(u)\, dx. \qquad \square$$

Lemma 3.2 suggests that λ can not be too large for (P_λ) to have a weak solution. In fact, it provides an upper bound on λ, if we know that $\int_\Omega u\, dx$ is bounded from below. This is proved in the next lemma:

Lemma 3.3 (Bound from below)**.** *Let u be a weak solution of (P_λ) for some $\lambda \geq 0$. Then*

$$u \geq \underline{u} \quad \text{in } \Omega$$

where $\underline{u}$ is the solution corresponding to $\lambda = 0$:

$$\left\{ \begin{array}{rcll} -\operatorname{div}(T\underline{u}) & = & H & \text{in } \Omega, \\ \underline{u} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \qquad (P_0)$$

Proof. For $\delta \geq 0$, let u_δ be the solution to the problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}(Tu) & = & (1-\delta)H & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \right. \qquad (P_\delta)$$

Problem (P_δ) has a solution $u_\delta \in \mathrm{Lip}(\Omega)$ (by Theorem 2.5) and (u_δ) is increasing to $\underline{u}$ when $\delta \downarrow 0$. We also recall [Giu76] that the function u_δ is the unique minimizer in $L^{p+1} \cap \mathrm{BV}(\Omega)$ of the functional

$$\mathscr{F}_\delta(u) = \int_\Omega \sqrt{1+|\nabla u|^2} - \int_\Omega (1-\delta)H(x)u(x)\,dx + \int_{\partial\Omega} |u|.$$

The lemma then follows easily from the comparison principle, Lemma A.1: Taking $G_-(x,s) = -(1-\delta)H(x)s,\ G_+(x,s) = -H(x)s - \lambda F(s) + \lambda G(u(x),s)$, $K_- = K_+ = L^{p+1} \cap \mathrm{BV}(\Omega)$, Lemma A.1 implies:

$$\begin{aligned} 0 &\le \int_\Omega -\delta H(\max(u_\delta,u)-u) + \lambda[F(u) - F(\max(u,u_\delta)) + G(u,\max(u,u_\delta))] \\ &= -\int_\Omega (\delta H + \lambda f(u))\,(u_\delta - u)_+, \end{aligned}$$

where $v_+ = \max(v,0)$. Since $H > 0$ and $u \ge 0$ in Ω, this implies $u_\delta \le u$ a.e. in Ω. Taking the limit $\delta \to 0$, we obtain $\underline{u} \le u$ a.e. in Ω. □

As a corollary to Lemma 3.2 and Lemma 3.3, we have the following a priori bound on λ:

Lemma 3.4 (A priori bound)**.** *If (P_λ) has a weak solution for some $\lambda \ge 0$, then*

$$\lambda \le \frac{P(\Omega) - \int_\Omega H\,dx}{\int_\Omega \underline{u}\,dx}$$

with $\underline{u}$ solution of (P_0).

4 Existence of minimal weak solutions for $\lambda \in [0,\lambda^*)$

In this section, we begin the proof of Theorem 2.7 by showing the following proposition:

Proposition 4.1. *Let Ω be a bounded subset of $\mathbb{R}^n$ such that $\partial\Omega$ is $\mathcal{C}^3$. Assume that $H(x)$ satisfies conditions (3), (15) and (16). Then, there exists $\lambda^* > 0$ such that:*

(i) For all $\lambda \in [0,\lambda^*)$, (P_λ) has one minimal weak solution u_λ.

(ii) For $\lambda > \lambda^*$, (P_λ) has no weak solution.

(iii) The application $\lambda \mapsto u_\lambda$ is non-decreasing.

To establish Theorem 2.7, it will thus only remain to show the existence of an extremal solution for $\lambda = \lambda^*$. This will be done in Section 6. To prove Proposition 4.1, we will first show that weak solutions exist for small values of λ. Then, we will prove that the set of the values of λ for which weak solutions exist is an interval.

4.1 Existence of weak solutions for small values of λ

We start with the following lemma:

Lemma 4.2. *Suppose that (3), (15) and (16) hold. Then there exists $\lambda_0 > 0$ such that (P_λ) has a weak solution for all $\lambda < \lambda_0$.*

Note that Lemma 4.2 is proved by Huisken in [Hui84] (see also [Sto94]) in the case $p = 1$. Our proof is slightly different from those two references and relies on the fact that $H > 0$.

Proof. We will show that for small λ, the functional $\mathscr{J}_\lambda$ has a local minimizer in $L^{p+1} \cap \mathrm{BV}(\Omega)$ that satisfies $u = 0$ on $\partial\Omega$. Such a minimizer is a critical point for $\mathscr{J}_\lambda$, and thus (see Remark 2.3) a weak solution of (P_λ).

Let δ be a small parameter such that $(1+\delta)(1-\varepsilon_0) < 1$ where ε_0 is defined by the conditions (3) and (15). Then there exists $\varepsilon' > 0$ such that

$$\left| \int_A (1+\delta) H \, dx \right| \leq (1+\delta)(1-\varepsilon_0)\mathscr{H}^{n-1}(\partial A) \leq (1-\varepsilon')P(A),$$

and

$$|(1+\delta)H(y)| \leq (1-\varepsilon')(n-1)\Gamma(y) \qquad \forall y \in \partial\Omega.$$

Theorem 2.5 thus gives the existence of $w \geq 0$ local minimizer in $\mathrm{BV}(\Omega)$ of

$$\mathscr{G}_\delta(u) = \mathscr{A}(u) - \int_\Omega (1+\delta)H(x)u \, dx + \int_{\partial\Omega} |u| \, d\sigma(x),$$

with $w \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ and $w = 0$ on $\partial\Omega$.

It is readily seen that the functional $\mathscr{J}_\lambda$ has a global minimizer u in

$$K = \{v \in L^{p+1} \cap \mathrm{BV}(\Omega)\, ;\, 0 \leq v \leq w + 1\}.$$

We are now going to show that if λ is small enough, then u satisfies

$$u(x) \leq w(x) \quad \text{ in } \Omega. \tag{25}$$

For this, we use the comparison principle (Lemma A.1) with $G_-(x,s) = -H(x)s - \lambda F(s)$ and $G_+(x,s) = -(1+\delta)H(x)s$ (i.e. $\mathscr{F}_- = \mathscr{J}_\lambda$ and $\mathscr{F}_+ = \mathscr{G}_\delta$), and $K_- = L^{p+1} \cap \mathrm{BV}(\Omega)$, $K_+ = K$. Since $\max(u,w) \in K$, we obtain

$$\begin{aligned} 0 &\leq \int_\Omega -\delta H(\max(u,w) - w) + \lambda(F(\max(u,w)) - F(w)) \, dx \\ &\leq \int_\Omega -\delta H(\max(u,w) - w) + \lambda \sup_{s \in [0, \|w\|_\infty + 1]} |f(s)|(\max(u,w) - w) \, dx \\ &\leq \int_\Omega -(u-w)_+ \left[\delta H - \lambda f(\|w\|_\infty + 1)\right] \, dx. \end{aligned}$$

Therefore, if we take λ small enough such that $\lambda < \delta \frac{\inf H}{f(\|w\|_\infty + 1)} = \delta \frac{H_0}{f(\|w\|_\infty + 1)}$, we deduce (25).

Finally, (25) implies that $u = 0$ on $\partial\Omega$ and that u is a critical point of $\mathscr{J}_\lambda$ in $L^{p+1} \cap \mathrm{BV}(\Omega)$, which completes the proof. □

4.2 Existence of u_λ for $\lambda < \lambda^*$

We now define

$$\lambda^* = \sup\{\lambda\,;\,(P_\lambda) \text{ has a weak solution}\}.$$

Lemmas 3.4 and 4.2 imply

$$0 < \lambda^* < \infty.$$

In order to complete the proof of Proposition 4.1, we need to show:

Proposition 4.3. *For all $\lambda \in [0, \lambda^*)$ there exists a minimal weak solution u_λ of (P_λ). Furthermore, the application $\lambda \mapsto u_\lambda$ is non-decreasing.*

Proof of Proposition 4.3. Let us fix $\lambda_1 \in [0, \lambda^*)$. By definition of λ^*, there exists $\overline{\lambda} \in (\lambda_1, \lambda^*]$ such that (P_λ) has a weak solution $\overline{u} \in L^{p+1} \cap \mathrm{BV}(\Omega)$ for $\lambda = \overline{\lambda}$.

We also recall that $\underline{u}$ denotes the solution to (P_0). We then define the sequence u_n as follows: We take

$$u_0 = \underline{u}$$

and for any $n \geq 1$, we set

$$I_n(v) = \mathscr{A}(v) - \int_\Omega [H + \lambda_1 f(u_{n-1})]v\, dx + \int_{\partial\Omega} |v|$$

and let u_n be the unique minimizer of I_n in $\mathrm{BV}(\Omega)$. In order to prove Proposition 4.3, we will show that this sequence (u_n) is well defined (i.e. that u_n exists for all n), and that it converges to a weak solution of (P_{λ_1}). This will be a consequence of the following Lemma:

Lemma 4.4. *For all $n \geq 1$, the functional I_n admits a global minimizer u_n on* $\mathrm{BV}(\Omega)$*. Moreover, $u_n \in \mathrm{Lip}(\Omega)$ satisfies*

$$\underline{u} \leq u_{n-1} < u_n \leq \overline{u} \text{ in } \Omega. \tag{26}$$

We can now complete the proof of Proposition 4.3: by Lebesgue's monotone convergence theorem, we get that (u_n) converges almost everywhere and in $L^{p+1}(\Omega)$ to a function u_∞ satisfying

$$0 \leq u_\infty \leq \overline{u}.$$

In particular, we have $u_\infty = 0$ on $\partial\Omega$. Furthermore, for every $n \geq 0$, we have

$$I_n(u_n) \leq I_n(0) = |\Omega|$$

and so by (7),

$$\int_\Omega |Du_n| \leq 2|\Omega| + \sup(H)\|\overline{u}\|_{L^1} + \lambda_1 \|\overline{u}\|^{p+1}_{L^{p+1}(\Omega)},$$

hence, by lower semi-continuity of the total variation, $u_\infty \in L^{p+1} \cap \mathrm{BV}(\Omega)$. Finally, for all $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$ and for all $n \geq 1$, we have

$$I_n(u_n) \leq I_n(v)$$

and using the lower semi-continuity of the perimeter, and the strong L^{p+1} convergence, we deduce

$$\mathscr{A}(u_\infty) - \int H u_\infty + \lambda_1 f(u_\infty) u_\infty \, dx \leq \mathscr{A}(v) - \int Hv + \lambda_0 f(u_\infty) v \, dx$$

We conclude, using Lemma 3.1 *(ii)*, that u_∞ is a solution of (P_{λ_1}). □

The rest of this section is devoted to the proof of Lemma 4.4:

Proof of Lemma 4.4. We recall that $\underline{u}$ denotes the unique minimizer of $\mathscr{F}_0$ in $\mathrm{BV}(\Omega)$ and that, by Lemma 3.3, we have the inequality $\underline{u} \leq \overline{u}$ a.e. on Ω.

Assume now that we constructed u_{n-1} satisfying $u_{n-1} \in \mathrm{Lip}(\Omega)$ and

$$\underline{u} \leq u_{n-1} \leq \overline{u}.$$

We are going to show that u_n exists and satisfies (26) (this implies Lemma 4.4 by first applying the result to $n = 1$ and proceeding from there by induction).

First of all, Lemma 3.2 implies

$$\int_A H + \overline{\lambda} f(\overline{u}) \, dx \leq P(A)$$

for all measurable sets $A \subset \Omega$. Since $u_{n-1} \leq \overline{u}$ and $\lambda_1 < \overline{\lambda}$, we deduce that

$$\int_A H + \lambda_1 f(u_{n-1}) \, dx < P(A) \tag{27}$$

for all measurable sets $A \subset \Omega$. Following Giusti [Giu78], we can then prove (a proof of this lemma is given at the end of this section):

Lemma 4.5. *There exists $\varepsilon > 0$ such that*

$$\int_A H + \lambda_1 f(u_{n-1}) \, dx < (1 - \varepsilon) P(A)$$

for all measurable sets $A \subset \Omega$. In particular (3) holds with $\overline{H} = H + \lambda_1 f(u_{n-1})$ instead of H

This lemma easily implies the existence of a minimizer u_n of I_n in $\mathrm{BV}(\Omega)$ (using Theorem 2.5 with $\overline{H}$ instead of H). Furthermore, since $u_{n-1} \in \mathrm{Lip}(\Omega)$ and $u_{n-1} = 0$ on $\partial\Omega$ condition (13) is satisfied with $\overline{H}$ instead of H and so (by Theorem 2.5):

$$u_n = 0 \text{ on } \partial\Omega$$

and

$$u_n \in \mathrm{Lip}(\Omega).$$

Finally, we check that the minimizer u_n satisfies

$$\underline{u} \leq u_n \leq \overline{u}.$$

Indeed, the first inequality is a consequence of the comparison Lemma A.1 applied to $\mathscr{F}_- = \mathscr{F}_0$, $\mathscr{F}_+ = I_n$, $K_+ = K_- = \mathrm{BV}(\Omega)$, which gives

$$0 \leq -\int_\Omega \lambda_1 f(u_{n-1})(\max(\underline{u}, u_n) - u_n)\, dx.$$

The second inequality is obtained by applying Lemma A.1 to $\mathscr{F}_- = I_n$, $\mathscr{F}_+ = \mathscr{F}^{[\overline{u}]}_{\overline{\lambda}}$, $K_+ = K_- = L^{p+1} \cap \mathrm{BV}(\Omega)$:

$$0 \leq \int_\Omega (\lambda_1 f(u_{n-1}) - \overline{\lambda} f(\overline{u}))(\max(\overline{u}, u_n) - \overline{u})\, dx$$

and using the fact that $u_{n-1} \leq \overline{u}$ and $\lambda_1 < \overline{\lambda}$.

Since $u_n \in \mathrm{Lip}(\Omega)$, u_n satisfies the Euler-Lagrange equation associated to the minimization of I_n: $-\mathrm{div}(Tu_n) = H + \lambda_1 f(u_{n-1})$. If $n \geq 2$ and $u_{n-1} \geq u_{n-2}$, we then obtain the inequality $u_n > u_{n-1}$ by the strong maximum principle (57) for Lipschitz continuous functions. □

Proof of Lemma 4.5. The proof of the lemma is similar to the proof of Lemma 1.1 in [Giu78]: Assuming that the conclusion is false, we deduce that there exists a sequence A_k of (non-empty) subsets of Ω satisfying $\displaystyle\int_{A_k} \overline{H} \geq (1 - k^{-1}) P(A_k)$, $\overline{H} := H + \lambda_1 f(u_{n-1})$. In particular $P(A_k) = \displaystyle\int_{\mathbb{R}^N} |D\varphi_{A_k}|$ is bounded, so there exists a Borel subset A of Ω such that, up to a subsequence, $\varphi_{A_k} \to \varphi_A$ in $L^1(\Omega)$ and, by lower semi-continuity of the perimeter, $\displaystyle\int_A \overline{H} \geq P(A)$. This is a contradiction to the strict inequality (27) except if A is empty. But the isoperimetric inequality gives

$$|A_k|^{\frac{n}{n-1}} \leq P(A_k) \leq (1 - k^{-1})^{-1} \int_{A_k} \overline{H} \leq (1 - k^{-1})^{-1} \|\overline{H}\|_{L^n(A_k)} |A_k|^{\frac{n}{n-1}}$$

hence

$$(1 - k^{-1}) \leq \|\overline{H}\|_{L^n(A_k)} \quad \text{for all } k \geq 2.$$

Since $\overline{H}$ is bounded (remember that $u_{n-1} \in \mathrm{Lip}(\Omega)$), we deduce

$$\frac{1}{2} \leq C |A_k|^{1/n}$$

and so $|A| > 0$ since $\varphi_{A_k} \to \varphi_A$ in $L^1(\Omega)$. Consequently, A cannot be empty and we have a contradiction. □

5 Uniform L^∞ bound for minimal weak solutions

The goal of this section is to establish the L^∞ estimate (Proposition 2.8) for $\lambda < \lambda^*$. More precisely, we show:

Proposition 5.1. *There exists a constant C depending only on Ω and H such that, for every $0 \le \lambda < \lambda^*$, the minimal weak solution u_λ to (P_λ) satisfies*

$$\|u_\lambda\|_{L^\infty(\Omega)} \le C.$$

This estimate will be used in the next section to show that u_λ converges to a weak solution of (P_λ) as $\lambda \to \lambda^*$.

The proof relies on an energy method à la DeGiorgi [DG57]. Note that, in general, weak solutions are not minimizers (not even local ones) of the energy functional $\mathscr{J}_\lambda$. But it is classical that the minimal solutions u_λ enjoy some semi-stability properties. More precisely, we will show that u_λ is a global minimizer of $\mathscr{J}_\lambda$ with respect to non-positive perturbations. We will then use classical calculus of variation methods to prove Proposition 5.1.

5.1 Minimal solutions as one-sided global minimizers

We now show the following lemma:

Lemma 5.2. *The minimal weak solution u_λ of (P_λ) is a global minimizer of the functional $\mathscr{J}_\lambda$ over the set $K_\lambda = \{v \in L^{p+1} \cap \mathrm{BV}(\Omega); 0 \le v \le u_\lambda\}$. Furthermore, u_λ is a semi-stable solution in the sense that, if $u_\lambda \in \mathrm{Lip}(\Omega)$, then $\mathscr{J}_\lambda''(u_\lambda) \ge 0$: for all φ in $\mathcal{C}^1(\Omega)$ satisfying $\varphi = 0$ on $\partial\Omega$, we have:*

$$Q_\lambda(\varphi) := \int_\Omega \frac{|\nabla\varphi|^2}{(1+|\nabla u_\lambda|^2)^{1/2}} - \frac{|\nabla\varphi \cdot \nabla u_\lambda|^2}{(1+|\nabla u_\lambda|^2)^{3/2}} - \lambda f'(u_\lambda)\varphi^2 \, dx \ge 0. \qquad (28)$$

Proof. It is readily seen that the functional $\mathscr{J}_\lambda$ admits a global minimizer $\tilde{u}_\lambda$ on K_λ. We are going to show that $\tilde{u}_\lambda = u_\lambda$ by proving, by recursion on n, that $\tilde{u}_\lambda \ge u_n$ for all n, where (u_n) is the sequence used to construct the minimal weak solution u_λ in the proof of Proposition 4.3, that is $u_0 = \underline{u}$ and $I_n(u_n) = \min_{v\in \mathrm{BV}(\Omega)} I_n(v)$ with, we recall,

$$I_n(v) = \mathscr{A}(v) - \int_\Omega (H + \lambda f(u_{n-1}))v + \int_{\partial\Omega} |v| d\mathscr{H}^{N-1}.$$

Set $u_{-1} = 0$, so that $u_0 = \underline{u}$ is the minimizer of I_0. Let $n \ge 0$. Applying Lemma A.1 to $\mathscr{F}_- = I_n$, $\mathscr{F}_+ = \mathscr{J}_\lambda$, $K_- = \mathrm{BV}(\Omega)$, $K_+ = K_\lambda$, we obtain

$$0 \le \lambda \int_\Omega F(\tilde{u}_\lambda) - F(\max(u_n, \tilde{u}_\lambda)) + f(u_{n-1})(\max(u_n, \tilde{u}_\lambda) - \tilde{u}_\lambda)\, dx. \qquad (29)$$

For $n = 0$, (29) reduces to:

$$0 \le -\int_\Omega F(\max(\underline{u}, \tilde{u}_\lambda)) - F(\tilde{u}_\lambda)\, dx,$$

which implies $\underline{u} \le \tilde{u}_\lambda$ a.e. in Ω since F is increasing.

For $n \ge 1$, assuming that we have proved that $u_{n-1} \le \tilde{u}_\lambda$ a.e. in Ω, we have $f(u_{n-1}) \le f(\tilde{u}_\lambda)$ and (29) implies

$$\begin{aligned} 0 &\le -\int_\Omega F(\max(u_n, \tilde{u}_\lambda)) - F(\tilde{u}_\lambda) - f(\tilde{u}_\lambda)(\max(u_n, \tilde{u}_\lambda) - \tilde{u}_\lambda)\, dx \\ &= -\int_\Omega G(\tilde{u}_\lambda, \max(u_n, \tilde{u}_\lambda))\, dx. \end{aligned}$$

The strict convexity of F implies $\tilde{u}_\lambda = \max(u_n, \tilde{u}_\lambda)$ and thus $u_n \le \tilde{u}_\lambda$ a.e. in Ω.

Passing to the limit $n \to \infty$, we deduce $u_\lambda \le \tilde{u}_\lambda$ in Ω and thus $u_\lambda = \tilde{u}_\lambda$, which completes the proof that u_λ is a one sided minimizer.

Next, we note that if φ is a non-positive smooth function satisfying $\varphi = 0$ on $\partial\Omega$, then $\mathscr{J}_\lambda(u_\lambda + t\varphi) \ge \mathscr{J}_\lambda(u_\lambda)$ for all $t \ge 0$. Letting t go to zero, and assuming that $u_\lambda \in \mathrm{Lip}(\Omega)$, we deduce that the second variation $Q_\lambda(\varphi)$ is non-negative. Since $Q_\lambda(\varphi) = Q_\lambda(-\varphi)$, it is readily seen that (28) holds true for non-negative functions. Finally decomposing φ into its positive and negative part, we deduce (28) for any φ. □

5.2 L^∞ estimate

We now prove:

Proposition 5.3 (L^∞ estimate)**.** *Let $\lambda \in (0, \lambda^*)$. There exists a constant C_1 depending on λ^{-1} and Ω such that the minimal weak solution u_λ satisfies $\|u_\lambda\|_{L^\infty(\Omega)} \le C_1$.*

Note that this implies Proposition 5.1: Proposition 5.3 gives the existence of C depending only on Ω such that $\|u_\lambda\|_{L^\infty(\Omega)} \le C$ for every $\min(1, \lambda^*/2) \le \lambda < \lambda^*$. And since $0 \le u_\lambda \le u_{\lambda'}$ if $\lambda < \lambda'$, the inequality is also satisfied when $0 \le \lambda \le \min(1, \lambda^*/2)$.

Proof. This proof is essentially a variation of the proof of Theorem 2.2 in Giusti [Giu76]. We fix $\lambda \in (0, \lambda^*)$ and set $u = u_\lambda$.

For some fixed $k > 1$, we set $v_k = \min(u, k)$ and $w_k = u - v_k = (u - k)_+$. The difference between the areas of the graphs of u and v_k can be estimated by below as follows ([Ger74]):

$$\int_\Omega |Dw_k| - |\{u > k\}| \le \mathscr{A}(u) - \mathscr{A}(v_k).$$

On the other hand, since $0 \le v_k \le u$, Lemma 5.2 gives $\mathscr{J}_\lambda(u) \le \mathscr{J}_\lambda(v_k)$, which implies

$$\mathscr{A}(u) - \mathscr{A}(v_k) \le \int_\Omega H(u - v_k) + \lambda[F(u) - F(v_k)]\, dx.$$

Writing

$$F(u) - F(v_k) = \int_0^1 f(su + (1-s)v_k)\, ds\, (u - v_k),$$

we deduce the following inequality

$$\int_\Omega |Dw_k| \le |\{u > k\}| + \int_\Omega \left(H + \lambda \int_0^1 f(su + (1-s)v_k)\, ds \right) w_k\, dx. \tag{30}$$

First, we will show that (30) implies the following estimate:

$$\|u\|_{L^q(\Omega)} \le C_1(q), \tag{31}$$

for every $q \in [1, +\infty)$, where $C_1(q)$ depends on q, Ω, λ^{-1}.

Indeed, by Lemma 3.2, we have $\displaystyle\int_A H + \lambda f(u)\, dx \le P(A)$ for all finite perimeter subset A of Ω. We deduce (using the coarea formula):

$$\begin{aligned} \int_\Omega (H + \lambda f(u)) w_k\, dx &= \int_0^{+\infty} \int_{\{w_k > t\}} H + \lambda f(u)\, dx\, dt \\ &\le \int_0^{+\infty} P(w_k > t) dt \\ &\le \int_\Omega |Dw_\lambda|. \end{aligned}$$

So (30) becomes

$$0 \le |\{u > k\}| - \lambda \int_{\{u \ge k\}} \left[f(u) - \int_0^1 f(su + (1-s)v_k)\, ds \right] w_k\, dx.$$

Since $u \ge 1$ and $v_k \ge 1$ on $\{u \ge k\}$, and since $f'(s) \ge 1$ for $s \ge 1$, we have

$$\begin{aligned} f(u) &\ge f(su + (1-s)v_k) + (u - su - (1-s)v_k) \\ &= f(su + (1-s)v_k) + (1-s)(u - v_k) \end{aligned}$$

on $\{u \ge k\}$. We deduce (recall that $w_k = u - v_k = (u-k)_+$):

$$\int_\Omega [(u-k)_+]^2\, dx \le \frac{2}{\lambda} |\{u > k\}|,$$

which implies, in particular, (31) for $q = 2$. Furthermore, integrating this inequality with respect to $k \in (k', +\infty)$, we get:

$$\int_\Omega [(u-k)_+]^3\, dx \le 3 \cdot \frac{2}{\lambda} \int_\Omega (u-k)_+\, dx,$$

and by repeated integration we obtain:

$$\int_\Omega [(u-k)^+]^q\, dx \le q(q-1) \frac{1}{\lambda} \int_\Omega [(u-k)^+]^{q-2}\, dx$$

for every $q \geq 3$, which implies (31) by induction on q.

Note however, that the constant $C_1(q)$ blows up as $q \to \infty$, and so we cannot obtain the L^∞ estimate that way. We thus go back to (30): using Poincaré's inequality for $\mathrm{BV}(\Omega)$ functions which vanish on $\partial\Omega$ and (30), we get

$$\begin{aligned}
\|w_k\|_{L^{\frac{n}{n-1}}(\Omega)} &\leq C(\Omega) \int_\Omega |Dw_k| \\
&\leq C(\Omega) \left(|\{u > k\}| + \int_\Omega (H + \lambda f(u))\, w_k \right) \\
&\leq C(\Omega) \left(|\{u > k\}| + \|H + \lambda f(u)\|_{L^n(\{w_k > 0\})} \|w_k\|_{L^{\frac{n}{n-1}}(\Omega)} \right)
\end{aligned}$$

Inequality (31) implies in particular that $H + \lambda f(u) \in L^n(\Omega)$ (with bound depending on Ω, λ^{-1}), so there exists $\varepsilon > 0$ such that $C(\Omega)\|H + \lambda f(u)\|_{L^n(A)} \leq 1/2$ for any subset $A \subset \Omega$ with $|A| < \varepsilon$. Moreover, Lemma 3.2 gives $\|u\|_{L^1(\Omega)} \leq P(\Omega)/\lambda$ and therefore

$$|\{w_k > 0\}| = |\{u > k\}| \leq \frac{1}{k} \frac{P(\Omega)}{\lambda}.$$

It follows that there exists k_0 depending on Ω, λ^{-1} such that

$$C(\Omega)\|H + \lambda f(u)\|_{L^n(\{w_k > 0\})} \leq 1/2$$

for $k \geq k_0$. For $k \geq k_0$, we deduce

$$\|w_k\|_{L^{\frac{n}{n-1}}(\Omega)} = \|(u-k)_+\|_{L^{\frac{n}{n-1}}(\Omega)} \leq 2C(\Omega)|\{u > k\}|.$$

Finally, for $k' > k$, we have $1_{|\{u > k'\}} \leq \left(\frac{(u-k)_+}{k'-k}\right)^{\frac{n}{n-1}}$ and so

$$|\{u > k'\}| \leq \frac{1}{(k'-k)^{\frac{n}{n-1}}} \|(u-k)_+\|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(\Omega)} \leq \frac{2C(\Omega)}{(k'-k)^{\frac{n}{n-1}}} |\{u > k\}|^{\frac{n}{n-1}}$$

which implies, by classical arguments (see [Sta66]) that $|\{u_\lambda > k\}|$ is zero for k large (depending on $|\Omega|$ and λ^{-1}). The proposition follows. $\square$

As a consequence, we have:

Corollary 5.4. *There exists a constant C depending only on Ω and H such that*

$$\int_\Omega |Du_\lambda| \leq C.$$

Proof. By Lemma 3.1 (ii) and Proposition 5.3, we get:

$$\begin{aligned}\mathcal{A}(u_\lambda) &\leq \mathcal{A}(v) - \int_\Omega (H + \lambda f(u_\lambda))v\,dx + \int_\Omega (H + \lambda f(u_\lambda))u_\lambda\,dx \\ &\leq \mathcal{A}(v) + C\int_\Omega |v|\,dx + C\end{aligned}$$

for any function $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$ such that $v = 0$ on $\partial\Omega$. Taking $v = 0$, the result follows immediately. $\square$

6 Existence of the extremal solution

We can now complete the proof of Theorem 2.7. The only missing piece is the existence of a weak solution for $\lambda = \lambda^*$, which is given by the following proposition:

Proposition 6.1. *There exists a function $u^* \in L^{p+1}(\Omega) \cap \mathrm{BV}(\Omega)$ such that*

$$u_\lambda \to u^* \quad \text{in } L^{p+1}(\Omega) \quad \text{as } \lambda \to \lambda^*.$$

Furthermore, u^* is a weak solution of (P_λ) for $\lambda = \lambda^*$.

Proof. Recalling that the sequence u_λ is non-decreasing with respect to λ, it is readily seen that Proposition 5.1 implies the existence of a function $u^* \in L^\infty(\Omega)$ such that

$$\lim_{\lambda\to\lambda^*} u_\lambda(x) = u^*(x).$$

Furthermore, by the Lebesgue dominated convergence theorem, u_λ converges to u^* strongly in $L^q(\Omega)$ for all $q \in [1, \infty)$.

Next, we have to check that u^* satisfies the boundary condition $u^* = 0$ on $\partial\Omega$. This follows from the following lemma, the proof of which relies heavily on Condition (15):

Lemma 6.2. *There exists a Lipschitz continuous function h defined in a neighborhood of $\partial\Omega$ and satisfying*

$$h = 0, \qquad \text{on } \partial\Omega$$

such that

$$u_\lambda \leq h, \qquad \forall \lambda \leq \lambda^*$$

in a neighborhood of $\partial\Omega$.

Proof. The barrier h will be such that for some small $\delta > 0$ and $\eta > 0$ we have:

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right) \geq H + \delta & \text{in } \Omega_\eta \\ h = 0 & \text{on } \partial\Omega \\ Th \cdot \nu = 1 & \text{on } \partial\Gamma_\eta \end{cases} \tag{32}$$

and

$$\lambda f\left(\sup_{\Omega_\eta} h\right) < \delta \tag{33}$$

where $\Omega_\eta = \{x \in \Omega, d(x,\partial\Omega) < \eta\}$ and $\Gamma_\eta = \{x \in \Omega, d(x,\partial\Omega) = \eta\}$.

The construction of h is fairly classical (see [Gia74, Giu84]): We look for h in the form

$$h(x) = \psi(d(x))$$

where $d(x)$ is the distance of x to $\partial\Omega$. When $\partial\Omega$ is of class $\mathcal{C}^2$, it is well know (see [Giu84] for instance) that d is of class $\mathcal{C}^2$ in Ω_η for η small enough. Furthermore, we have $|\nabla d| = 1$ and $-\Delta d(y) = (n-1)\Gamma(y)$ for $y \in \partial\Omega$. Using Conditions (15) and (16), it is easy to see that for η small enough, there exists $\varepsilon > 0$ such that

$$-\Delta d(x) \geq H(x) + \varepsilon \qquad \text{in } \Omega_\eta.$$

We can then check that h satisfies (32) as soon as ψ satisfies

$$\psi(0) = 0, \qquad \psi'(\eta) = +\infty,$$

$$\psi'(s) \geq 2\sqrt{M/\varepsilon} \quad 0 \leq s \leq \eta$$

where $M = \sup_{\Omega_\varepsilon} |\Delta d|$ and

$$\frac{\psi''}{(1+\psi'^2)^{3/2}} \leq \frac{\varepsilon}{4}.$$

This last condition says that the mean curvature of ψ (as a function of one variable) must be less than $\varepsilon/4$. We thus take for ψ a piece of a circle with radius $\frac{8}{\varepsilon}$ passing through $(0,0)$ and tangent to the line $s = \eta$. By taking η small enough, it is easy to see that we can have $\psi(\eta)$ as small as we want, thus satisfying condition (33).

We can now prove, by induction on n, that

$$u_n \leq h, \qquad \text{in } \Omega_\varepsilon \text{ and } \forall n \geq 0 \tag{34}$$

where u_n is the sequence of functions used in the construction of u_λ in Section 4.2. Indeed, assuming that

$$-\mathrm{div}(Tu_n) \leq -\mathrm{div}(Th)$$

(this is obviously satisfied for u_0 and it will be satisfied for u_n using (33) and the induction hypothesis), we have, for any test function $\varphi \geq 0$

$$\begin{aligned}
\int_{\Omega_\eta} (Th - Tu_n)\cdot\nabla\varphi\, dx &\geq \int_{\partial\Omega} (Th - Tu_n)\cdot\nu\varphi d\sigma(x) + \int_{\Gamma_\eta} (Th - Tu_n)\cdot\nu\varphi d\sigma(x) \\
&\geq \int_{\partial\Omega} (Th - Tu_n)\cdot\nu\varphi d\sigma(x) + \int_{\Gamma_\eta} (1 - Tu_n)\cdot\nu\varphi d\sigma(x) \\
&\geq \int_{\partial\Omega} (Th - Tu_n)\cdot\nu\varphi d\sigma(x).
\end{aligned}$$

Now, taking $\varphi = (u_n - h)_+$, we note that $\varphi = 0$ on $\partial\Omega$ and thus

$$\int_{\Omega_\eta \cap \{u_n \geq h\}} (Th - Tu_n) \cdot (\nabla u_n - \nabla h) \, dx \geq 0.$$

We deduce (34). Passing to the limit $n \to \infty$, we deduce

$$u_\lambda \leq h \qquad \text{in } \Omega_\varepsilon \quad \forall \lambda < \lambda^*.$$

Passing to the limit $\lambda \to \lambda^*$, we obtain Lemma 6.2.

$\square$

We can now complete the proof of Proposition 6.1: It only remains to check that u^* is a critical point of the energy function: By lower semi-continuity of the area functional $\mathscr{A}(u)$ and Corollary 5.4, we have

$$\mathscr{A}(u^*) \leq \liminf_{\lambda \to \lambda^*} \mathscr{A}(u_\lambda) < \infty.$$

So, if we write

$$\lambda \int F(u_\lambda) \, dx - \lambda^* \int F(u^*) \, dx = (\lambda - \lambda^*) \int F(u_\lambda) \, dx + \lambda^* \int F(u_\lambda) - F(u^*) \, dx,$$

it is readily seen that

$$\mathscr{J}_{\lambda^*}(u^*) \leq \liminf_{\lambda \to \lambda^*} \mathscr{J}_\lambda(u_\lambda).$$

Using Lemma 3.1, we have, for any $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$ with $v = 0$ on $\partial\Omega$:

$$\mathscr{J}_\lambda(u_\lambda) \leq \mathscr{J}_\lambda(v) + \lambda \int_\Omega G(u_\lambda, v) \, dx$$

which yields, as $\lambda \to \lambda^*$:

$$\mathscr{J}_{\lambda^*}(u^*) \leq \mathscr{J}_{\lambda^*}(v) + \lambda^* \int_\Omega G(u^*, v) \, dx.$$

for any $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$ with $v = 0$ on $\partial\Omega$. Lemma 3.1 implies that u^* is a weak solution of (P_λ) for $\lambda = \lambda^*$. $\square$

Note that Lemma 6.2 also implies that u_λ is Lipschitz at the boundary $\partial\Omega$ for all $\lambda \in [0, \lambda^*]$.

7 Regularity of the minimal solution in the radial case

7.1 Proof of Theorem 2.9

Throughout this section, we assume that $\Omega = B_R$ and that H depends on $r = |x|$ only. Then, for any rotation T that leaves B_R invariant, we see that the function $u_\lambda^T(x) = u_\lambda(Tx)$ is a weak solution of (P_λ), and the minimality of u_λ implies

$$u_\lambda \leq u_\lambda^T \text{ in } \Omega.$$

Taking the inverse rotation T^{-1}, we get the opposite inequality and so $u_\lambda^T = u_\lambda$, i.e. u_λ is radially (or spherically) symmetric. Furthermore, equation (P_λ) reads:

$$-\frac{1}{r^{n-1}} \frac{d}{dr} \left(\frac{r^{n-1} u_r}{(1+u_r^2)^{1/2}} \right) = H + \lambda f(u). \tag{35}$$

or

$$-\left[\frac{u_{rr}}{(1+u_r^2)^{3/2}} + \frac{n-1}{r} \frac{u_r}{(1+u_r^2)^{1/2}} \right] = H + \lambda f(u) \tag{36}$$

together with the boundary conditions

$$u_r(0) = 0, \qquad u(R) = 0.$$

Note that, by integration of (35) over $(0, r)$, $0 < r < R$, we obtain

$$\frac{-r^{n-1} u_r(r)}{(1+u_r(r)^2)^{1/2}} = \int_0^r [H + \lambda f(u)] r^{n-1} dr, \tag{37}$$

which gives $u_r \leq 0$, provided u is Lipschitz continuous in Ω at least.

It is classical that the solutions of (4) can blow up at $r = 0$. In our case however, the functions u_λ are bounded in L^∞. We deduce the following result:

Lemma 7.1 (Bound on the gradient near the origin)**.** *There exists $r_1 \in (0, R)$ and $C_1 > 0$ such that for any $\lambda \in [0, \lambda^*]$, we have*

$$|\nabla u_\lambda(x)| \leq C_1 \textit{ for a.a. } x \textit{ such that } |x| \leq r_1.$$

Proof. First, we assume that u_λ is smooth. Then, integrating (P_λ) over B_r, we get:

$$\int_{\partial B_r} \frac{\nabla u_\lambda \cdot \nu}{\sqrt{1 + |\nabla u_\lambda|^2}} \, dx = \int_{B_r} H + \lambda f(u_\lambda) \, dx.$$

Since u_λ is spherically symmetric, this implies:

$$\frac{|(u_\lambda)_r|}{\sqrt{1 + |(u_\lambda)_r|^2}}(r) = \frac{1}{P(B_r)} \int_{B_r} H + \lambda f(u_\lambda) \, dx \tag{38}$$

and the L^∞ bound on u_λ yields:

$$\frac{|(u_\lambda)_r|}{\sqrt{1+|(u_\lambda)_r|^2}}(r) \le C\frac{|B_r|}{P(B_r)} \le Cr.$$

In particular, there exists r_1 such that $Cr \le 1/2$ for $r \le r_1$ and so

$$|(u_\lambda)_r|(r) \le C_1 \qquad \text{for } r \le r_1. \tag{39}$$

Of course, these computations are only possible if we already know that u_λ is a classical solution of (P_λ). However, it is always possible to perform the above computations with the sequence (u_n) used in the proof of Proposition 4.3 to construct u_λ. In particular, we note that we have $\underline{u} \le u_n \le u_\lambda$ for all n and

$$-\mathrm{div}(Tu_n) = H + \lambda f(u_{n-1}) \text{ in } \Omega$$

so the same proof as above implies that there exists a constant C independent of n or λ such that

$$|\nabla u_n| \le C_1 \text{ for all } x \text{ such that } |x| \le r_1.$$

The lemma follows by taking the limit $n \to \infty$ (recall that the whole sequence u_n converges in a monotone fashion to u_λ). $\square$

Proof of Theorem 2.9. We now want to prove the gradient estimate (17). Thanks to Lemma 7.1, we only have to show the result for $r \in [r_1, R]$. We denote $u^* = u_{\lambda^*}$. Since u^* is a weak solution of (P_λ), Lemma 3.2 with $A = B_r$ ($r \in [0, R]$) implies

$$\int_{B_r} H + \lambda^* f(u^*)\, dx \le P(B_r)$$

and so, using the fact that $u^* \ge u_\lambda \ge \underline{u}$, we have

$$\int_{B_r} H + \lambda f(u_\lambda)\, dx \le P(B_r) - \int_{B_r} (\lambda^* - \lambda) f(u_\lambda) \le P(B_r) - (\lambda^* - \lambda)\int_{B_r} f(\underline{u})\, dx.$$

Hence (38) becomes:

$$\frac{|(u_\lambda)_r|}{\sqrt{1+|(u_\lambda)_r|^2}}(r) \le 1 - \frac{(\lambda^* - \lambda)}{P(B_r)}\int_{B_r} f(\underline{u})\, dx.$$

For $r \in (r_1, R)$, we have

$$\frac{(\lambda^* - \lambda)}{r^{n-1}}\int_{B_r} f(\underline{u})\, dx \ge (\lambda^* - \lambda)\delta > 0$$

for some universal δ and so

$$|(u_\lambda)_r|(r) \le \frac{C}{\lambda^* - \lambda} \qquad \text{for } r \in [r_1, R].$$

Together with (39), this gives the result.

Note once again that these computations can only be performed rigorously on the functions (u_n), which satisfy in particular $\underline{u} \leq u_n \leq u^*$ for all n. So (17) holds for u_n instead of u_λ. The result follows by passing to the limit $n \to \infty$. □

Remark 7.2. *We point out that the Lipschitz regularity near the origin $r = 0$ is a consequence of the L^∞ estimate (it is in fact enough to have $f(u_\lambda) \in L^n$), while the gradient estimate away from the origin only requires $f(u_\lambda)$ to be integrable.*

7.2 Regularity of the extremal solution

In this section, we prove Theorem 2.10, that is the regularity of the extremal solution u^*. The proof is divided in two parts: boundary regularity and interior regularity.

7.2.1 Boundary regularity

We have the following a priori estimate:

Lemma 7.3 (Bound on the gradient at the boundary)**.** *Assume that $\Omega = B_R$, that H depends on r only and that conditions (3), (15) and (16) are fulfilled. Let u be any classical solution of (P_λ). Then there exists a constant C depending only on R, ε_0 and n such that*

$$|u_r(R)| \leq C(1 + \lambda).$$

Since we know that $u_\lambda \in \mathrm{Lip}(\Omega)$ for $\lambda < \lambda^*$, Proposition 2.4 implies that u_λ is a classical solution, so Lemma 7.3 yields

$$|(u_\lambda)_r(R)| \leq C(1 + \lambda) \quad \text{ for all } \lambda < \lambda^*.$$

Passing to the limit, we obtain:

$$|u^*_r(R)| \leq C(1 + \lambda^*). \tag{40}$$

Proof of Lemma 7.3: In this proof, Assumption (15) plays a crucial role. When Ω is a ball of radius R and using the fact that $H \in \mathrm{Lip}(\Omega)$, it implies:

$$H(r) \leq (1 - \varepsilon_0)\frac{n-1}{R} \tag{41}$$

in a neighborhood of $\partial\Omega$ (with a slightly smaller ε_0). The argument of our proof is similar to the proof of Theorem 2.5 (ii) (to show that u satisfies the Dirichlet condition), and relies on the construction of an appropriate barrier. Actually, whenever we have $H(y) \leq (n-1)\Gamma(y)$, $y \in \partial\Omega$, there is a a natural barrier at the boundary given by the cylinder generated by ∂B_R. Here, we modify this cylinder by slightly bending it along its generating straight line. The generating straight line thus becomes a circle of radius ε^{-1} and condition (41) implies that this hypersurface is a supersolution for (P_λ). By radial symmetry, this amounts to consider a circle of radius ε^{-1} (ε to be determined) centered at (M, δ) with δ small and $M > R$ chosen such that the circle passes through the point $(R, 0)$ (see Figure 1). We define the function $h(r)$ in $[M - \varepsilon^{-1}, R]$ such that $(r, h(r))$ lies on the circle (with $h(r) < \delta$).

Then, we note that for $r \in [M - \varepsilon^{-1}, R]$ and $\varepsilon\delta \leq 1$, we have

$$\frac{h'(r)}{(1 + h'(r)^2)^{1/2}} \leq \frac{h'(R)}{(1 + h'(R)^2)^{1/2}} = -(1 - (\delta\varepsilon)^2)^{1/2} \leq -1 + (\delta\varepsilon)^2$$

(this quantity can be interpreted as the horizontal component of the normal vector to the circle), and

$$\frac{d}{dr}\left(\frac{h'(r)}{(1 + h'(r)^2)^{1/2}}\right) = \varepsilon$$

(this quantity is actually the one-dimensional curvature of the curve $r \mapsto h(r)$). Hence we have:

$$\begin{aligned}
\frac{1}{r^{n-1}}\frac{d}{dr}\left(\frac{r^{n-1}h'(r)}{(1 + h'(r)^2)^{1/2}}\right) &= \frac{d}{dr}\left(\frac{h'(r)}{(1 + h'(r)^2)^{1/2}}\right) + \frac{n-1}{r}\frac{h'(r)}{(1 + h'(r)^2)^{1/2}} \\
&\leq \varepsilon + \frac{n-1}{r}(-1 + (\delta\varepsilon)^2) \\
&\leq \varepsilon + \frac{n-1}{R}(-1 + (\delta\varepsilon)^2)
\end{aligned}$$

We now use a classical sliding method: Let

$$\eta^* = \inf\{\eta > 0\,;\, u(r) \leq h(r - \eta) \text{ for } r \in [M - \varepsilon^{-1} + \eta, R]\}.$$

If $\eta^* > 0$, then $h(r + \eta^*)$ touches u from above at a point in $(M - \varepsilon^{-1} + \eta, R)$ such that $u < \delta$ (recall that u is Lipschitz continuous so it cannot touch $h(r - \eta)$ at $M - \varepsilon^{-1} + \eta$ since $h = \delta$ and $h' = \infty$ at that point). At that contact point, we must thus have

$$\begin{aligned}
\frac{1}{r^{n-1}}\frac{d}{dr}\left(\frac{r^{n-1}h'(r)}{(1 + h'(r)^2)^{1/2}}\right) &\geq \frac{1}{r^{n-1}}\frac{d}{dr}\left(\frac{r^{n-1}u_r(r)}{(1 + u_r(r)^2)^{1/2}}\right) \\
&\geq -(H + \lambda f(u)) \\
&\geq -(1 - \varepsilon_0)\frac{n-1}{R} - \lambda\delta^p.
\end{aligned}$$

We will get a contradiction if ε and δ are such that

$$\varepsilon + \frac{n-1}{R}(-1 + (\delta\varepsilon)^2) < -(1 - \varepsilon_0)\frac{n-1}{R} - \lambda\delta^p$$

which is equivalent to

$$\varepsilon + \lambda\delta^p + \frac{n-1}{R}(\varepsilon\delta)^2 < \frac{n-1}{R}\varepsilon_0.$$

This can be achieved easily by choosing ε and δ small enough.

It follows that $\eta^* = 0$ and so $u \le h$ in the neighborhood of R. Since $u(R) = h(R) = 0$, we deduce:

$$|u'(R)| \le |h'(R)| \le C(R,n)(\varepsilon\delta)^{-1} \le C(R,n)\frac{1+\lambda}{\varepsilon_0^2}.$$

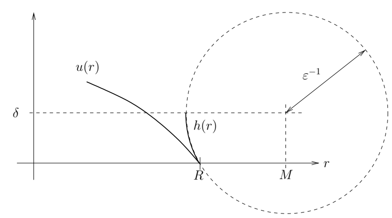

Figure 1: Construction of a barrier

□

Corollary 7.4 (Bound on the gradient near the boundary)**.** *Under the hypotheses of Lemma 7.3, there exist $\eta \in (0,R)$ and $C > 0$ depending on R, ε_0 and n only such that*

$$|u_r(r)| \le C \qquad \textit{for all } r \in [R-\eta, R].$$

Proof. The same proof as that of Lemma 7.3 shows that there exists $\delta > 0$ and $C > 0$ such that:

$$\text{If } u(r) \le \delta \text{ for all } r \in [r_0, R] \text{ with } R - r_0 \le \delta \text{ then } |u_r(r_0)| \le C. \tag{42}$$

Furthermore, the proof of Lemma 7.3 implies that $u(r) \le h(r)$ in a neighborhood of R, and so for some small η we have:

$$u(r) \le \delta \text{ for all } r \in [R-\eta, R].$$

The result follows. □

7.2.2 Interior regularity

We now show the following interior regularity result:

Proposition 7.5 (Interior bound on the gradient)**.** *Let $\eta \in (0, R/2)$. There exists $C_\eta > 0$ depending only on η, n and $\int_\Omega |Du_\lambda|$ such that, for all $0 \le \lambda < \lambda^*$,*

$$|\nabla u_\lambda(x)| \le C_\eta \textit{ for all } x \textit{ in } \Omega \textit{ with } \eta < |x| < R - \eta.$$

Using Lemma 7.1 (regularity for r close to 0), Corollary 7.4 (regularity for r close to R), and Proposition 7.5 (together with Corollary 5.4 which give the BV estimate uniformly with respect to λ), we deduce that there exists C depending only on H and n such that

$$|\nabla u_\lambda(x)| \leq C \text{ for all } x \text{ in } \Omega,$$

for all $\lambda \in [0, \lambda^*)$. Theorem 2.10 then follows by passing to the limit $\lambda \to \lambda^*$.

Proof of Proposition 7.5. It is sufficient to prove the result for $\frac{\lambda^*}{2} < \lambda < \lambda^*$. Throughout the proof, we fix $\lambda \in (\frac{\lambda^*}{2}, \lambda^*)$, $r_0 \in (\eta, R - \eta)$ and we denote

$$u = u_\lambda \qquad \text{and} \qquad v = \sqrt{1 + u_r^2}.$$

Idea of the proof: Let $\varphi_0 = \varphi_{B_{r_0}}$ (the characteristic function of the set B_{r_0}). Then by definition of $\mathscr{J}_\lambda$, we have for all $t \geq 0$:

$$\mathscr{J}_\lambda(u + t\varphi_0) \leq \mathscr{J}_\lambda(u) + t \int_\Omega |D\varphi_0| - t \int_\Omega H\varphi_0 \, dx - \lambda \int_\Omega F(u + t\varphi_0) - F(u) \, dx$$

Furthermore, since $u \geq \underline{u}$, we have $u \geq \mu > 0$ in B_{r_0} and so

$$F(u + t\varphi_0) - F(u) \geq f(u)t\varphi_0 + \frac{\alpha}{2} t^2 \varphi_0^2 \quad \text{ for all } x \in \Omega$$

(with α such that $f'(s) \geq \alpha$ for all $s \geq \mu$). It follows:

$$\begin{aligned}
\mathscr{J}_\lambda(u + t\varphi_0) &\leq \mathscr{J}_\lambda(u) + t \int_\Omega |D\varphi_0| - t \int_\Omega (H + \lambda f(u))\varphi_0 \, dx - t^2 \frac{\alpha\lambda}{2} \int_\Omega \varphi_0^2 \, dx \\
&= \mathscr{J}_\lambda(u) + tP(B_{r_0}) - t \int_{B_{r_0}} H + \lambda f(u) \, dx - t^2 \frac{\alpha\lambda}{2} |B_{r_0}| \\
&= \mathscr{J}_\lambda(u) + tP(B_{r_0}) \left(1 - \frac{|u_r(r_0)|}{v(r_0)}\right) - t^2 \frac{\alpha\lambda}{2} |B_{r_0}|.
\end{aligned}$$

where we used the following equality, obtained by integration of (P_λ) over B_{r_0}:

$$-P(B_{r_0}) \frac{u_r(r_0)}{v(r_0)} = \int_{B_{r_0}} H + \lambda f(u) \, dx.$$

This would imply $\frac{|u_r|}{v} \leq 1 - \delta$ and yield Proposition 7.5 if we had $\mathscr{J}_\lambda(u) \leq \mathscr{J}_\lambda(u + t\varphi_0)$ for some $t > 0$. Unfortunately, $u = u_\lambda$ is only a minimizer with respect to negative perturbations. The proof of Proposition 7.5 thus consists in using the semi-stability to show that u is almost a minimizer (up to some term of order 3) with respect to some positive perturbations.

Step 1: First of all, the function φ_0 above is not smooth, so we need to consider the following piecewise linear approximation of φ_0:

$$\varphi_\varepsilon = \begin{cases} 1 & \text{if } r \leq r_0 - \varepsilon \\ \varepsilon^{-1}(r_0 - r) & \text{if } r_0 - \varepsilon \leq r \leq r_0 \\ 0 & \text{if } r \geq r_0. \end{cases}$$

We then have (using Equation (P_λ) and denoting by ω_n the volume of the unit ball in $\mathbb{R}^n$):

$$\begin{aligned}
\mathscr{J}_\lambda(u+t\varphi_\varepsilon) &\le \mathscr{J}_\lambda(u) + t\int_\Omega |\nabla\varphi_\varepsilon|\,dx - t\int_\Omega (H+\lambda f(u))\varphi_\varepsilon\,dx - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2\,dx \\
&= \mathscr{J}_\lambda(u) + t\int_\Omega |\nabla\varphi_\varepsilon|\,dx - t\int_\Omega \frac{(u)_r(\varphi_\varepsilon)_r}{v}\,dx - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2\,dx \\
&= \mathscr{J}_\lambda(u) + t\int_\Omega \left(1-\frac{|u_r|}{v}\right)|\nabla\varphi_\varepsilon|\,dx - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2\,dx \\
&= \mathscr{J}_\lambda(u) + t\omega_n\int_{r_0-\varepsilon}^{r_0} \left(1-\frac{|u_r|}{v}\right)\varepsilon^{-1}r^{n-1}\,dr - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2\,dx \\
&\le \mathscr{J}_\lambda(u) + t\omega_n\varepsilon^{-1}\int_{r_0-\varepsilon}^{r_0} \frac{1}{v^2}r^{n-1}\,dr - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2\,dx
\end{aligned}$$

and so if we denote $\rho(\varepsilon) = \sup_{r\in(r_0-\varepsilon,r_0)}\frac{1}{v^2}$, we deduce:

$$\mathscr{J}_\lambda(u+t\varphi_\varepsilon) \le \mathscr{J}(u)_\lambda + t\omega_n r_0^{n-1}\rho(\varepsilon) - t^2\frac{\alpha\lambda}{2}\omega_n\left(\frac{r_0}{2}\right)^n \tag{43}$$

for all $\varepsilon < r_0/2$.

Step 2: Since our goal is to show that $\rho(\varepsilon)$ is cannot be too small, we need to control $\mathscr{J}(u+t\varphi_\varepsilon)$ from below: for a smooth radial function φ, we denote

$$\theta(t) = \mathcal{A}(u+t\varphi) = \int_\Omega L(u_r+t\varphi_r),$$

where $L(s) = (1+s^2)^{1/2}$. Then

$$\theta^{(3)}(t) = \int_\Omega L^{(3)}(u_r+t\varphi_r)\varphi_r^3\,dx$$

where

$$L^{(3)}\colon s\mapsto \frac{-3s}{(1+s^2)^{5/2}}$$

satisfies

$$|L^{(3)}(s)| \le \frac{3}{(1+s^2)^2}, \quad \forall s\ge 0.$$

When $\varphi=\varphi_\varepsilon$, we have $|u_r+t\varphi_r|\ge|u_r|$ for all $t\ge 0$ and therefore:

$$\begin{aligned}
|\theta^{(3)}(t)| &\le \int_\Omega \frac{3}{v^4}(|(\varphi_\varepsilon)_r|^3\,dx \\
&\le \varepsilon^{-3}\omega_n\int_{r_0-\varepsilon}^{r_0}\frac{3}{v^4}r^{n-1}\,dr \\
&\le \varepsilon^{-2}\omega_n\,\rho(\varepsilon)^2\,r_0^{n-1}
\end{aligned}$$

for all $t \geq 0$.

Since the second variation $Q_\lambda(\varphi_\varepsilon)$ is non-negative by Lemma 5.2 (recall that u_λ is a semi-stable solution), we deduce that for some $t_0 \in (0,t)$ we have:

$$\begin{aligned}
\mathscr{J}_\lambda(u+t\varphi_\varepsilon) &= \mathscr{J}_\lambda(u) + \frac{t^2}{2}Q_\lambda(\varphi_\varepsilon) + \theta^{(3)}(t_0)\frac{t^3}{6} - \lambda\int_\Omega \frac{f''(u+t_0\varphi_\varepsilon)}{6}t^3\varphi^3\,dx \\
&\geq \mathscr{J}_\lambda(u) - \frac{t^3}{2}|\theta^{(3)}(t_0)| - \|f''(u+t_0\varphi_\varepsilon)\|_{L^\infty(B_{r_0})}\lambda t^3\omega_n r_0^n \\
&\geq \mathscr{J}_\lambda(u) - \frac{t^3}{2}\varepsilon^{-2}\omega_n\,\rho(\varepsilon)^2\,r_0^{n-1} - C\lambda t^3\omega_n r_0^n, \qquad (44)
\end{aligned}$$

where we used the fact that $f''(u+t_0\varphi_\varepsilon) \in L^\infty(B_{r_0})$ (if $p \geq 2$, this is a consequence of the L^∞ bound on u, if $p \in (1,2)$, then this follows from the fact that $u + t_0\varphi_\varepsilon \geq \underline{u} > 0$ in B_{r_0}).

Step 3: Inequalities (43) and (44) yield:

$$\lambda\frac{t^2}{2}\omega_n r_0^n \leq t\omega_n r_0^{n-1}\rho(\varepsilon) + \frac{t^3}{2}\varepsilon^{-2}\omega_n\,\rho(\varepsilon)^2\,r_0^{n-1} + C\lambda t^3\omega_n r_0^n$$

and so

$$\frac{\lambda r_0}{2}(1-2Ct)t \leq \rho(\varepsilon) + \frac{\varepsilon^{-2}t^2}{2}\,\rho(\varepsilon)^2$$

for all $t \geq 0$. If $t \leq 1/(4C)$, we deduce

$$\mu t \leq \rho(\varepsilon) + \frac{\varepsilon^{-2}t^2}{2}\,\rho(\varepsilon)^2$$

with $\mu = \lambda r_1/4$ (recall that $r_0 > r_1$).

Let now $t = M\varepsilon$ (M to be chosen later), then we get

$$\mu M\varepsilon \leq \rho(\varepsilon) + \frac{M^2}{2}\,\rho(\varepsilon)^2.$$

If $\rho(\varepsilon) \leq \frac{\mu M\varepsilon}{2}$, then

$$\rho(\varepsilon) + \frac{M^2}{2}\,\rho(\varepsilon)^2 \leq \frac{\mu M\varepsilon}{2} + \frac{\mu^2M^4\varepsilon^2}{8}$$

and we get a contradiction if $\frac{\mu^2M^4\varepsilon^2}{8} < \frac{\mu M\varepsilon}{2}$. It follows that

$$\rho(\varepsilon) \geq \frac{\mu M\varepsilon}{2} \qquad \text{for all } \varepsilon < \frac{4}{\mu M^3}. \qquad (45)$$

Step 4: Since $\rho(\varepsilon) = \sup_{r\in(r_0-\varepsilon,r_0)}\frac{1}{v^2}$, (45) yields

$$\inf_{r\in(r_0-\varepsilon,r_0)} v^2 \leq \frac{2}{\mu M\varepsilon} \qquad \text{for all } \varepsilon < \frac{4}{\mu M^3}.$$

In order to conclude, we need to use some type of Harnack inequality to control $\sup_{r\in(r_0-\varepsilon,r_0)} v^2$. This will follow from the following lemma:

Lemma 7.6. *Let $v=\sqrt{1+u_r^2}$. Then v solves the following equation in $(0,R)$:*

$$-\frac{1}{r^{n-1}}\left(\frac{r^{n-1}v_r}{v^3}\right)_r + c^2 = H_r\frac{u_r}{v}+\lambda f'(u)\frac{u_r^2}{v}. \tag{46}$$

where

$$c^2=\frac{n-1}{r^2}\frac{u_r^2}{v^2}+\frac{u_{rr}^2}{v^6}$$

is the sum of the square of the curvatures of the graph of u.

We postpone the proof of this lemma to the end of this section. Clearly, the equation (46) is degenerate elliptic. In order to write a Harnack inequality, we introduce $w=\frac{1}{v^2}$, solution of the following equation

$$\frac{1}{r^{n-1}}\left(r^{n-1}w_r\right)_r = 2H_r\frac{u_r}{v}+2\lambda f'(u)\frac{u_r^2}{v}-2c^2$$

which is a nice uniformly elliptic equation in a neighborhood of $r_0\in(0,R)$. In particular, if $\varepsilon\le R-r_0$, Harnack's inequality [GT01] yields:

$$\sup_{r\in(r_0-\varepsilon,r_0)} w \le C\inf_{r\in(r_0-\varepsilon,r_0)} w + C\varepsilon\|g\|_{L^1(r_0-2\varepsilon,r_0+\varepsilon)} \tag{47}$$

where

$$g=2H_r\frac{u_r}{v}+2\lambda f'(u)\frac{u_r^2}{v}-2c^2.$$

Next, we note that

$$|g|\le 2|H_r|+C\lambda|u_r|+2c^2.$$

It is readily seen that the first $(n-1)$ curvatures $\frac{1}{r}\frac{u_r}{v}$ are bounded in a neighborhood of $r_0\neq 0$. Furthermore, since the mean curvature is in L^∞, it is easy to check that the last curvature is also bounded: more precisely, (36) gives

$$\frac{u_{rr}}{v^3}=-H-\lambda f(u)-\frac{n-1}{r}\frac{u_r}{v}\in L^\infty.$$

We deduce that $c^2\in L^\infty$ and since $u\in \mathrm{BV}(\Omega)$, we get

$$\|g\|_{L^1(r_0-2\varepsilon,r_0+\varepsilon)}\le C\int_\Omega |Du|+C$$

Together with (47) and (45) (and recalling that $\rho(\varepsilon)=\sup_{r\in(r_0-\varepsilon,r_0)} w^2$), we deduce:

$$\frac{\mu M\varepsilon}{2}\le C\inf_{r\in(r_0-\varepsilon,r_0)} w + C\left(\int_\Omega |Du|+1\right)\varepsilon \qquad \text{for all } \varepsilon<\frac{4}{\mu M^3}.$$

With M large enough ($M \geq \frac{4C}{\mu}\left(\int_\Omega |Du| + 1\right)$), it follows that

$$\frac{\mu M \varepsilon}{4} \leq C \inf_{r\in(r_0-\varepsilon,r_0)} w \qquad \text{for all } \varepsilon < \frac{4}{\mu M^3}$$

and thus (with $\varepsilon = \min(\frac{2}{\mu M^3}, (R-r_0)/4, \frac{1}{4MC})$):

$$v(r_0)^2 \leq \sup_{r\in(r_0-\varepsilon,r_0)} v^2 \leq C((\lambda r_0)^{-1}, (R-r_0)^{-1}, \int_\Omega |Du|, \|u\|_{L^\infty(\Omega)}),$$

which completes the proof. □

Proof of Lemma 7.6. Taking the derivative of (35) with respect to r and multiplying by u_r, we get:

$$\frac{n-1}{r^n}\left(\frac{r^{n-1}u_r}{v}\right)_r u_r - \frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_r}{v}\right)_{rr} u_r = H_r u_r + \lambda f'(u) u_r^2.$$

Using the fact that

$$\left(\frac{u_r}{v}\right)_r = \frac{u_{rr}}{v^3} \qquad \text{and} \qquad v_r = \frac{u_r u_{rr}}{v},$$

we deduce:

$$\frac{(n-1)^2}{r^n}\frac{r^{n-2}u_r^2}{v} + \frac{n-1}{r}\frac{u_r u_{rr}}{v^3} - \frac{n-1}{r^{n-1}}\left(\frac{r^{n-2}u_r}{v}\right)_r u_r - \frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_{rr}}{v^3}\right)_r u_r$$
$$= H_r u_r + \lambda f'(u) u_r^2$$

and so (simplifying and dividing by v):

$$\frac{(n-1)^2}{r^2}\frac{u_r^2}{v^2} - \frac{(n-1)(n-2)}{r^{n-1}}\frac{r^{n-3}u_r^2}{v^2} - \frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_{rr}}{v^3}\right)_r \frac{u_r}{v} = H_r\frac{u_r}{v} + \lambda f'(u)\frac{u_r^2}{v}.$$

This yields

$$\frac{(n-1)}{r^2}\frac{u_r^2}{v^2} - \frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_{rr}u_r}{v^4}\right)_r + \frac{1}{r^{n-1}}\frac{r^{n-1}u_{rr}}{v^3}\left(\frac{u_r}{v}\right)_r = H_r\frac{u_r}{v} + \lambda f'(u)\frac{u_r^2}{v}$$

hence

$$\frac{(n-1)}{r^2}\frac{u_r^2}{v^2} - \frac{1}{r^{n-1}}\left(\frac{r^{n-1}v_r}{v^3}\right)_r + \frac{u_{rr}^2}{v^6} = H_r\frac{u_r}{v} + \lambda f'(u)\frac{u_r^2}{v}$$

which is the desired equation. □

7.2.3 Proof of Theorem 2.11

In this section, we adapt the continuation method of [CR75] to prove Theorem 2.11.

First, we need to introduce some notations: Let $\alpha \in (0,1)$ and, for $k \in \mathbb{N}$, let $\mathcal{C}_0^{k,\alpha}(\overline{\Omega})$ be the set of functions $u \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$ that satisfy $u = 0$ on $\partial\Omega$. Let $\mathscr{T} : \mathcal{C}_0^{2,\alpha}(\overline{\Omega}) \times \mathbb{R} \to \mathcal{C}_0^{\alpha}(\overline{\Omega})$ be defined by

$$\mathscr{T}(u,\lambda) = -\mathrm{div}(Tu) - H - \lambda f(u).$$

The function $\mathscr{T}$ is twice continuously differentiable and, at any point $(u,\lambda) \in \mathcal{C}_0^{2,\alpha}(\overline{\Omega}) \times \mathbb{R}$, has first derivatives

$$\mathscr{T}_u(u,\lambda)\colon v \mapsto -\partial_i(a^{ij}(\nabla u)\partial_j v) - \lambda f'(u)v, \quad \mathscr{T}_\lambda(u,\lambda) = -f(u),$$

where we use the convention of summation over repeated indices and set, for $\mathbf{p} \in \mathbb{R}^n$,

$$a^i(\mathbf{p}) = \frac{\mathbf{p}_i}{(1+|\mathbf{p}|^2)^{1/2}}, \quad a^{ij}(\mathbf{p}) = \frac{\partial a^i}{\partial \mathbf{p}_j}(\mathbf{p}) = \frac{1}{(1+|\mathbf{p}|^2)^{1/2}}\left(\delta_{ij} - \frac{\mathbf{p}_i\mathbf{p}_j}{1+|\mathbf{p}|^2}\right).$$

The second derivatives of $\mathscr{T}$ at any point $(u,\lambda) \in \mathcal{C}_0^{2,\alpha}(\overline{\Omega}) \times \mathbb{R}$ are

$$\mathscr{T}_{uu}(u,\lambda)(v,w) = -\partial_i(a^{ijk}(\nabla u)\partial_j v \partial_k w) - \lambda f''(u)vw$$

and $\mathscr{T}_{u\lambda}(u,\lambda)(v,\mu) = -\mu f'(u)v$, $\mathscr{T}_{\lambda\lambda}(u,\lambda) = 0$, where

$$a^{ijk}(\mathbf{p}) = \frac{\partial a^{ij}}{\partial \mathbf{p}_k}(\mathbf{p}) = 3\frac{\mathbf{p}_i\mathbf{p}_j\mathbf{p}_k}{(1+|\mathbf{p}|^2)^{5/2}} - \frac{1}{(1+|\mathbf{p}|^2)^{3/2}}(\delta_{ij}\mathbf{p}_k + \delta_{ik}\mathbf{p}_j + \delta_{kj}\mathbf{p}_i).$$

We note for further use that, given $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$,

$$a^{ijk}(\mathbf{p})\mathbf{q}_i\mathbf{q}_j\mathbf{q}_k = 3\frac{\mathbf{p}\cdot\mathbf{q}}{(1+|\mathbf{p}|^2)^{5/2}}((\mathbf{p}\cdot\mathbf{q})^2 - |\mathbf{q}|^2(1+|\mathbf{p}|^2)),$$

and thus, in particular,

$$\mathbf{p}\cdot\mathbf{q} \geq 0 \quad \Longrightarrow \quad a^{ijk}(\mathbf{p})\mathbf{q}_i\mathbf{q}_j\mathbf{q}_k \leq 0. \tag{48}$$

Next, we note that for any u, v, w radially symmetric function, non-increasing with respect to r, we have

$$[a^i(\nabla u) - a^i(\nabla v) - a^{ij}(\nabla u)\partial_j(u-v)]\partial_i w \geq 0, \tag{49}$$

or, equivalently, setting $A(\nabla u) := (a^{ij}(\nabla u))_{ij}$:

$$(Tu - Tv - A(\nabla u)\nabla(u-v))\cdot\nabla w \geq 0. \tag{50}$$

Indeed, the left-hand side of (49) rewrites

$$(h(p) - h(q) - h'(p)(p-q))s, \qquad \text{where } h(p) = \frac{p}{(1+p^2)^{1/2}},$$

where $p = \partial_r u \leq 0$, $q = \partial_r v \leq 0$, $s = \partial_r w \leq 0$ and h is convex on $\mathbb{R}_-$.

Recall that $\underline{u} \in \mathcal{C}_0^{2,\alpha}(\overline{\Omega})$ is the solution to $\mathscr{T}(\underline{u}, 0) = 0$. In particular $\underline{u}$ is radially symmetric and non-increasing with respect to r. At $\lambda = 0$, the map $\mathscr{T}_u(\underline{u}, 0) \colon \mathcal{C}_0^{2,\alpha}(\overline{\Omega}) \to \mathcal{C}_0^{\alpha}(\overline{\Omega})$ is invertible since it defines a uniformly elliptic operator with no zero-th order terms. By the Implicit Function Theorem, we obtain the existence of $a > 0$ and of a $\mathcal{C}^2$ curve $\lambda \mapsto u(\lambda)$ from $[0, a]$ to $\mathcal{C}_0^{2,\alpha}(\overline{\Omega})$ of solutions to $\mathscr{T}(u(\lambda), \lambda) = 0$ such that $u(0) = \underline{u}$.

Let now $\overline{\lambda} \in (0, \lambda^*]$ be the largest $b > 0$ such that this curve can be continued to $[0, b)$ under the additional constraint that for all $\lambda \in [0, b)$, $\mathscr{T}_u(u(\lambda), \lambda)$ is invertible. We denote by L_λ the elliptic operator $L_\lambda = \mathscr{T}_u(u(\lambda), \lambda)$ and by

$$\mu_1(\lambda) < \mu_2(\lambda) \leq \mu_3(\lambda) \ldots$$

its eigenvalues. It is readily seen that $\mu_1(0) > 0$ (since there are no zero-th order terms in L_0). Since $\lambda \mapsto \mu_1(\lambda)$ is continuous[1] and $\mu_1(\lambda) \neq 0$ on $[0, \overline{\lambda})$, we see that $\mu_1(\lambda) > 0$ for all $\lambda \in [0, \overline{\lambda})$.

Note also that the function $u(\lambda)$ is a radially symmetric[2], and that L_λ therefore admits a first eigenvector $w_\lambda^1 > 0$ associated to the eigenvalue $\mu_1(\lambda)$ which is also a radially symmetric function. Furthermore, one can check that w_λ^1 is non-increasing with respect to r: As in (35)-(37), this follows directly from the equation $L_\lambda w_\lambda^1 = \mu_1(\lambda) w_\lambda^1$ written in terms of the r-variable, *i.e.*

$$-\frac{1}{r^{n-1}} \partial_r \left(\frac{r^{n-1}}{(1 + |\partial_r u(\lambda)|^2)^{3/2}} \partial_r w_\lambda^1 \right) = \lambda f'(u(\lambda)) w_\lambda^1 + \mu_1(\lambda) w_\lambda^1 \geq 0.$$

We can now prove that $u(\lambda)$ and u_λ coincide.

Lemma 7.7. *We have $\overline{\lambda} = \lambda^*$, $u(\lambda) = u_\lambda$ (the minimal solution), $\mu_1(\lambda) > 0$ for all $\lambda \in [0, \lambda^*)$ and $\mu_1(\lambda^*) = 0$.*

Proof. We adapt the proof of Theorem 3.2 in [KK74]. Let $\lambda \in [0, \overline{\lambda})$, $\nu \in [0, \lambda^*]$. Using the fact that $u(\lambda)$ and u_ν are solutions to (P_λ), we get:

$$\begin{aligned} L_\lambda(u(\lambda) - u_\nu) =& - \operatorname{div}(A(\nabla u(\lambda)) \nabla (u(\lambda) - u_\nu)) - \lambda f'(u(\lambda))(u(\lambda) - u_\nu) \\ =& \lambda f(u(\lambda)) - \nu f(u_\nu) - \lambda f'(u(\lambda))(u(\lambda) - u_\nu) \\ & + \operatorname{div}[T u(\lambda) - T u_\nu - A(\nabla u(\lambda)) \nabla (u(\lambda) - u_\nu)]. \end{aligned}$$

Since f is convex, we have

$$\lambda f(u(\lambda)) - \nu f(u_\nu) - \lambda f'(u(\lambda))(u(\lambda) - u_\nu) \leq (\lambda - \nu) f(u_\nu)$$

and it follows from (49) that

$$\int_\Omega L_\lambda(u(\lambda) - u_\nu) w dx \leq (\lambda - \nu) \int_\Omega f(u_\nu) w dx \tag{51}$$

[1]this follows from the continuity of the map $\lambda \mapsto u(\lambda)$ valued in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$ and from the characterization of $\mu_1(\lambda)$ as the supremum over non-trivial $\varphi \in \mathcal{C}^2(\overline{\Omega})$ of the Rayleigh quotients $\frac{(L_\lambda \varphi, \varphi)}{(\varphi, \varphi)}$ where $(\cdot, \cdot)$ is the canonical scalar product over $L^2(\Omega)$

[2]this is the case of every terms in the iterative sequence $u_n(\lambda)$ converging to $u(\lambda)$ that is constructed by application of the Implicit Function Theorem

for any radially symmetric non-negative non-increasing function $w \in \mathcal{C}^{2,\alpha}(\Omega)$. Taking $\nu = \lambda$ and $w = w_\lambda^1$, the positive eigenvector corresponding to the first eigenvalue $\mu_1(\lambda)$, we deduce:

$$\mu_1(\lambda) \int_\Omega (u(\lambda) - u_\lambda) w_\lambda^1 dx \leq 0.$$

We have $u(\lambda) - u_\lambda \geq 0$ since u_λ is the minimal solution to (P_λ) and $\mu_1(\lambda) > 0$, $w_\lambda^1 > 0$ in Ω, hence $u(\lambda) = u_\lambda$ in Ω.

We now extend the definition of L_λ to the whole interval $[0, \lambda^*]$ by setting $L_\lambda = \mathscr{T}_u(u_\lambda, \lambda)$. In particular, $\mu_1(\overline{\lambda}) = 0$ and (51) is valid for λ in the whole range $[0, \lambda^*]$. To prove the second part of Lemma 7.7, assume by contradiction $\overline{\lambda} < \nu < \lambda^*$. Taking $\lambda = \overline{\lambda}$ and $w = w_{\overline{\lambda}}^1$ in (51), we obtain

$$0 \leq (\overline{\lambda} - \nu) \int_\Omega f(u_\nu) w_{\overline{\lambda}}^1 dx.$$

This is impossible since $\overline{\lambda} < \nu$ and $\int_\Omega f(u_\nu) w_{\overline{\lambda}}^1 dx > 0$. $\square$

We can now complete the proof of Theorem 2.11. Let $w_* \in \mathcal{C}_0^{2,\alpha}(\overline{\Omega})$ be the first eigenvector of L_{λ^*}: $L_{\lambda^*} w_* = 0$, $w_* > 0$ in Ω, w_* is radial non-increasing with respect to r. Let $Z \subset \mathcal{C}_0^{2,\alpha}(\overline{\Omega})$ be the closed subspace of elements $z \in \mathcal{C}_0^{2,\alpha}(\overline{\Omega})$ orthogonal (for the $L^2(\Omega)$ scalar product) to w_*. Let $\mathscr{T}^*$ be the $\mathcal{C}^2$ map $\mathbb{R} \times Z \times \mathbb{R} \to \mathcal{C}_0^\alpha(\overline{\Omega})$ defined by

$$\mathscr{T}^*(s, z, \lambda) = \mathscr{T}(u_* + s w_* + z, \lambda).$$

The derivative $\mathscr{T}^*_{z,\lambda}(0, 0, \lambda_*)$ is invertible. Indeed, given $v \in \mathcal{C}_0^\alpha(\overline{\Omega})$, $(\zeta, \mu) \in Z \times \mathbb{R}$ is solution to

$$\mathscr{T}^*_{z,\lambda}(0, 0, \lambda_*) \cdot (\zeta, \mu) = v$$

if

$$L_{\lambda^*} \zeta + \mu f(u_*) = v. \tag{52}$$

By the Fredholm alternative (and the Schauder regularity theory for elliptic PDEs), (52) has a unique solution $\zeta \in Z$ provided

$$\mu \int_\Omega f(u_*) w_* dx = \int_\Omega v w_* dx.$$

This condition uniquely determines μ since $f(u_*), w_* > 0$ in Ω and, in particular, $\int_\Omega f(u_*) w_* dx > 0$. By the Implicit Function Theorem, it follows that there is an $\varepsilon > 0$ and a $\mathcal{C}^2$-curve $(-\varepsilon, \varepsilon) \to Z \times \mathbb{R}$, $s \mapsto (z(s), \lambda(s))$ such that

$$(z, \lambda)(0) = (0, \lambda^*), \quad \mathscr{T}^*(s, z(s), \lambda(s)) = 0, \quad \forall |s| < \varepsilon. \tag{53}$$

By derivating once with respect to s in (53), we obtain

$$L_{\lambda^*} w_* + \mathscr{T}^*_{z,\lambda}(0, 0, \lambda^*) \cdot (z'(0), \lambda'(0)) = 0, \tag{54}$$

hence $z'(0) = 0$, $\lambda'(0) = 0$. We set $u(s) = u_* + sw_* + z(s)$. Then $u'(0) = w_* > 0$ in Ω. To show the effective bending of the curve $s \mapsto (u(s), \lambda(s))$, there remains to prove that $\lambda''(0) < 0$. Let us differentiate twice with respect to s in (53): we obtain

$$
\begin{aligned}
&-\partial_i(a^{ij}(\nabla u)\partial_j u'') - \lambda f'(u)u'' - \partial_i(a^{ijk}(\nabla u)\partial_k u'\partial_j u') \\
&\qquad\qquad\qquad = \lambda'' f(u) + 2\lambda' f'(u)u' + \lambda f''(u)|u'|^2.
\end{aligned}
$$

At $s = 0$, this gives

$$
L_{\lambda^*}u''(0) - \partial_i(a^{ijk}(\nabla u)\partial_k w_* \partial_j w_*) = \lambda''(0)f(u_*) + \lambda^* f''(u_*)|w_*|^2.
$$

Integrating the result against w_* over Ω, we deduce that

$$
\int_\Omega a^{ijk}(\nabla u)\partial_k w_* \partial_j w_* \partial_i w_* dx = \lambda''(0)\int_\Omega f(u_*)w_* dx + \lambda^* \int_\Omega f''(u_*)w_*^3 dx. \quad (55)
$$

Since $\nabla u \cdot \nabla w_* = \partial_r u \partial_r w_* \geq 0$, (48) shows that the left-hand side in (55) is non-positive. Finally, since $f(u_*), f''(u_*), w_* > 0$, we get $\lambda''(0) < 0$. $\square$

A Comparison principles

It is well known that classical solutions of (P_λ) satisfy a strong comparison principle, namely, if $u, v \in \mathrm{Lip}(\Omega)$ satisfy

$$
-\mathrm{div}(Tu) \leq -\mathrm{div}(Tv) \text{ in } \Omega, \qquad u \leq v \text{ on } \partial\Omega \quad (56)
$$

with $u \neq v$, then

$$
u < v \text{ in } \Omega. \quad (57)
$$

If u, v are in $W^{1,1}(\Omega)$ and satisfy (56), then we still have a weak comparison principle, *i.e.* $u \leq v$ a.e. in Ω (see [Giu84]). But no such principle holds for functions that are only in $\mathrm{BV}(\Omega)$ (even if one of the function is smooth). This is due to the lack of strict convexity of the functional $\mathscr{A}$ on $\mathrm{BV}(\Omega)$ that is affine on any interval $[0, \varphi_A]$ (in particular, we have $\mathcal{L}(\varphi_A) = \mathcal{L}(-\varphi_A) = \mathcal{L}(0) = 0$ for any finite perimeter set A).

Throughout the paper, we consider weak solutions to (P_λ) which are, *a priori*, not better (with respect to integrability properties of the gradient) than $\mathrm{BV}(\Omega)$. In order to derive comparison results, we use Lemma 3.1, which allows us to interpret weak solutions as global minimizers of an accurate functional and the following lemma.

Lemma A.1 (Comparison principle)**.** *Let $q \geq 1$. Let $G_\pm \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the growth condition $|G_\pm(x, s)| \leq C_1(x)|s|^q + C_2(x)$ where $C_1 \in L^\infty(\Omega)$ and $C_2 \in L^1(\Omega)$. Let $\mathscr{F}_\pm$ be the functional defined on $L^q \cap \mathrm{BV}(\Omega)$ by*

$$
\mathscr{F}_\pm(v) = \mathscr{A}(v) + \int_{\partial\Omega} |v| d\mathscr{H}^{N-1} + \int_\Omega G_\pm(x, v)\, dx.
$$

Suppose that $u_\pm$ is a global minimizer of $\mathscr{F}_\pm$ on a set $K_\pm$ and suppose that

$$\min(u_+, u_-) \in K_-, \quad \max(u_+, u_-) \in K_+,$$

Then we have

$$0 \leq \Delta(\max(u_+, u_-)) - \Delta(u_+), \quad \Delta(v) := \int_\Omega G_+(x, v) - G_-(x, v)\, dx.$$

Proof of Lemma A.1. We need to recall the inequality

$$\int_Q |D\varphi_{E\cup F}| + \int_Q |D\varphi_{E\cap F}| \leq \int_Q |D\varphi_E| + \int_Q |D\varphi_F|, \tag{58}$$

which holds for any open set $Q \subset \mathbb{R}^m$ ($m \geq 1$) and any sets E, F with locally finite perimeter in $\mathbb{R}^m$. Applied to $Q = \Omega \times \mathbb{R}$ and to the characteristic functions of the subgraphs of u and v, Inequality (58) gives:

$$\mathscr{A}(\max(u, v)) + \mathscr{A}(\min(u, v)) \leq \mathscr{A}(u) + \mathscr{A}(v), \quad u, v \in \mathrm{BV}(\Omega). \tag{59}$$

Since $\int_\Omega |Du| \leq \mathscr{A}(u)$, this shows in particular that $\max(u, v)$, $\min(u, v)$ and $(u - v)_+ = \max(u, v) - v = u - \min(u, v) \in \mathrm{BV}(\Omega)$ whenever u and $v \in \mathrm{BV}(\Omega)$.

Since $u \mapsto \int_\Omega G_\pm(u)$ is invariant by rearrangement, we deduce:

$$\mathscr{F}_-(\max(u_+, u_-)) + \mathscr{F}_-(\min(u_+, u_-)) \leq \mathscr{F}_-(u_+) + \mathscr{F}_-(u_-). \tag{60}$$

Furthermore, we have $\min(u_+, u_-) \in K_-$, and so $\mathscr{F}_-(u_-) \leq \mathscr{F}_-(\min(u_+, u_-))$. Therefore, (60) implies that $\mathscr{F}_-(\max(u_+, u_-)) \leq \mathscr{F}_-(u_+)$, which, by definition of Δ also reads:

$$\mathscr{F}_+(\max(u_+, u_-)) - \Delta(\max(u_+, u_-)) \leq \mathscr{F}_+(u_+) - \Delta(u_+).$$

Finally, we recall that u_+ is the global minimizer of $\mathscr{F}_+$ on K_+ and that $\max(u_+, u_-) \in K_+$, and so $\mathscr{F}_+(u_+) \leq \mathscr{F}_+(\max(u_+, u_-))$. We conclude that $\Delta(\max(u_+, u_-)) - \Delta(u_+) \geq 0$. ☐

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