EXISTENCE AND ASYMPTOTICS OF FRONTS IN NON LOCAL COMBUSTION MODELS

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Abstract. We prove the existence and give the asymptotic behavior of non local fronts in homogeneous media.

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1. Introduction

This paper is devoted to the study of fronts propagation in homogeneous media for a fractional reaction-diffusion equation appearing in combustion theory. More precisely, we consider the following classical scalar model for the combustion of premixed gas with ignition temperature:

$$u_t + (-\partial_{xx})^\alpha u = f(u) \quad \text{ in } \mathbb{R}\times\mathbb{R}, \tag{1}$$

where the function f satisfies:

$$\begin{cases} f:\mathbb{R}\to\mathbb{R} \text{ continuous function} \\ f(u)\geq 0 \text{ for all } u\in\mathbb{R} \text{ and } \operatorname{supp} f=[\theta,1] \\ f'(1)<0 \end{cases} \tag{2}$$

where $\theta\in(0,1)$ is a fixed number (usually referred to as the ignition temperature).

The operator $(-\partial_{xx})^\alpha$ denotes the fractional power of the Laplace operator in one dimension (with $\alpha\in(0,1]$). It can be defined by the following singular integral

$$(-\partial_{xx})^\alpha u(x) = c_\alpha \, \mathrm{PV}\int_{\mathbb{R}} \frac{u(x)-u(z)}{|x-z|^{1+2\alpha}}\,dz \tag{3}$$

where PV stands for the Cauchy principal value. This integral is well defined, for instance, if u belongs to $C^2(\mathbb{R})$ and satisfies

$$\int_{\mathbb{R}} \frac{|u(x)|}{(1+|x|)^{1+2\alpha}}\, dx < +\infty$$

(in particular, smooth bounded functions are admissible). Alternatively, the fractional Laplace operator can be defined as a pseudo-differential operator with symbol $|\xi|^{2\alpha}$. We refer the reader to the book by Landkof where an extensive study of $(-\partial_{xx})^\alpha$ is performed by means of harmonic analysis techniques (see [Lan72]).

In this paper, we will always take $\alpha \in (1/2, 1]$, and we are interested in particular solutions of (1) which describe transition fronts between the stationary states 0 and 1 (traveling fronts). These traveling fronts are solutions of (1) that are of the form

$$u(t,x) = \phi(x+ct) \tag{4}$$

with

$$\begin{cases} \lim_{x\to-\infty} \phi(x) = 0 \\ \lim_{x\to+\infty} \phi(x) = 1. \end{cases}$$

The number c is the speed of propagation of the front. It is readily seen that ϕ must solve

$$(-\partial_{xx})^\alpha \phi + c\,\phi' = f(\phi) \qquad \text{for all } x \in \mathbb{R}$$

When $\alpha = 1$ (standard Laplace operator), it is well known that there exists a unique speed c and a unique profile ϕ (up to translation) that correspond to a traveling front solution of (1) (see e.g. [BLL90, BN92, BNS85]). The goal of this paper is to generalize these results to the case $\alpha \in (1/2, 1)$. We are thus looking for ϕ and c satisfying

$$\begin{cases} (-\partial_{xx})^\alpha \phi + c\,\phi' = f(\phi) \qquad \text{for all } x \in \mathbb{R} \\ \lim_{x\to-\infty} \phi(x) = 0 \\ \lim_{x\to+\infty} \phi(x) = 1 \\ \phi(0) = \theta \end{cases} \tag{5}$$

(the last condition is a normalization condition which ensures the uniqueness of ϕ). Our main theorem is the following:

Theorem 1.1. *Let $\alpha \in (1/2, 1)$ and assume that f satisfies (2), then there exists a unique pair (ϕ_0, c_0) solution of* (5)*. Furthermore, $c_0 > 0$ and ϕ_0 is monotone increasing.*

We will also obtain the following result, which describes the asymptotic behavior of the front at $-\infty$:

Theorem 1.2. *Let $\alpha \in (1/2, 1)$ and assume that f satisfies (2). Let ϕ_0 be the unique solution of* (5) *provided by Theorem 1.1. Then there exist m, M such that*

$$\phi_0(x) \leq \frac{M}{|x|^{2\alpha-1}} \qquad \text{for } x \leq -1$$

and

$$\phi_0'(x) \geq \frac{m}{|x|^{2\alpha}} \qquad \text{for } x \leq -1.$$

The proof of Theorem 1.1 follows classical arguments developed by Berestycki-Larrouturou-Lions [BLL90] (see also Berestycki-Nirenberg [BN92]): Truncation of the domain, construction of sub- and super-solutions and passage to the limit. As usual, one of the main difficulty is to make sure that we recover a finite, non trivial speed of propagation at the limit. The main novelty (compared with similar results when $\alpha = 1$) is the construction of sub- and super-solutions where the classical exponential profile is replaced by power tail functions.

2. Truncation of the domain

The first step is to truncate the domain: for some $b > 0$, we consider the following problem:

$$\begin{cases} (-\partial_{xx})^\alpha \phi_b + c_b \phi_b' = f(\phi_b) & \text{for all } x \in [-b, b] \\ \phi_b(x) = 0 & \text{for } s \leq -b \\ \phi_b(x) = 1 & \text{for } s \geq b \\ \phi_b(0) = \theta. \end{cases} \tag{6}$$

The goal of this section is to prove that this problem has a solution for b large enough. More precisely, we are now going to prove:

Proposition 2.1. *Assume $\alpha \in (1/2, 1)$ and that f satisfies (2). Then there exists a constant M such that if $b > M$ the truncated problem* (6) *has a unique solution (ϕ_b, c_b). Furthermore, the following properties hold:*

(i) *There exists K independent of b such that $-K \leq c_b \leq K$.*
(ii) *ϕ_b is non-decreasing with respect to x and satisfies $0 < \phi_b(x) < 1$ for all $x \in (-b, b)$.*

Before we can prove this Proposition, we need to detail the construction of sub- and super-solutions.

2.1. **Construction of sub- and super-solutions.**

In the proof of the existence of traveling waves for the standard Laplace operator ($\alpha = 1$), sub- and super-solution of the form $e^{\gamma x}$ play a crucial role, in particular in the determination of the asymptotic behavior of the traveling waves as $x \to -\infty$. These particular functions are replaced, in the case of the fractional Laplace operator, by functions with polynomial tail. In what follows, we will rely on two important lemmas:

Lemma 2.2. *Let $\beta \in (0,1)$ and define*

$$\varphi(x) = \begin{cases} \frac{1}{|x|^\beta} & \text{if } x < -1 \\ 1 & \text{if } x > -1. \end{cases}$$

Then φ satisfies

$$(-\partial_{xx})^\alpha \varphi + c\varphi'(x) = \frac{-c_\alpha}{2\alpha |x|^{2\alpha}} + c\frac{\beta}{|x|^{\beta+1}} + O\left(\frac{1}{|x|^{\beta+2\alpha}}\right)$$

when $x \to -\infty$.

and

Lemma 2.3. *Let $\beta > 1$ and define*

$$\bar\varphi(x) = \begin{cases} \frac{1}{|x|^\beta} & x < -1 \\ 0 & x > -1 \end{cases}$$

then

$$(-\partial_{xx})^\alpha \bar\varphi + c\bar\varphi'(x) = \frac{-c_\alpha}{\beta - 1}\frac{1}{|x|^{2\alpha+1}} + c\frac{\beta}{|x|^{\beta+1}} + O\left(\frac{1}{|x|^{\beta+2\alpha}}\right)$$

when $x \to -\infty$.

Proof of Lemma 2.2. We want to estimate $(-\partial_{xx})^\alpha\varphi$ for $x < -1$. We have:

$$(-\partial_{xx})^\alpha \varphi(x) = -c_\alpha \mathrm{PV} \int_{\mathbb{R}} \frac{\varphi(x+y) - \varphi(x)}{|y|^{1+2\alpha}}\, dy,$$

which we decompose as follow:

$$\begin{aligned}(-\partial_{xx})^\alpha \varphi(x) &= c_\alpha \int_{-\infty}^{-1-x} \frac{\varphi(x) - \varphi(x+y)}{|y|^{1+2\alpha}}\, dy + c_\alpha \int_{-1-x}^{+\infty} \frac{\varphi(x) - \varphi(x+y)}{|y|^{1+2\alpha}}\, dy \\ &= I + II\end{aligned}$$

A simple explicit computation yields:

$$II = \left(\frac{1}{|x|^\beta} - 1\right)\frac{c_\alpha}{2\alpha|x+1|^{2\alpha}}.$$

Performing the change of variables $y = xz$, one gets

$$I = \frac{c_\alpha}{|x|^{\beta+2\alpha}} \int_{+\infty}^{-\frac{1}{x}-1} \frac{|z+1|^\beta - 1}{|z+1|^\beta |z|^{1+2\alpha}}\, dz.$$

Note that the integrand has a singularity at $z = 0$, and this integral has to be understood as a principal value. We also observe that the integrand has a singularity at $z = -1$, but since $\beta < 1$, this singularity is integrable, and thus

$$I \sim -c_\alpha \frac{1}{|x|^{\beta+2\alpha}} \mathrm{PV} \int_{-1}^{+\infty} \frac{|z+1|^\beta - 1}{|z+1|^\beta|z|^{1+2\alpha}}\, dz. \qquad \text{as } x \to -\infty.$$

We deduce:

$$(-\partial_{xx})^\alpha \varphi(x) = \frac{-c_\alpha}{2\alpha|x|^{2\alpha}} + O\left(\frac{1}{|x|^{\beta+2\alpha}}\right)$$

when $x \to -\infty$, and the result follows. □

Proof of Lemma 2.3. Again, we decompose $(-\partial_{xx})^\alpha \bar\varphi$ as follow:

$$\begin{aligned}(-\partial_{xx})^\alpha \bar\varphi(x) &= c_\alpha \int_{-\infty}^{-1-x} \frac{\bar\varphi(x)-\bar\varphi(x+y)}{|y|^{1+2\alpha}}\,dy + c_\alpha \int_{-1-x}^{+\infty} \frac{\bar\varphi(x)-\bar\varphi(x+y)}{|y|^{1+2\alpha}}\,dy \\ &= I + II\end{aligned}$$

Now, a simple explicit computation yields:

$$II = \frac{c_\alpha}{|x|^\beta} \frac{1}{2\alpha|x+1|^{2\alpha}}.$$

And performing the change of variables $y = xz$, one gets

$$I \quad = \quad \frac{c_\alpha}{|x|^{\beta+2\alpha}} \int_{+\infty}^{-\frac{1}{x}-1} \frac{|z+1|^\beta - 1}{|z+1|^\beta |z|^{1+2\alpha}}\,dz.$$

Note that the integrand as a singularity at $z = 0$, and this integral has to be understood as a principal value. We also observe that the integrand has a singularity at $z = -1$ and since $\beta > 1$, this singularity is divergent and thus

$$I \sim \frac{-c_\alpha}{\beta - 1}|x|^{\beta-1}.$$

We deduce:

$$(-\partial_{xx})^\alpha \bar\varphi(x) = \frac{-c_\alpha}{\beta-1}\frac{1}{|x|^{2\alpha+1}} + O\left(\frac{1}{|x|^{\beta+2\alpha}}\right)$$

which yields the result. □

2.2. **Proof of Proposition 2.1.** We now turn to the proof of Proposition 2.1. First, we fix $c \in \mathbb{R}$ and consider the following problem:

$$\begin{cases} (-\partial_{xx})^\alpha \phi + c\,\phi' = f(\phi) & \text{for all } x \in [-b, b] \\ \phi(x) = 0 \quad \text{for } x \le -b \\ \phi(x) = 1 \quad \text{for } x \ge b \end{cases} \tag{7}$$

We have:

Lemma 2.4. *For any $c \in \mathbb{R}$, Equation (7) has a unique solution ϕ_c. Furthermore ϕ_c is non-decreasing with respect to x and $c \to \phi_c$ is continuous.*

Proof. Since 1 and 0 are respectively super- and sub-solutions, we can use Perron's method (recall that the fractional laplacian enjoys a comparison principle) to prove the existence of a solution $\phi_c(x)$ for any $c \in \mathbb{R}$. By a sliding argument, we can show that ϕ_c is unique and non-decreasing with respect to x. The fact that the function $c \to \phi_c$ is continuous follows from classical arguments (see [BN92] for details). □

We now have to show that there exists a unique $c = c_b$ such that $\phi_{c_b}(0) = \theta$. This will be a consequence of the following lemma:

Lemma 2.5. *There exist constants M, K such that for $b > M$ the followings hold:*

(1) *if $c > K$ then the solution of (7) satisfies $\phi_c(0) < \theta$,*
(2) *if $c < -K$ then the solution of (7) satisfies $\phi_c(0) > \theta$.*

Together with the fact that $\phi_c(0)$ is continuous with respect to c, Lemma 2.5 implies that there exists $c_b \in [-K, -K]$ such that ϕ_{c_b} satisfies $\phi_{c_b}(0) = \theta$ and is thus a solution of (6). This completes the proof of Proposition 2.1.

Proof of Lemma 2.5. We consider the function

$$\varphi(x) = \begin{cases} \frac{1}{|x|^{2\alpha-1}} & x < -1 \\ 1 & x \geq -1 \end{cases} \tag{8}$$

and note that Lemma 2.2 (with $\beta = 2\alpha - 1$) yields that if c is large enough ($c \geq \frac{c_\alpha}{2\alpha(2\alpha-1)}$), then

$$(-\partial_{xx})^\alpha \varphi(x) + c\varphi'(x) \geq 0$$

for $x \leq -A$ (for some A large enough). We can also assume that $\varphi(x) \leq \theta$ for $x \leq -A$, and so

$$(-\partial_{xx})^\alpha \varphi(x) + c\varphi'(x) \geq f(\varphi) = 0 \qquad \text{for } x \leq -A.$$

Furthermore, for $-A < x < -1$, $(-\partial_{xx})^\alpha \varphi(x)$ is bounded while

$$c\varphi'(x) \geq c\frac{2\alpha - 1}{A^{2\alpha}}.$$

For c large enough, we thus have

$$(-\partial_{xx})^\alpha \varphi(x) + c\varphi'(x) \geq \sup f \geq f(\varphi) \qquad \text{for } -A < x < -1.$$

We deduce that there exists K such that if $c \geq K$ then

$$(-\partial_{xx})^\alpha \varphi(x) + c\varphi'(x) \geq f(\varphi) \qquad \text{for } x < -1$$

and so φ is a supersolution for (7).

Choosing M such that $\varphi(-M) < \theta$, we now see that if $c \geq K$ and $b > M$, then $\varphi(x - M)$ is a super-solution for (7). By a sliding argument, we deduce that $\phi_c(x) \leq \varphi(x - M)$ and so $\phi_c(0) \leq \varphi(-M) < \theta$.

For the lower bound, we define $\varphi_1(x) = 1 - \varphi(-x)$. Then we we have, if $-c \geq K$ ($c \leq -K$) and for $x > 1$

$$(-\partial_{xx})^\alpha \varphi_1(x) + c\varphi_1'(x) = -[(-\partial_{xx})^\alpha \varphi(-x) + (-c)\varphi'(-x)] \leq 0 \leq f(\varphi).$$

Moreover, we have $\varphi_1(x) = 0$ for $x \leq 1$. Proceeding as above, we deduce that if $c \leq -K$, then $\phi_c(0) > \theta$, which concludes the proof. □

3. Proof of Theorem 1.1

In order to complete the proof of Theorem 1.1, we have to prove that we can pass to the limit $b \to \infty$ in the truncated problem. More precisely, Theorem 1.1 follows from the following proposition:

Proposition 3.1. *Under the conditions of Proposition 2.1, there exists a subsequence $b_n \to \infty$ such that $\phi_{b_n} \longrightarrow \phi_0$ and $c_{b_n} \longrightarrow c_0$. Furthermore, $c_0 \in (0, K]$ and ϕ_0 is a monotone increasing solution of (5).*

Proof of Proposition 3.1. We recall that $c_b \in [-K, K]$, and classical elliptic estimates (see [BCP68]) yield:

$$||\phi_b||_{\mathcal{C}^{2,\gamma}} \leq C$$

for some $\gamma \in (0, 1)$. Thus there exists a subsequence $b_n \to \infty$ such that

$$c_n := c_{b_n} \longrightarrow c_0 \in [-K, K]$$

$$\phi_n := \phi_{b_n} \longrightarrow \phi_0$$

as $n \to \infty$. It is readily seen that ϕ_0 solves

$$(-\partial_{xx})^\alpha \phi_0 + c_0\, \phi_0' = f(\phi_0) \qquad \text{for all } x \in \mathbb{R}. \tag{9}$$

It is also readily seen that $\phi_0(x)$ is monotone increasing, $\phi_0(0) = \theta$ and ϕ_0 is bounded. By a standard compactness argument, there exists γ_0, γ_1 such that $\lim_{x\to-\infty} \phi_0(x) = \gamma_0$ and $\lim_{x\to+\infty} \phi_0(x) = \gamma_1$ with

$$0 \leq \gamma_0 \leq \theta \leq \gamma_1 \leq 1.$$

It remains to prove that $c_0 > 0$, $\gamma_0 = 0$ and $\gamma_1 = 1$. For that, we will mainly follow classical arguments (see [BLL90], [BH07]).

First, we have the following lemma:

Lemma 3.2. *The function ϕ_0 satisfies*

$$\int_{\mathbb{R}} (-\partial_{xx})^\alpha \phi_0(x)\, dx = 0.$$

Proof of Lemma 3.2. The result follows formally by integrating formula (3) with respect to x and using the antisymmetry with respect to the variables x and z. However, because of the principal value, one has to be a little bit careful with the use of Fubini's theorem.

To avoid this difficulty, we will use instead the equivalent formula for the fractional laplacian:

$$\begin{aligned}(-\partial_{xx})^\alpha \phi_0(x) \;=\;& c_\alpha \int_{\mathbb{R}\setminus[x-\varepsilon, x+\varepsilon]} \frac{\phi_0(x) - \phi_0(z)}{|x - z|^{1+2\alpha}}\, dz \\ &+ c_\alpha \int_{[x-\varepsilon, x+\varepsilon]} \frac{\phi_0(x) - \phi_0(z) + \phi_0'(x)(z - x)}{|x - z|^{1+2\alpha}}\, dz\end{aligned} \tag{10}$$

which is valid for all $\varepsilon > 0$ and does not involve singular integrals. Integrating the first term with respect to $x \in \mathbb{R}$, and using Fubini's theorem, we get

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{\phi_0(x)-\phi_0(z)}{|x-z|^{1+2\alpha}}\,dz\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}\setminus[z-\varepsilon,z+\varepsilon]} \frac{\phi_0(x)-\phi_0(z)}{|x-z|^{1+2\alpha}}\,dx\,dz \\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{\phi_0(x)-\phi_0(z)}{|x-z|^{1+2\alpha}}\,dz\,dx\end{aligned}$$

and so this integral vanishes. Using Taylor's theorem, the second term in (10) can be rewritten as

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{|x-z|^{1+2\alpha}} \int_x^z (z-t)\phi_0''(t)\,dt\,dz \quad = \quad \int_{-\varepsilon}^{\varepsilon} \frac{1}{|y|^{1+2\alpha}} \int_x^{x+y} (y+x-t)\phi_0''(t)\,dt\,dy.$$

Integrating with respect to x and using (twice) Fubini's theorem, we deduce

$$\begin{aligned}&\int_{\mathbb{R}}\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{|x-z|^{1+2\alpha}} \int_x^z (z-t)\phi_0''(t)\,dt\,dz\,dx \\ &\quad= \int_{-\varepsilon}^{\varepsilon} \frac{1}{|y|^{1+2\alpha}} \int_{-\infty}^{+\infty}\int_x^{x+y} (y+x-t)\phi_0''(t)\,dt\,dx\,dy \\ &\quad= \int_{-\varepsilon}^{\varepsilon} \frac{1}{|y|^{1+2\alpha}} \int_{-\infty}^{+\infty}\int_{t-y}^{t} (y+x-t)\phi_0''(t)\,dx\,dt\,dy \\ &\quad= \int_{-\varepsilon}^{\varepsilon} \frac{y^2}{2|y|^{1+2\alpha}} \int_{-\infty}^{+\infty} \phi_0''(t)\,dt\,dy \\ &\quad= 0,\end{aligned}$$

where we used the fact that $\lim_{x\to\pm\infty}\phi_0'(x) = 0$ and so $\int_{-\infty}^{+\infty}\phi_0''(t)\,dt = 0$. The lemma follows. □

Now, we can integrate equation (9) with respect to $x \in \mathbb{R}$, and using Lemma 3.2, we get:

$$\int_{\mathbb{R}} f(\phi_0(x))\,dx = c_0(\gamma_1-\gamma_0) < \infty. \tag{11}$$

In particular, we observe that (11) implies that

$$f(\gamma_0) = f(\gamma_1) = 0,$$

otherwise the integral would be infinite.

Next, we prove:

Lemma 3.3. *The limiting speed satisfies:*

$$c_0 > 0.$$

Proof. First of all, we note that for all n, there exists $a_n \in (0, b_n)$ such that $\phi_n(a_n) = \frac{1+\theta}{2}$. Furthermore, up to another subsequence, by elliptic estimates, the function $\psi_n(x) = \phi_{b_n}(a_n + x)$ converges to a function ψ_0. Note that since $\psi_0 \in \mathcal{C}^\gamma$, there exists $r > 0$ such that

$$\psi_0(x) \in \left[\frac{3+\theta}{4}, \frac{1+3\theta}{4}\right] \qquad \text{for } x \in [-r, r]$$

and so there exists $\kappa_0 > 0$ such that

$$\int_{\mathbb{R}} f(\psi_0)\, dx > \kappa_0. \tag{12}$$

Up to a subsequence, we can assume that $b_n + a_n$ is either convergent or goes to $+\infty$. We need to distinguish the two cases:

Case 1: $b_n + a_n \to +\infty$: In that case, ψ_0 solves

$$(-\partial_{xx})^\alpha \psi_0 + c_0 \psi_0' = f(\psi_0) \qquad \text{for all } x \in \mathbb{R}. \tag{13}$$

Furthermore, $\psi_0(0) = \frac{1+\theta}{2}$ and ψ_0 is monotone increasing. In particular, it is readily seen that there exists $\bar{\gamma}_0$ and $\bar{\gamma}_1$ such that $\lim_{x\to-\infty} \psi_0(x) = \bar{\gamma}_0$ and $\lim_{x\to+\infty} \psi_0(x) = \bar{\gamma}_1$ with

$$0 \le \bar{\gamma}_0 \le \frac{1+\theta}{2} \le \bar{\gamma}_1 \le 1.$$

Integrating (13) over $\mathbb{R}$, and using the fact that

$$\int_{\mathbb{R}} (-\partial_{xx})^\alpha \psi_0(x)\, dx = 0$$

(the proof is the same as in Lemma 3.2) we deduce

$$c_0(\bar{\gamma}_1 - \bar{\gamma}_0) = \int_{\mathbb{R}} f(\psi_0)\, dx < \infty \tag{14}$$

and so

$$f(\bar{\gamma}_0) = f(\bar{\gamma}_1) = 0.$$

This implies that

$$\bar{\gamma}_1 = 1 \qquad \text{and} \qquad \bar{\gamma}_0 \le \theta.$$

Finally, (14) and (12) yields

$$c_0(1-\theta) \ge \int_{\mathbb{R}} f(\psi_0)\, dx \ge \kappa_0$$

which gives the result.

Case 2: $a_n + b_n \to \bar{a} < \infty$: In that case, ψ_0 solves

$$(-\partial_{xx})^\alpha \psi_0 + c_0 \psi_0' = f(\psi_0) \qquad \text{for all } x \in (-\infty, \bar{a}) \tag{15}$$

and we need to modify the proof slightly. First, we notice that $\psi_0(x) = 1$ for $x \ge \bar{a}$, and we observe that $(-\partial_{xx})^\alpha \psi_0(x) \ge 0$ for $x \ge \bar{a}$. In particular

$$\int_{-\infty}^{\bar{a}} (-\partial_{xx})^\alpha \psi_0(x)\, dx \le \int_{\mathbb{R}} (-\partial_{xx})^\alpha \psi_0(x)\, dx = 0$$

Proceeding as above, we check that $\lim_{x\to-\infty}\psi_0(x)=\bar\gamma_0\leq\theta$ and integrating (15) over $(-\infty,\bar a)$, we deduce

$$c_0(1-\theta)\geq\int_{\mathbb{R}} f(\psi_0)\,dx>0.$$

□

The positivity of the speed, together with the sub-solution constructed in Lemma 2.2 will now give $\gamma_0=0$. More precisely, we now prove:

Lemma 3.4. *The function ϕ_0 satisfies:*

$$\lim_{x\to-\infty}\phi_0(x)=0.$$

Proof. Let $c_1=c_0/2>0$ and take n large enough so that $c_{b_n}\geq c_1$.

We recall that by Lemma 2.2 (see also the proof of Lemma 2.5) that the function

$$\varphi(x)=\begin{cases}\frac{1}{|x|^{2\alpha-1}} & x<-1\\ 1 & x>-1\end{cases}$$

satisfies

$$(-\partial_{xx})^\alpha\varphi+K\varphi'\geq 0 \qquad \text{in } \{\varphi<1\}$$

for some K large enough. Introducing $\varphi_\varepsilon(x)=\varphi(\varepsilon x)$, we deduce

$$(-\partial_{xx})^\alpha\varphi_\varepsilon+\varepsilon^{2\alpha-1}K\varphi_\varepsilon'(x)\geq 0 \qquad \text{in } \{\varphi_\varepsilon(x)<1\}$$

and taking ε small enough (recalling that $2\alpha>1$), we get

$$(-\partial_{xx})^\alpha\varphi_\varepsilon+c_1\varphi_\varepsilon'(x)\geq 0 \qquad \text{in } \{\varphi_\varepsilon<1\}.$$

Furthermore, $\varphi_\varepsilon=1$ for $x\geq 0$, and so by a sliding argument, we deduce $\phi_{b_n}(x)\leq\varphi_\varepsilon(x)$ for all n such that $c_{b_n}\geq c_1$ and thus

$$\phi_0(x)\leq\varphi_\varepsilon(x)$$

which implies in particular that $\gamma_0=0$. □

Finally, we conclude the proof of Proposition 3.1 by proving that $\gamma_1=1$:

Lemma 3.5. *The function ϕ_0 satisfies:*

$$\lim_{x\to+\infty}\phi_0(x)=1$$

Proof. We recall that (11) implies that either $\gamma_1=\theta$ or $\gamma_1=1$ (otherwise the integral is infinite). Furthermore, if $\gamma_1=\theta$, then $\phi_0\leq\theta$ on $\mathbb{R}$ and so $\int_{\mathbb{R}} f(\phi_0(x))\,dx=0$. Since $\gamma_0=0<\theta$, (11) implies $c_0=0$, which is a contradiction. Hence $\gamma_1=1$. □

□

4. Asymptotic behavior

We now prove Theorem 1.2, which further characterizes the behavior of ϕ_0 as $x \to -\infty$. We recall that in the case of the regular Laplacian ($\alpha = 1$), ϕ_0 and its derivatives decrease exponentially fast to 0 as $x \to -\infty$. When $\alpha \in (1/2, 1)$, it is readily seen that the proof of Lemma 3.4 actually implies:

Proposition 4.1 (Asymptotic behavior of ϕ_0)**.** *There exists M such that*

$$\phi_0(x) \leq \frac{M}{|x|^{2\alpha-1}} \qquad \text{for } x \leq -1$$

Noticing that $\phi_0' > 0$ solves

$$(-\partial_{xx})^\alpha \phi_0'' + c_0(\phi_0')' = 0 \qquad \text{for } x \leq 0,$$

we can also prove:

Proposition 4.2 (Asymptotic behavior of ϕ_0')**.** *There exists a constant m such that*

$$\phi_0'(x) \geq \frac{m}{|x|^{2\alpha}} \qquad \text{for } x \leq -1.$$

Proof. Lemma 2.3 implies that the function

$$\bar{\varphi}(x) = \begin{cases} \frac{1}{|x|^{2\alpha}} & x < -1 \\ 0 & x > -1 \end{cases}$$

satisfies

$$(-\partial_{xx})^\alpha \bar{\varphi} + c\bar{\varphi}'(x) = -\frac{c_\alpha}{2\alpha - 1}\frac{1}{|x|^{2\alpha+1}} + c\frac{2\alpha}{|x|^{2\alpha+1}} + O\left(\frac{1}{|x|^{4\alpha}}\right)$$

when $x \to \infty$, and so

$$(-\partial_{xx})^\alpha \bar{\varphi} + k\bar{\varphi}'(x) \leq 0 \qquad \text{for } x \leq -A$$

if k is small enough and A is large.

We introduce $\varphi_\varepsilon(x) = \bar{\varphi}(\varepsilon x)$, which satisfies

$$(-\partial_{xx})^\alpha \varphi_\varepsilon + \varepsilon^{1-2\alpha} k \varphi_\varepsilon' \leq 0 \qquad \text{for } x < -\varepsilon^{-1}A$$

hence

$$(-\partial_{xx})^\alpha \varphi_\varepsilon + c_0 \varphi_\varepsilon' \leq 0 \qquad \text{for } x < -\varepsilon^{-1}A$$

provided we choose ε small enough.

Finally, we take r so that

$$\phi_0'(x) \geq r\varphi_\varepsilon(x) \qquad \text{for } -\varepsilon^{-1}A < x < -\varepsilon^{-1}.$$

Proposition 4.2 now follows from the maximum principle and a sliding argument using the fact that $\varphi_\varepsilon(x) = 0$ for $x \geq -\varepsilon^{-1}$. □

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