Anomalous transport of particles in Plasma physics

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Abstract

We investigate the long time/small mean-free-path asymptotic behavior of the solutions of a Vlasov-Lévy-Fokker-Planck equation and show that the asymptotic dynamics for the VLFP are described by an anomalous diffusion equation.

Keywords: Kinetic equations, Fokker-Planck operator, Lévy statistic, anomalous diffusion limit, fractional heat equation.
2010 MSC: 82D10, 35Q84, 35R11.

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1. Introduction

1.1. The Vlasov-Lévy-Fokker-Planck equation

In this note, we investigate the long time/small mean-free-path asymptotic behavior of the solutions of the following Vlasov-Lévy-Fokker-Planck equation:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \nu \mathrm{div}_v(vf) - (-\Delta_v)^s f & \text{in } \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty) \\ f(x,v,0) = f_0(x,v) & \text{in } \mathbb{R}^N \times \mathbb{R}^N \end{cases} \tag{1}$$

for $s \in (0,1)$, $\nu > 0$. Such an equation models the evolution of the distribution function $f(x,v,t)$ of a cloud of particles in a Plasma: The left hand side of (1) models the free transport of the particles, while the Lévy-Fokker-Planck operator, in the right hand side:

$$\mathcal{L}^s(f) = \nu \mathrm{div}_v(vf) - (-\Delta_v)^s f \tag{2}$$

describes the interactions of the particles with the background. It can be interpreted as a deterministic description of the Langevin equation for the velocity of the particles, $\dot{v}(t) = -\nu v(t) + A(t)$, where ν is the friction coefficient and $A(t)$ is a white noise. The

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[1]Partially supported by NSF Grant DMS-0901340
[2]Partially supported by NSF Grant DMS-1109397

classical Fokker-Planck operator corresponds to $s = 1$ and arises when $A(t)$ is a Gaussian white noise. In that case, equilibrium distributions (solutions of $\mathcal{L}^1(M) = 0$) are Maxwellian (or Gaussian) velocity distributions $M = C\exp(-\frac{\nu|v|^2}{2})$. However, some experimental measurements of particles and heat fluxes in confined plasma point to non-local features and non-Gaussian distribution functions (see for instance [7]). The introduction of fractional Lévy statistic in the velocity equation (replacing the Gaussian white noise by Lévy white noise in Langevin equation) can be seen as an attempt at taking into account these non-local effects in plasma turbulence.

The operator $\mathcal{L}^s$ for $s \in (0,1)$ has been studied in particular by Gentil-Imbert [4] (see also Proposition 1.1 below). Instead of Maxwellian velocity distribution functions, thermodynamical equilibriums for $\mathcal{L}^s$ are described by Lévy stable distribution functions. These are power law (or heavy tail) functions, which are characterized in particular by infinite second moment of velocity (or infinite variance).

In this paper, we show that the long time/small mean-free-path limit for the kinetic equation (1) leads to an anomalous (or fractional) diffusion equation for the density of particles. For the classical Vlasov-Fokker-Planck equation, a similar asymptotic leads to a standard diffusion equation. The derivation of anomalous diffusion regimes for kinetic equations has been recently investigated in the framework of linear Boltzmann equation (in which the Fokker-Planck operator is replaced by an integral operator), see [6, 5, 1, 2]. The common theme between these works and the present paper is the fact that the equilibrium distribution functions are heavy tail functions with infinite second moment of velocity. However, the different nature of the Fokker-Planck operator requires the introduction of *new* techniques for the investigation of this limit.

1.2. Properties of $\mathcal{L}^s$

We recall that the fractional Laplace operator $(-\Delta_v)^s$ can be defined using the Fourier transform, by

$$\widehat{(-\Delta_v)^s f}(\xi) = |\xi|^{2s}\widehat{f}(\xi) \tag{3}$$

or as the following singular integral

$$(-\Delta_v)^s f(v) = c_s \text{P.V.} \int [f(v) - f(w)] \frac{1}{|v-w|^{N+2s}}\, dw \tag{4}$$

for some constant c_s depending on s and the dimension N. We have the following result (see [4]):

Proposition 1.1. *When $s \in (0,1)$, there exists a unique normalized equilibrium distribution function $F(v)$, solution of*

$$\mathcal{L}^s(F) = \nu\, div_v(vF) - (-\Delta_v)^s F = 0, \qquad \int_{\mathbb{R}^N} F(v)\, dv = 1. \tag{5}$$

Furthermore, $F(v) > 0$ for all v, and F is a heavy-tail distribution function:

$$F(v) \sim \frac{C}{|v|^{N+2s}} \qquad as\ |v| \to \infty.$$

This proposition is proved in [4]. An explicit formula can easily be found for the Fourier transform of F. Indeed, $F(v)$ satisfies (5) if and only if its Fourier transform $\widehat{F}(\xi)$ satisfies (using (3)):

$$-|\xi|^{2s}\widehat{F}(\xi) - \nu\xi\cdot\nabla_\xi\widehat{F}(\xi) = 0, \qquad \text{and } \widehat{F}(0) = 1,$$

which yields the symmetric Lévy distribution in Fourier space $\widehat{F}(\xi) = e^{-\frac{1}{s\nu}|\xi|^{2s}}$ (note that when $s = 1$, we recover Maxwell's distribution function).

1.3. Main results

We now turn to our main goal, which is the investigation of the long time/small mean-free-path-limit of (1). As in [6, 5, 1, 2], we start by rescaling the space and time variable as follows:

$$x \mapsto \varepsilon x, \qquad t \mapsto \varepsilon^{2s} t.$$

To simplify the notations, we also take $\nu = 1$ so that (1) becomes

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v\cdot\nabla_x f^\varepsilon = \operatorname{div}_v(vf^\varepsilon) - (-\Delta_v)^s f^\varepsilon & \text{in } \mathbb{R}^N\times\mathbb{R}^N\times(0,\infty) \\ f^\varepsilon(x,v,0) = f_0(x,v) & \text{in } \mathbb{R}^N\times\mathbb{R}^N. \end{cases} \tag{6}$$

The existence of a unique solution satisfying appropriate a priori estimates can be established exactly as in the case of the usual Vlasov-Fokker-Planck equation. We refer the reader to the article of Degond [3] for further details. Note that since there is no acceleration field here, this is relatively easy. We do not dwell on this issue, which is not the focus of this paper.

Our main result is the following:

Theorem 1.2. *Assume that $f_0 \in L^2(\mathbb{R}^{2N}, F(v)^{-1}dvdx)$. Then, up to a subsequence, the solution f^ε of (6) converges weakly in $L^\infty(0,T;L^2(\mathbb{R}^{2N},F(v)^{-1}dvdx))$, as $\varepsilon\to 0$ to $\rho(x,t)F(v)$ where $\rho(x,t)$ solves*

$$\begin{cases} \partial_t\rho + (-\Delta_x)^s\rho = 0 & \text{in } \mathbb{R}^N\times(0,\infty) \\ \rho(x,0) = \rho_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{7}$$

with $\rho_0(x) = \int f_0(x,v)\,dv$.

2. Proof of Theorem 1.2

2.1. A priori estimates

In order to prove Theorem 1.2, we need some a priori estimates for f^ε, which are consequence of the dissipation properties of the operator $\mathcal{L}^s$. These properties have been studied in particular in [4]. The following result will be of use in the sequel. We present the proof here for the sake of completeness.

Proposition 2.1. *For all f smooth enough,*

$$-\int_{\mathbb{R}^N}\mathcal{L}^s(f)\frac{f}{F}\,dv = \mathcal{D}(f) := \frac{c_s}{2}\int_{\mathbb{R}^{2N}}\left[\frac{f(w)}{F(w)} - \frac{f(v)}{F(v)}\right]^2\frac{F(v)}{|v-w|^{N+2s}}\,dv\,dw. \tag{8}$$

Furthermore, there exists $\theta > 0$ such that

$$\mathcal{D}(f) \geq \theta \int_{\mathbb{R}^N} |f(v) - \rho F(v)|^2 \frac{1}{F(v)}\, dv \tag{9}$$

Proof. Inequality (8) is proved in [4] in a more general setting. Let $g = f/F$. We write

$$\begin{aligned}
-\int_{\mathbb{R}^N} \mathcal{L}^s(f)(f/F)\, dv &= -\int_{\mathbb{R}^N} \nu \mathrm{div}_v(vF) g^2/2 - (-\Delta_v)^s(gF) g\, dv \\
&= -\int_{\mathbb{R}^N} (-\Delta_v)^s(F) g^2/2 - (-\Delta_v)^s(gF) g\, dv \\
&= \int_{\mathbb{R}^N} -\frac{1}{2} F((-\Delta_v)^s(g^2) + Fg(-\Delta_v)^s(g)\, dv
\end{aligned}$$

where we used (5). It follows (using (4)):

$$\begin{aligned}
&-\int_{\mathbb{R}^N} \mathcal{L}^s(f)(f/F)\, dv \\
&= c_s \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left\{ -\frac{1}{2}[g(v)^2 - g(w)^2] + g(v)^2 - g(w)g(v) \right\} \frac{F(v)}{|v-w|^{N+2s}}\, dv\, dw \\
&= \frac{c_s}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(v) \frac{[g(v) - g(w)]^2}{|v-w|^{N+2s}}\, dv\, dw = \mathcal{D}(f)
\end{aligned}$$

which gives (8).

Finally, we write $f = \rho F + h$, with $\int h\, dv = 0$. We then have

$$\mathcal{D}(f) = \frac{c_s}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(w) \left[\frac{h(v)}{F(v)} - \frac{h(w)}{F(w)} \right]^2 \frac{1}{|v-w|^{N+2s}}\, dv\, dw$$

and since $\int h(v)\, dv = 0$, Poincare's inequality (see for instance Mouhot-Russ-Sire [8]) yields

$$\mathcal{D}(f) \geq \theta \int_{\mathbb{R}^N} \left(\frac{h(v)}{F(v)} \right)^2 F(v)\, dv$$

for some $\theta > 0$, which gives (9). □

Corollary 2.2. *Assume that f_0 satisfies the conditions of Theorem 1.2. Then the solution $f^\varepsilon(x, v, t)$ of (6) satisfies*

$$\sup_{t\in[0,T]} \int \frac{(f^\varepsilon)^2}{F}\, dx\, dv + \varepsilon^{-2s}\theta \int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f^\varepsilon - \rho^\varepsilon F|^2}{F}\, dv\, dx\, dt \leq \int \frac{(f_0)^2}{F}\, dx\, dv$$

where $\rho^\varepsilon = \int f^\varepsilon\, dv$. In particular, there exists a function $\rho(x,t) \in L^2(\mathbb{R}^N \times (0,\infty))$ such that

$$f^\varepsilon(x,v,t) \rightharpoonup \rho(x,t)F(v) \qquad \textit{weakly in } L^2_{F^{-1}}(\mathbb{R}^N \times (0,\infty)).$$

Proof. Multiplying (6) by f^ε/F and integrating with respect to x and v, we get

$$\varepsilon^{2s} \frac{d}{dt} \int \frac{(f^\varepsilon)^2}{F}\, dx\, dv = \int \mathcal{L}^s(f^\varepsilon) f^\varepsilon/F\, dx\, dv$$

and Proposition 2.1 gives the result. □

2.2. Proof of the main result.

We can now prove our main result:

Proof of Theorem 1.2. Let $\varphi(x,t)$ be a test function in $\mathcal{D}(\mathbb{R}^N \times [0,\infty))$. Multiplying (6) by $\phi^\varepsilon(x,v,t) = \varphi(x+\varepsilon v, t)$, we get:

$$\int f^\varepsilon \Big[\varepsilon^{2s}\partial_t \phi^\varepsilon + \varepsilon v \cdot \nabla_x \phi^\varepsilon - v \cdot \nabla_v \phi^\varepsilon + (-\Delta)^s \phi^\varepsilon\Big]\, dx\, dv\, dt$$
$$+\varepsilon^{2s} \int f_0(x,v)\phi^\varepsilon(x,v,0)\, dx\, dv = 0.$$

Next, we note that

$$v \cdot \nabla_v \phi^\varepsilon = \varepsilon v \cdot \nabla_x \phi^\varepsilon \qquad \text{and} \qquad (-\Delta)^s \phi^\varepsilon = \varepsilon^{2s} (-\Delta)^s \varphi(x+\varepsilon v, t).$$

We deduce

$$\int f^\varepsilon \Big[\partial_t \varphi + (-\Delta)^s \varphi\Big](x+\varepsilon v, t)\, dx\, dv\, dt + \int f_0(x,v)\varphi(x+\varepsilon v, 0)\, dx\, dv = 0 \qquad (10)$$

and we conclude thanks to the following lemma:

Lemma 2.3. *For all test function $\psi \in \mathcal{D}(\mathbb{R}^d \times [0,\infty))$, we have*

$$\lim_{\varepsilon \to 0} \int f^\varepsilon \psi(x+\varepsilon v, t)\, dx\, dv\, dt = \int \rho(x,t)\psi(x,t)\, dx\, dt$$

Indeed, passing to the limit in (10), Lemma 2.3 gives

$$\int \rho(x,t) \Big[\partial_t \varphi + (-\Delta)^s \varphi\Big](x,t)\, dx\, dt + \int \rho_0(x)\varphi(x,0)\, dx = 0$$

which is the weak formulation of (7). □

Proof of Lemma 2.3. We write

$$\int f^\varepsilon \psi(x+\varepsilon v, t)\, dx\, dv\, dt = \int f^\varepsilon \psi(x,t)\, dx\, dv\, dt + \int f^\varepsilon \Big[\psi(x+\varepsilon v, t) - \psi(x)\Big]\, dx\, dv\, dt.$$

The first term converges to

$$\int \rho(x,t) F(v) \psi(x,t)\, dx\, dv\, dt = \int \rho(x,t)\psi(x,t)\, dx\, dt.$$

For the second term, we note that

$$\int_{|v|\leq R} \Big| f^\varepsilon \Big[\psi(x+\varepsilon v, t) - \psi(x)\Big]\Big|\, dx\, dv\, dt \leq \varepsilon C R ||D\psi||_{L^\infty} \int |f^\varepsilon|\, dx\, dv\, dt$$

and so

$$\limsup_{\varepsilon \to 0} \int_{|v|\leq R} \Big| f^\varepsilon \Big[\psi(x+\varepsilon v, t) - \psi(x)\Big]\Big|\, dx\, dv\, dt = 0.$$

Furthermore,

$$\int_{|v|\geq R} \left| f^\varepsilon \Big[\psi(x+\varepsilon v,t)-\psi(x)\Big]\right| \,dx\,dv\,dt$$
$$\leq C\left(\int_{|v|\geq R} \frac{(f^\varepsilon)^2}{F}\,dx\,dv\,dt\right)^{1/2}\left(\int_{|v|\geq R}\Big[\psi(x+\varepsilon v,t)-\psi(x)\Big]^2 F(v)\,dx\,dv\,dt\right)^{1/2}$$

and so

$$\limsup_{\varepsilon\to 0}\int_{|v|\geq R} \left| f^\varepsilon \Big[\psi(x+\varepsilon v,t)-\psi(x)\Big]\right| \,dx\,dv\,dt$$
$$\leq C||\psi||_{L^2\mathbb{R}^N\times(0,T)}\left(\int \frac{(f_0)^2}{F}\,dx\,dv\,dt\right)^{1/2}\left(\int_{|v|\geq R} F(v)\,dv\right)^{1/2}$$
$$\leq C||\psi||_{L^2\mathbb{R}^N\times(0,T)}\left(\int \frac{(f_0)^2}{F}\,dx\,dv\,dt\right)^{1/2} R^{-\alpha/2}.$$

We deduce

$$\limsup_{\varepsilon\to 0}\int \left| f^\varepsilon \Big[\psi(x+\varepsilon v,t)-\psi(x)\Big]\right| \,dx\,dv\,dt \leq CR^{-\alpha/2}$$

and since this holds for all $R>0$, the result follows. □

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