THE MODULI SPACE OF WEIGHTED CONFIGURATIONS ON PROJECTIVE SPACE

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Contents

1. Introduction

In this paper we study certain symplectic quotients of n-fold products of $\mathbb{CP}^m$ by the unitary group U(m+1) acting diagonally. After clarifying some basic properties of these quotients—when they are nonempty and nonsingular—, we construct the action-angle variables, defined on a dense open subset, of an integrable Hamiltonian system. The system generalizes the "bending flows" on the polygon space of [KM96], and its semiclassical quantization reproduces formulas from the representation theory of U(m+1).

Think of a point of $\mathbb{CP}^m$ as the line through a unit vector $w \in \mathbb{C}^{m+1}$, and identify this line with the hermitean projection $w \otimes w^*$ that maps $v \in \mathbb{C}^{m+1}$ to $(v, w)w$. Pick $n \geq m+1$, and for $i = 1, \ldots, n$ fix numbers $r_i > 0$; the reason for the restriction $n \geq m+1$ becomes clear in equation (1.1) below. Let $w_i \in \mathbb{C}^{m+1}$ be unit vectors, and set $e_i = r_i w_i \otimes w_i^*$. These hermitean matrices have rank=1 (with eigenvalues $r_i, 0, \ldots, 0$), and may be thought of either as weighted points in $\mathbb{CP}^m$ or as elements in an orbit $\mathcal{O}_{r_i}$ of U(m+1) acting by conjugation. We largely use the second interpretation. The orbits carry the Kostant-Kirillov symplectic form, call it ω_i, which is $2r_i$ times the usual Fubini-Study form on $\mathbb{CP}^m$. The (diagonal) action of U(m+1) on $\Pi_{i=1}^n (\mathbb{CP}^m, \omega_i)$ is Hamiltonian, and its momentum map is given by

$$\mu : (e_1, \ldots, e_n) \mapsto e_1 + \cdots + e_n.$$

With $\Lambda = \frac{1}{m+1}(r_1 + \cdots + r_n)$, the symplectic quotient $\mu^{-1}(\Lambda\mathbb{I})/\mathrm{U(m+1)}$ turns out to be

$$\{(e_1, \ldots, e_n) \mid e_1 + \cdots + e_n = \Lambda\mathbb{I}\}/\mathrm{U(m+1)}. \tag{1.1}$$

We denote this symplectic quotient by $M_{\mathbf{r}}$ ($\mathbf{r}$ stands for the n-tuple $(r_1, \ldots, r_n)$).

Our goal, as mentioned already, is to construct an integrable Hamiltonian system on $M_{\mathbf{r}}$, possessing action-angle coordinates defined almost everywhere, and to develop some connections with representation theory. We now describe the content of our paper in more detail.

The paper [KM96] by Kapovich and Millson provides motivation and an appealing geometrical setting. They work in the Lie algebra $\mathfrak{su}(2)$. That is, they take $m = 1$ and use $e_i = r_i(w_i \otimes w_i^* - \frac{1}{2}\mathbb{I})$. Then e_i may be thought of as a vector in $\mathbb{R}^3$, of length r_i, and $M_{\mathbf{r}}$ becomes the space of closed polygons with prescribed side lengths r_i. By analogy, we continue to refer to an n-tuple $\mathbf{e} = (e_1, \ldots, e_n)$ as an *n-gon*, to e_i as the i-th *edge*, and to the partial sums $\sum_1^{i+1} e_j = A_i$ as the *diagonals* of the n-gon (cf. Figure 1 in §6). Each A_i is, generically, a Hermitean matrix of rank $\min\{i+1, m+1\}$. Thus we may think of an n-gon as a sequence of Hermitean matrices (the diagonals), each A_i formed from its predecessor A_{i-1} by making a rank-1 perturbation with the nonzero eigenvalue r_{i+1} of the perturbing matrix fixed in advance.

We first give necessary and sufficient conditions on $\mathbf{r}$ for the spaces $M_{\mathbf{r}}$ to be nonempty. Following [LM], we find that $M_{\mathbf{r}}$ is nonempty if and only if $\mathbf{r}$ satisfies the *strong triangle inequalities*

$$mr_i \leq r_1 + \cdots + \widehat{r_i} + \cdots + r_n, \quad 1 \leq i \leq n.$$

Let $C(n, m+1) \subset \mathbb{R}^n_+$ be the polyhedral cone defined by these inequalities. If we normalize $\mathbf{r}$ by requiring $\sum_i r_i = m+1$ we find that $M_{\mathbf{r}}$ is nonempty if and only if $\mathbf{r}$ is an element of a certain convex polytope known as the *hypersimplex* and denoted by $\Delta_{n-1}(m+1)$, see [Zi] and [GGMS]. Thus, $C(n, m+1)$ is the cone on $\Delta_{n-1}(m+1)$. (The strong triangle inequalities are, of course, a special case of the Klyachko inequalities on the eigenvalues of sums of hermitean matrices [Kly98]).

We also explain, very briefly, how to identify the space $M_{\mathbf{r}}$ with a weighted complex analytic quotient of the n-fold product $\Pi_1^n \mathbb{CP}^m$. The existence of the structure of a complex analytic space on such quotients is a special case of [HL94] and [Sj95]. Deligne and Mostow in [DM86] constructed the weighted complex analytic quotients of $\mathbb{CP}^1$, and the connection with the symplectic quotient of products of $\mathbb{CP}^1$ (spatial polygons) was found independently in [KM96] and [Kly92].

The description of $M_{\mathbf{r}}$ concludes with the observation that it is smooth if and only if $\mathbf{r}$ does not lie on certain hyperplane sections of the cone $C(n, m+1)$. The subsequent discussion involving our integrable Hamiltonian system will be restricted to this generic case.

In the three-dimensional setting of [KM96], the action variables of their integrable system are the lengths of the diagonal vectors A_i, or equivalently, the positive eigenvalues of the $\mathfrak{su}(2)$ matrices representing those vectors. The corresponding Hamiltonian flows, the "bending flows", rotate half of the polygon rigidly about a fixed diagonal at constant speed, while leaving the other half of the polygon fixed. The analogous action variables in the higher-dimensional setting are still the eigenvalues of the diagonals. Let λ_{ij} be the j-th eigenvalue of the i-th diagonal. A subset of the λ_{ij} will be generically functionally independent, defining a real Lagrangean polarization on an open dense subset of $M_{\mathbf{r}}$. Generically again, λ_{ij} has multiplicity one. Let P_{ij} be the orthogonal projection on the corresponding eigenline. Then the Hamiltonian flow of λ_{ij} is obtained by conjugating the first i edges by the one-parameter group $\exp(\sqrt{-1}\, tP_{ij})$ and leaving the last $n-i$ edges fixed. Since "half" the polygon moves by a rigid motion and the "other half" remains fixed we still call these flows "bending flows". Because $P_{ij}^2 = P_{ij}$, the bending flows are clearly periodic with period 2π. The commutativity of the flows is made plausible by the geometric picture; a proof by calculation is also easy.

We then turn to the momentum polyhedron and the angle variables. A critical role in identifying the image of the momentum map is played by the Weinstein-Aronszajn formula from perturbation theory; the simple version we need shows that the eigenvalues of A_{i-1} and A_i interlace. As a consequence, one finds that the momentum polytope is defined by certain Gel'fand-Tsetlin patterns (see §§ 8, 9). The angle variables also make their appearance at this stage. One sees from the Weinstein-Aronszajn formula that if $u_{i-1,j}$ is a unit eigenvector of the diagonal A_{i-1}, corresponding to the eigenvalue $\lambda_{i-1,j}$, and if $A_i = A_{i-1} + r_{i+1} w_{i+1} \otimes w_{i+1}^*$ is the next diagonal, then the modulus of the inner product $(u_{i-1,j}, w_{i+1})$ is left constant by all bending flows. One therefore expects the collection of numbers $\arg(u_{i-1,j}, w_{i+1})$ to lead to the angle variables. This is almost correct. There are arbitrary phases in the choices of the unit eigenvectors $u_{i-1,j}$ and the unit vectors w_{i+1}, which would affect the arguments of the inner products. It is therefore necessary to combine these inner products into "four point functions" $(w,x)(x,y)(y,z)(z,w)$ (the terminology comes from [BeSch]) in order to produce an angle that is independent of choices. Because of this somewhat subtle definition, the computation of Poisson brackets amongst the actions and the angles is not straightforward.

The occurrence and special form of Gel'fand-Tsetlin patterns in the description of the momentum polytope for the bending Hamiltonians is explained by a basic observation of [HK97]. They discovered that Gel'fand-MacPherson duality [GGMS, p.305], intertwines the bending Hamiltonians on $M_{\mathbf{r}}$ and the Gel'fand-Tsetlin Hamiltonians on a symplectic quotient of the Grassmannian $G(m+1, \mathbb{C}^n)$ by the maximal torus of $PU(n)$. Their ideas easily extend to our setting, see §11.

We conclude by relating our system to representation theory. Assume that the r_i are positive integers, and that $\sum_1^n r_i$ is divisible by $m+1$ (so that Λ in (1.1) is an integer). This quantization of our system yields the Pieri formula for decomposing the n-fold tensor product of symmetric powers $\bigotimes \mathcal{S}^{r_i}$ of the basic representation of U(m+1) on $\mathbb{C}^{m+1}$. Indeed, Pieri's formula is just the Weinstein-Aronszajn formula, and the decomposition of the tensor product is indexed by lattice points in the momentum polytope.

The duality of Hausmann and Knutson also has a quantum analogue. It asserts the equality of the multiplicity of the representation $\det^{\Lambda}$ of U(m+1) in the above n-fold tensor product and the multiplicity of the weight $\mathbf{r}$ of the maximal torus of U(n) in the Λ-th Cartan power of the $m+1$-st exterior power of the standard representation of U(n). This quantum duality is a reflection of the rule for associating a semistandard Young tableau to a Gel'fand-Tsetlin pattern (but for the special patterns described above) [GZ86], see below, §12, Remark 10.

It is our hope that there are analogous results for all symplectic quotients of products of flag manifolds. For general such products, one can find integrable systems that reduce to ours in the case of projective space, but it appears very hard to construct an explicit family of Hamiltonians *with periodic flows*, i.e. action variables. If such a construction could be carried out and the momentum polytope could be computed, then by counting lattice points in the momentum polytope one could find information on decomposing tensor products of irreducible representations. Many deep connections are now known between tensor product decompositions and convex polyhedra; these polytopes, however, do not seem to arise as images of momentum mappings. One of the main motivations for our paper is that the special case treated here of is probably the only case where everything can be worked out with simple explicit formulas.

2. The moduli space of n-gons in $\mathcal{H}_{m+1}$

In this section, we collect the notation used throughout, and in particular, introduce the moduli space of n-gons with which we will be concerned.

(1) Let $\mathcal{H}_{m+1}$ be the vector space of $m+1$ by $m+1$ Hermitean matrices. We identify it with the Lie algebra $\mathfrak{u}(m+1)$ by the linear map $\phi : \mathfrak{u}(m+1) \to \mathcal{H}_{m+1}$ given by $x \mapsto X = \sqrt{-1}\,x$. This makes $\mathcal{H}_{m+1}$ into a Lie algebra, but we shall not need to refer to the induced bracket (which is $[\![X,Y]\!] = -\sqrt{-1}\,(XY-YX)$). The symbol $[\cdot,\cdot]$ will continue to denote the matrix commutator $XY-YX$.

(2) $\mathcal{H}^0_{m+1} = \{X \in \mathcal{H}_{m+1} \mid \mathrm{Tr}X = 0\}$.

(3) We identify the Lie algebra $\mathcal{H}_{m+1}$ with its dual via the bilinear form $(X,Y) = \mathrm{Tr}XY$. A U(m+1)-orbit $\mathcal{O}$ then carries the Kostant-Kirillov symplectic form. The Poisson bracket is defined by

$$\{f,g\}(X) = \sqrt{-1}\,\mathrm{Tr}\Big([\nabla f(X), \nabla g(X)]\,X\Big), \tag{2.1}$$

and the Hamiltonian equation generated by f is

$$\dot{X} = \sqrt{-1}\,[\nabla f(X), X]. \tag{2.2}$$

With these conventions, $\mathcal{H}_2^0$ and its Poisson bracket may be identified with Euclidean space $\mathbb{R}^3$ and its standard bracket, see §10.

(4) For $r > 0$, we let $\mathcal{O}_r$ denote the orbit through $\mathrm{diag}(r, 0, \dots, 0)$. It is diffeomorphic to $\mathbb{CP}^m$, and the Kostant-Kirillov form is $4r$ times the Fubini-Study form on $\mathbb{CP}^m$. The elements of $\mathcal{O}_r$ are denoted by the letter e (for "edge", see below), usually with subscript.

(5) Let $w \in \mathbb{C}^{m+1}$ be a unit vector. Define $w \otimes w^* \in \mathcal{H}_{m+1}$ by $w \otimes w^*(v) = (v, w)w$. The elements of $\mathcal{O}_r$ are precisely the matrices of the form $rw \otimes w^*$. Given $e \in \mathcal{O}_r$, the unit vector w_i is determined only up to multiplication by a complex number of modulus one.

(6) Let $\mathbf{r} = (r_1, r_2, \dots, r_n)$ be an n-tuple of positive numbers. We define a *(closed) n-gon with side-lengths* $\mathbf{r}$ to be an n-tuple $\mathbf{e} = (e_1, e_2, \dots, e_n)$ such that for all $i, 1 \le i \le n$ we have
   (a) $e_i \in \mathcal{O}_{r_i}$,
   (b) $\sum_1^n e_i = \Lambda \mathbb{I}$; then $\Lambda = \frac{1}{m+1} \sum_1^{m+1} r_i$ follows from equality of traces.

   We call the matrices e_i the *edges* of the n-gon $\mathbf{e}$ and r_i the *length* of the edge e_i. Condition (b) says that the n-gon $\mathbf{e}$ is closed (modulo the center of $\mathcal{H}_{m+1}$).

(7) When $\mathbf{r}$ is given Λ always stands for $\frac{1}{m+1} \sum r_i$. Sometimes the notation $\Lambda_{\mathbf{r}}$ is used to emphasize the dependence of Λ on $\mathbf{r}$.

(8) Given $\mathbf{r}$, define $\widetilde{N}_{\mathbf{r}}$ to be the product symplectic manifold $\Pi_1^n \mathcal{O}_{r_i}$. The diagonal action of U(m+1) on $\widetilde{N}_{\mathbf{r}}$ is Hamiltonian with momentum map μ given by

$$\mu(\mathbf{e}) = \sum_1^n e_i.$$

(9) Given $\mathbf{r}$, let

$$\widetilde{M}_{\mathbf{r}} = \mu^{-1}(\Lambda \mathbb{I}) = \{\mathbf{e} \in \widetilde{N}_{\mathbf{r}} \mid \sum_{i=1}^n e_i = \Lambda \mathbb{I}\}.$$

   This is the space of closed n-gons. The unitary group acts diagonally on $\widetilde{M}_{\mathbf{r}}$.

(10) Finally, we define the moduli space, $M_{\mathbf{r}}$, of n-gons (with side-lengths $\mathbf{r}$) to be the quotient of $\widetilde{M}_{\mathbf{r}}$ by the diagonal action of U(m+1).

Because the stabilizer of the scalar matrix $\Lambda \mathbb{I}$ is all of U(m+1), we obtain

Lemma 2.1. *$M_{\mathbf{r}}$ is the symplectic quotient of $\widetilde{N}_{\mathbf{r}}$ corresponding to the (one-point) orbit $\Lambda\mathbb{I} \in \mathcal{H}_{m+1}$ under the diagonal action of U(m+1).*

3. Nonemptiness of the moduli spaces

In this section we will prove one implication in the following theorem.

Theorem 3.1. *The moduli space $M_{\mathbf{r}}$ is nonempty if and only if $\mathbf{r}$ satisfies the system of strong triangle inequalities*

$$mr_i \leq r_1 + r_2 + \cdots + \widehat{r_i} + \cdots + r_n.$$

Here $\widehat{r_i}$ means that r_i has been omitted in the summation.

The full theorem is a consequence of the inequalities of [Kly98], see also [Bel]. It is proved explicitly in [LM], Theorem 4.7. We will give an elementary proof here of the necessity of the inequalities.

Definition 3.1. Let $X \in \mathcal{H}^0_{m+1}$. We will say X is *maximally singular* if X is conjugate to a diagonal matrix with eigenvalues $(r, -\frac{r}{m}, \dots, -\frac{r}{m})$. We note that the orbit $\mathcal{O}^0_r$ under U(m+1) of such an X is the projection onto tracefree matrices of the orbit $\mathcal{O}_r$ through $\text{diag}(r, 0, \dots, 0)$.

Lemma 3.1. *Suppose $X_1, X_2 \in \mathcal{H}^0_{m+1}$ are distinct, maximally singular, and satisfy $tr(X_j^2) = 1$. Then $tr(X_1X_2) \geq -1/m$, with equality if and only if X_1 and X_2 commute.*

Proof. We may write

$$X_j = \sqrt{\frac{m+1}{m}}(w_j \otimes w_j^* - \frac{1}{m+1}\mathbb{I}),$$

where $\|w_j\| = 1$, $j = 1, 2$.

Then

$$\begin{aligned}\text{Tr}X_1X_2 =& \frac{m+1}{m}\text{Tr}[(w_1 \otimes w_1^* - \frac{1}{m+1}\mathbb{I})(w_2 \otimes w_2^*)] \\ =& \frac{m+1}{m}[|(w_1, w_2)|^2 - \frac{1}{m+1}] \geq \frac{m+1}{m} \cdot - \frac{1}{m+1} \\ =& -\frac{1}{m+1}.\end{aligned}$$

Clearly we have equality if and only if $(w_1, w_2) = 0$ if and only if X and Y commute. □

Proposition 3.1. *Suppose that $M_{\mathbf{r}}$ is nonempty. Then $\mathbf{r}$ satisfies the strong triangle inequalities.*

Proof. Choose $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$. Then $e_1 + \cdots + e_n = \Lambda\mathbb{I}$ is equivalent to $r_1 X_1 + \cdots + r_n X_n = 0$, where the matrices

$$X_j = \sqrt{\frac{m+1}{m}}(w_j \otimes w_j^* - \frac{1}{m+1}\mathbb{I})$$

satisfy the hypotheses of Lemma 3.1. Alternatively,

$$r_i X_i = -r_1 X_1 - \cdots - \widehat{r_i X_i} - \cdots - r_n X_n.$$

Multiply each side by X_i and take the trace to obtain

$$r_i^2 = -\sum_{j(\neq i)} r_i r_j \mathrm{Tr}(X_j X_i) \leq \frac{1}{m} \sum_{j(\neq i)} r_i r_j.$$

Now divide both sides by r_i to obtain the result. □

Definition 3.2.

$$C(n, m+1) = \{\mathbf{r} \in (\mathbb{R})_+^n \mid M_{\mathbf{r}} \neq \emptyset\}.$$

As mentioned in the introduction, the intersection of $C(n, m+1)$ with the hyperplane $\sum r_i = m+1$ is known in the literature as the *hypersimplex*.

4. The space of n-gon linkages and the moduli spaces of weighted configurations on $\mathbb{CP}^m$

In [Sj95] and [HL94] the authors constructed the analytic quotient of a (not necessarily projective) Kähler manifold M by the action of a complex reductive group G. It is assumed that some maximal compact subgroup $K \subset G$ acts in a Hamiltonian fashion on M with momentum map μ. In their theory, a point $m \in M$ is defined to be *semistable* if the closure of the orbit $G \cdot m$ intersects the subset $\mu^{-1}(0)$ of M. The set of semistable points is denoted by M^{sst}; it is open in M. A point of M is defined to be *nice semistable* if the orbit itself intersects the zero momentum set. Define an equivalence relation, called *extended orbit equivalence*, by declaring two points to be related if their orbit closures intersect. (That this is indeed an equivalence relation follows from a theorem assetring that each equivalence class of semistable points contains a unique nice semistable orbit).

The *analytic quotient* of M by G, denoted $M//G$, is then defined to be the quotient of M^{sst} by extended orbit equivalence. Since any momentum zero point is nice semistable, there is an induced map from the symplectic quotient $\mu^{-1}(0)/K$ to the analytic quotient. The above authors prove that this map is a homeomorphism. These results were proved earlier for the case that the quotient is smooth in [Ki].

When M is a product of (partial) flag manifolds (so in particular of projective spaces), the authors in [LM] gave a characterization of semistable points using convex function theory on the associated symmetric space. In this section, we review their theory for the case of weighted configurations of points on $\mathbb{CP}^m$. We will need a brief review of the compactification of a symmetric space X of noncompact type. In what follows, we let G be the connected component of the identity of the isometry group of X, choose a basepoint o in X, and let K be the isotropy subgroup of o. For the purpose of understanding the rest of this paper, the reader may specialize X to the case of $X = \mathrm{SL(m+1,\mathbb{C})}/\,\mathrm{SU(m+1)}$. This may be realized as the set $\mathbf{P}$ of positive definite Hermitean matrices of determinant 1; then the basepoint is $\mathbb{I}$ and $K = \mathrm{SU(m+1)}$.

4.1. **The ideal boundary of a symmetric space of noncompact type.** We will briefly summarize the material in [E, §1.7].

Definition 4.1. Two unit-speed geodesics σ and γ are said to be asymptotes, or be asymptotic, if the Riemannian distance between $\sigma(t)$ and $\gamma(t)$ remains bounded for $t \geq 0$.

The asymptote relation between unit speed geodesics is an equivalence relation, and the set of equivalence classes will be denoted (anticipating later developments) $\partial_\infty X$. In what follows it will be more convenient to replace unit-speed geodesics by their restrictions to $[0, \infty]$. These restrictions will be referred to geodesic *rays*. Then two geodesic rays will be equivalent if they remain a bounded distance apart. Every ray has an origin (its value at 0) and an initial direction (the value of its derivative at 0). The set of rays has a topology, the "cone topology". Roughly speaking, two rays are close in the cone topology if their initial points are close and their initial directions are close. For a precise statement see [E]. We give $\partial_\infty X$ the quotient topology. We let $\overline{X}$ be the set which is the disjoint union of X and $\partial_\infty X$. The set $\overline{X}$ has a natural topology, again referred to as the cone topology, in which the induced topology on X is the natural one and the induced topology on $\partial_\infty X$ is the one just described. A sequence of points $\{x_n\}$ in X converges to the class of a ray $\sigma \in \partial_\infty X$, if the Riemannian distance from o to x_n goes to ∞ and the sequence of initial directions of the geodesic segments $\overline{\sigma(0)x_n}$ converges to the initial direction of the ray σ in $T_{\sigma(0)}(X)$. In particular, if σ is a geodesic ray, then $\lim_{t\to\infty} \sigma(t)$ is the class of σ in $\partial_\infty X$. We note that G acts on $\overline{X}$ and on $\partial_\infty X$.

Let S_o be the unit sphere in the tangent space $T_o(X)$. Define the "radial projection to infinity" $\phi : S_o \to \partial_\infty X$ by

$$\phi(u) = \lim_{t\to\infty} \exp tu.$$

We then have

Lemma 4.1. *The map ϕ is a K-equivariant homeomorphism. In particular, each equivalence class of rays contains a unique representative which emanates from o.*

Thus the space of ideal points $\partial_\infty X$ is a sphere, and the space $\overline{X}$ is homeomorphic to a closed ball. The main gain in passing from S_o to $\partial_\infty X$ is that one has a G-action extending the K-action. The G-orbits are compact in the cone topology and are (partial) flag manifolds. In order to relate $G\cdot\xi, \xi \in \partial_\infty X$, to a flag manifold, it suffices to compute the parabolic group G_ξ which stabilizes ξ. The rule for computing G_ξ is given in Proposition 2.17.3 of [E].

Proposition 4.1. *Let $u \in S_o$ be such that $\phi(u) = \xi$. Then*

$$G_\xi = \{g \in G : \lim_{t\to\infty} e^{-tu} g e^{tu} \text{ is finite}\}.$$

This proposition translates into a very simple formula for finding the flag in the case of interest to us, see [E], §2.13.8 and 2.17.27. Let ξ and u be as in the Proposition. So now $u \in \mathcal{H}_{m+1} - \{0\}$. Suppose it has ℓ distinct eigenvalues. Arrange them in decreasing order and define a partial flag $F_\cdot$ by letting $F_i, 1 \le i \le \ell - 1$, be the sum of the first i eigenspaces.

Proposition 4.2. *The flag $F_\cdot$ just described is the flag associated to the boundary point $\xi \in \partial_\infty X$.*

We note that $\mathbb{CP}^m \subset \partial_\infty X$ corresponds to the flags $F_\cdot$ consisting of one proper subspace, a line, and that u has exactly two distinct eigenvalues, the large one with multiplicity 1 and the small one with multiplicity m. Thus by Proposition 4.2, $\phi(u) = L$, where L is the eigenline belonging to the large eigenvalue of u.

4.2. **The characterization of semistability via convex function theory.** ¿From now on, we take X = SL(m+1,$\mathbb{C}$)/SU(m+1), and identify it with $\mathbf{P}$ as above. That is the special case required in this paper; the general X is treated in [LM]. We define the space of projective configurations $\mathcal{C}$ to be the n-fold product $\Pi_1^n \mathbb{CP}^m \subset \Pi_1^n \partial_\infty X$. We assume we are give $\mathbf{r}$ as above. We will define an open subset $\mathcal{C}_{\mathbf{r}}^{sst}$

of $\mathcal{C}$, the set of semistable weighted configurations (thinking of the i-th point as having weight r_i). First, we associate to a configuration $\xi = (\xi_1, \cdots, \xi_n)$ the atomic measure

$$\nu = \sum_1^n r_i \delta_{\xi_i},$$

where δ_{ξ_i} is the atomic measure with a single atom of mass 1 located at ξ_i. Next, we introduce the *Busemann function* $b(x, \xi)$, defined on the product $X \times \partial_\infty X$. If $\xi = \phi(u)$, then

$$b(P, \xi) := \lim_{t \to \infty} (d(P, e^{tu}) - d(\mathbb{I}, e^{tu})).$$

The limit exists, and is a convex Lipschitz function of P. In [LM], the authors define the *weighted Busemann function* b_ν to be the integral over $\partial_\infty X$ with respect to the measure ν. Thus

$$b_\nu(x) = \sum_1^n r_i b(x, \xi_i).$$

Of course, since ν is supported on $\mathbb{CP}^m$, we could just as well define b_ν to be the integral over $\mathbb{CP}^m$.

There is a particularly simple explicit formula for $b(P, \xi)$ for $\xi \in \mathbb{CP}^m$.

Lemma 4.2. *Let $P \in \mathbb{P}$ and $w \in \mathbb{C} - \{0\}$. Then*

$$b(P, w \otimes w^*) = \sqrt{(m+1)/m}\ \ln (\|P^{-1/2} w\|^2 / \|w\|^2).$$

It is proved in [LM] that a weighted configuration is semistable if and only if the weighted Busemann function is bounded below, and nice semistable if and only if the weighted Busemann function has a minimum. One can use Lemma 4.2 to relate stability properties and the strong triangle inequalities.

Remark 1. When the r_i's are positive integers, b_ν is essentially the natural logarithm of the function studied by Kempf and Ness, [KN79] and Ness, [Ne84]. In these papers, P is fixed and the w_i's vary (more precisely the w_i's are coded into a decomposable tensor which varies). Thus, the above results are the analogues for general weights of those of Kempf and Ness.

4.3. **The analytic quotient and its relation to the symplectic quotient.** We now indicate how the theory of [LM] allows one to visualize the relation between the symplectic and analytic quotients as a passage from S_o to the ideal boundary $\partial_\infty X$.

We extend ϕ to a map from n-gons to configurations by $\phi(\mathbf{e}) = \xi$ where, if $e_i = r_i w_i \otimes w_i^*$, then $\xi_i := \phi(w_i \otimes w_i^*)$. We then have ([LM])

Lemma 4.3. *If* $\mathbf{e} \in (\mathcal{H}_{m+1})^n$ *satisfies* $\mu(\mathbf{e}) = 0$*, then* $\phi(\mathbf{e})$ *is a nice semistable configuration.*

We obtain an inclusion $\iota : \widetilde{M_{\mathbf{r}}} \to \mathcal{C}_{\mathbf{r}}^{sst}$. We note that Proposition 4.2 gives the explicit formula

$$\iota(\mathbf{e}) = ([w_1], [w_2], \cdots, [w_n]), \text{where } e_i = r_i w_i \otimes w^*.$$

Here we have used $[w_i]$ to denote the image of the unit vector w_i in $\mathbb{CP}^m$.

The following theorem is then a special case of the general result relating symplectic quotients and analytic quotients proved in [Ki], [Sj95] and [HL94].

Theorem 4.1. *The inclusion* ι *induces a homeomorphism* $\iota : M_{\mathbf{r}} \to \mathcal{M}_{\mathbf{r}}$*. When* $M_{\mathbf{r}}$ *is a smooth manifold, so is* $\mathcal{M}_{\mathbf{r}}$*, and* ι *is a diffeomorphism.*

Thus the moduli space of n-gons $M_{\mathbf{r}}$ always has the structure of a complex analytic space, and when $M_{\mathbf{r}}$ is smooth it has the structure of a complex manifold. In fact, the symplectic structure and the complex structure are compatible, and accordingly when $M_{\mathbf{r}}$ is a smooth manifold, it has the structure of a Kähler manifold.

5. Smoothness of the Moduli Spaces

In this section we give a sufficient condition in terms of $\mathbf{r}$ for the space $M_{\mathbf{r}}$ to be smooth. For $m = 1$, it was shown in [KM96] that $M_{\mathbf{r}}$ will have singularities if, and only if, the index set $\{1, \dots, n\}$ can be partitioned into proper subsets I, J so that

$$\sum_{i \in I} r_i = \sum_{j \in J} r_j. \tag{5.1}$$

There then exists a polygon (in Euclidean space), with the given side lengths, that is contained in a line segment, and such polygons are the singular points of $M_{\mathbf{r}}$. For $m \geq 1$, we adapt (5.1) as follows.

Definition 5.1. For $1 \leq k \leq m$ and $I \cup J$ a proper partition of $\{1, \dots, n\}$, set

$$H_{I,J,k} = \{\mathbf{r} \in \mathbb{R}_+^n \mid k \sum_{i \in I} r_i = (m - k + 1) \sum_{j \in J} r_j\}.$$

The *wall* corresponding to this hyperplane is the intersection

$$W_{I,J,k} = H_{I,J,k} \cap C(n, m + 1)$$

(cf. Definition 3.2).

We will see that if $\mathbf{r}$ does not lie on a wall, then $M_{\mathbf{r}}$ is smooth. To this end, we need the higher-dimensional analog of degenerate polygons.

Let $\mathbf{r} \in W_{I,J,k}$. Suppose $I = \{i_1, \ldots, i_p\}$, $J = \{j_1, \ldots, j_q\}$ (so that $p + q = n$), and take I and J to be ordered, $i_1 < i_2 < \ldots$, $j_1 < j_2 < \ldots$. Set $\mathbf{r}_I = (r_{i_1}, \ldots, r_{i_p})$, and likewise for J. Choose an orthogonal decomposition $\mathbb{C}^{m+1} = V_1 \oplus V_2$ with $\dim V_1 = m - k + 1$ and $\dim V_2 = k$. Let $\mathcal{H}(V_i)$ denote the Hermitean endomorphisms of V_i. We have inclusions $\alpha_i : \mathcal{H}(V_i) \to \mathcal{H}_{m+1}$.

Lemma 5.1. *Write $\rho_I = \sum_I r_i$, $\rho_J = \sum_J r_j$, $\Lambda_I = \rho_I/(m-k+1)$, $\Lambda_J = \rho_J/k$, $\rho = \sum_1^n r_i$, and $\Lambda = \rho/(m+1)$ as usual. Then $\Lambda_I = \Lambda_J = \Lambda$.*

Proof. Because $\mathbf{r} \in W_{I,J,k}$, we have $k\rho_I = (m - k + 1)\rho_J$, which implies $\Lambda_I = \Lambda_J$. Furthermore,

$$k\rho = k\rho_I + k\rho_J = (m - k + 1)\rho_J + k\rho_J = (m + 1)\rho_J,$$

whence $\Lambda = \rho/(m + 1) = \rho_J/k = \Lambda_J$. □

Define a map

$$\iota_{I,J,V_1,V_2} : \widetilde{M}_{\mathbf{r}_I}(\mathcal{H}(V_1)) \times \widetilde{M}_{\mathbf{r}_J}(\mathcal{H}(V_2)) \to \widetilde{N}_{\mathbf{r}}$$

by

$$(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}) \mapsto (\alpha_\varepsilon(e_1^{(\varepsilon)}), \ldots, \alpha_\varepsilon(e_n^{(\varepsilon)})),$$

where in the ℓ^{th} entry, $\varepsilon = 1$, resp. 2, if $\ell \in I$, resp. $\ell \in J$. Lemma 5.1 shows that the image of ι_{I,J,V_1,V_2} in fact lies in $\widetilde{M}_{\mathbf{r}}$, i. e. , consists of closed polygons.

Definition 5.2. We say that $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$ is *decomposable* if it lies in the image of the map ι_{I,J,V_1,V_2} for some choice of I, J, V_1, V_2 as above.

Lemma 5.2. *$\widetilde{M}_{\mathbf{r}}$ contains a decomposable polygon if, and only if,* $\mathbf{r}$ *lies on a wall.*

We now turn to the smoothness of $M_{\mathbf{r}}$. Let $\widetilde{\Sigma}_{\mathbf{r}} \subset \widetilde{M}_{\mathbf{r}}$ be the set of decomposable polygons. We note that $\widetilde{\Sigma}_{\mathbf{r}}$ is invariant under U(m+1), and let $\Sigma_{\mathbf{r}}$ be the image of $\widetilde{\Sigma}_{\mathbf{r}}$ in $M_{\mathbf{r}}$.

Theorem 5.1. *(i) $\widetilde{M}_{\mathbf{r}} - \widetilde{\Sigma}_{\mathbf{r}}$ is a smooth manifold. (ii) The group SU(m+1) acts freely on $\widetilde{M}_{\mathbf{r}} - \widetilde{\Sigma}_{\mathbf{r}}$, hence the quotient $M_{\mathbf{r}} - \Sigma_{\mathbf{r}}$ is a smooth manifold.*

Proof. First an observation. If $w(t)$ is a smooth curve in $\mathbb{C}^{m+1}$, with $\|w(t)\| \equiv 1$, then $\mathrm{Tr} w(t) \otimes w(t)^* \equiv 1$ implies $\mathrm{Tr}\frac{d}{dt}(w(t) \otimes w(t)^*) \equiv 0$. Hence the derivative of the momentum map $\mu : \widetilde{N}_{\mathbf{r}} \to \mathcal{H}_{m+1}$ maps into $\mathcal{H}^0_{m+1}$.

We now prove (i). The following fact is standard.

Lemma 5.3. *Let $Z(e_i)$ be the centralizer of e_i in $\mathcal{H}^0_{m+1}$. Then*

$$d\mu\big|_{\mathbf{e}} : T_{\mathbf{e}}(\widetilde{N}_{\mathbf{r}}) \to \mathcal{H}^0_{m+1} \text{ is not onto } \iff \bigcap_{i=1}^{n} Z(\mathbf{e}_i) \neq \{0\}.$$

Indeed, the differential $d\mu\big|_{\mathbf{e}}$ will be onto if, and only if,

$$T_{e_1}(\mathcal{O}_{r_1}) + \cdots + T_{e_n}(\mathcal{O}_{r_n}) = \mathcal{H}^0_{m+1}.$$

Letting $^{\perp}$ denote orthogonal complement in $\mathcal{H}^0_{m+1}$, we see that $d\mu\big|_{\mathbf{e}}$ is onto if, and only if,

$$T_{e_1}(\mathcal{O}_{r_1})^{\perp} \cap \ldots \cap T_{e_n}(\mathcal{O}_{r_n})^{\perp} = \{0\}.$$

But $T_{e_i}(\mathcal{O}_{r_i})^{\perp} = Z(e_i)$, and the lemma follows.

Suppose now that $d\mu\big|_{\mathbf{e}}$ is not onto. Choose a nonzero $X \in \cap_{i=1}^{n} Z(e_i)$. Suppose that X has ℓ distinct eigenvalues, so that $\mathbb{C}^{m+1}$ is the orthogonal sum of the corresponding eigenspaces W_j. For each $e_i = r_i w_i \otimes w_i^*$, we have $\mathbb{C}w_i = \ker(e_i - r_i\mathbb{I})$. Since X and e_i commute, w_i is also an eigenvector of X. Hence $w_i \in W_{j_i}$ for some j_i. Now set $V_1 = W_1 + \cdots + W_{\ell-1}$, $V_2 = W_\ell$. Define $I = \{i \mid w_i \in V_1\}$, $J = \{j \mid w_j \in V_2\}$. It follows that $\mathbf{e}$ lies in the image of the map ι_{I,J,V_1,V_2}. Thus, if $\mathbf{e}$ is not decomposable, then $\Lambda\mathbb{I}$ is a regular value of $\mu_{\mathbf{r}}$. This proves part (i). For (ii), we need to check that if $\mathbf{e}$ is not decomposable, then the stabilizer of $\mathbf{e}$ under the action of U(m+1) is trivial. The argument just given works, because we deal with matrix groups. If $\kappa\mathbf{e}\kappa^{-1} = \mathbf{e}$, we write $\mathbb{C}^{m+1}$ as sum of eigenspaces of κ, and proceed as before. This completes the proof of Theorem 5.1. □

Corollary 5.1. *If $\mathbf{r}$ does not lie on a wall, then $M_{\mathbf{r}}$ is a smooth manifold.*

We conclude this section by identifying the critical sidelengths of closed polygons. We define the space of closed n-gons (with *arbitrary side-lengths*) by

$$CPol(n, m+1) = \{\mathbf{e} \in (\mathcal{H}_{m+1})^n : \sum_i e_i = \Lambda_{\mathbf{r}}\mathbb{I}\}.$$

Theorem 5.2. *Let $\mathbf{s}$: $\mathrm{CPol}(n, m+1) \to \mathbb{R}^n$ be the side-length map. The set of critical values of $\mathbf{s}$ is the union of the walls.*

Proof. We have seen that $\mathbf{r}$ lies on a wall if, and only if, $\Lambda_{\mathbf{r}}\mathbb{I}$ is a critical value of $\mu : \widetilde{N}_{\mathbf{r}} \to \mathcal{H}_{m+1}$. Let

$$\mathbf{e} \in \widetilde{M}_{\mathbf{r}} = \mathrm{CPol}(n, m+1) \cap \widetilde{N}_{\mathbf{r}}.$$

The result will follow once we prove that $T_{\mathbf{e}}\mu$ is onto if, and only if, $T_{\mathbf{e}}\mathbf{s}$ is onto.

To show this, consider a curve $\mathbf{e}(t) = (r_1(t)w_1(t) \otimes w_1(t)^*, \ldots)$ in $\mathrm{CPol}(n, m+1)$, with $\mathbf{e}(0) = \mathbf{e}$. We have $\dot{e}_i(0) = \beta_i w_i \otimes w_i^* + X_i$, where $\beta_i = \dot{r}_i(0)$ and, as noted above,

$$X_i = r_i(w_i \otimes \dot{w}_i(0)^* + \dot{w}_i(0) \otimes w_i^*) \in \mathcal{H}^0_{m+1}.$$

Moreover, since $\mathbf{e}$ is closed,

$$\sum_{i=1}^{n} \beta_i w_i \otimes w_i^* - \frac{1}{m+1} \sum_{i=1}^{n} \beta_i \mathbb{I} = -\sum_{i=1}^{n} X_i. \tag{5.2}$$

Now $T_{\mathbf{e}}\mathbf{s}$ is surjective exactly when for every $\boldsymbol{\beta} \in \mathbb{R}^n$ there exist X_i for which (5.2) holds. The left side of (5.2) runs over all of $\mathcal{H}^0_{m+1}$, hence so must the right side. This happens precisely when $T_{\mathbf{e}}\mu$ is surjective. Indeed, a curve $\mathbf{e}(t)$ in $\widetilde{N}_{\mathbf{r}}$ has $r_i \equiv$ constant, or $\boldsymbol{\beta} = \mathbf{0}$, and the tangent map is just $T_{\mathbf{e}}\mu(X_1, \ldots, X_n) = \sum_{i=1}^n X_i$. □

6. Bending Hamiltonians

Kapovich and Millson ([KM96]) studied an integrable Hamiltonian system on $\widetilde{M}_{\mathbf{r}}$ in the case $\mathfrak{su}(2)$, which corresponds to Euclidean space $\mathbb{E}^3$. To describe their system and its rank-one generalization, we fix some notation. Taking $m = 1$, we have $e_i = r_i w_i \otimes w_i^* \in \mathcal{H}_2, i = 1, \ldots, n$. In §1 we introduced the *diagonals* $A_0 = e_1$ and $A_i = e_1 + \cdots + e_{i+1}, i = 1, \ldots, n-2$. Note that for a closed polygon, $A_{n-1} = \Lambda\mathbb{I}$, which is indicated by a dashed line in Figure 1 (which would be absent in $\mathfrak{su}(m+1)$).

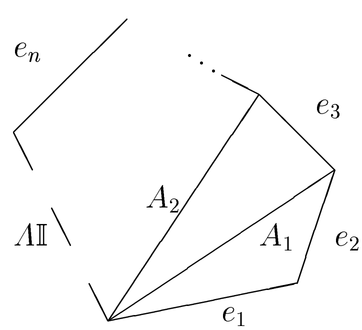

Figure 1. A polygon in $\mathfrak{u}(m+1)$

It was shown in [KM96] that (for $\mathfrak{su}(2)$) the functions $f_i(\mathbf{e}) = \|A_i\|$ Poisson commute. The diagonal A_i divides the polygon into two "flaps", and the flow generated by f_i is 2π-periodic, consisting of a rigid rotation of one flap about the diagonal. In this case, $\|A_i\|$ is the positive eigenvalue of A_i. The analogs of "bending Hamiltonians" for $m > 1$ are again the eigenvalues of the diagonals.

Notation 6.1. The eigenvalues of A_i are denoted by λ_{ij} in decreasing order, $\lambda_{i1} \geq \ldots \geq \lambda_{i,m+1}$.

We note that $A_{n-2} = \Lambda\mathbb{I} - e_n$, which has eigenvalues Λ (multiplicity m) and $\Lambda - r_n$, and those are fixed. Thus only the λ_{ij} for $1 \leq i \leq n-3$ are of possible interest. Furthermore, it will be seen in §7 that off submanifolds of $\widetilde{M}_{\mathbf{r}}$ of lower dimension, the nontrivial λ_{ij} (those not identically 0 or Λ) are simple. In that case, they will be smooth functions of $\mathbf{e}$, which is assumed throughout the present section.

6.1. **Bending Flows.** We want to calculate the Hamiltonian vector fields and flows generated by the λ_{ij}. By analogy with the case of $\mathbb{E}^3$, we call them "bending flows".

On a product of orbits, the Poisson bracket is the sum of the orbit brackets, and the next formula is evident from (2.2):

Proposition 6.1. *Suppose $f : \widetilde{M}_{\mathbf{r}} \to \mathbb{C}$ is smooth and depends only on $e_1, \ldots, e_{i+1}$. Then the Hamiltonian system generated by f is*

$$\dot{e}_k = \begin{cases} \sqrt{-1}\,[\nabla_k f(e_1, \ldots, e_{i+1}), e_k], & \text{if } 1 \leq k \leq i+1, \\ 0, & \text{if } i+1 < k \leq n, \end{cases} \tag{6.1}$$

where ∇_k denotes gradient with respect to e_k, all other e_j being held fixed.

To solve these equations when $f = \lambda_{ij}$, we need a standard lemma from perturbation theory.

Lemma 6.1. *Let λ be an isolated eigenvalue of $A \in \mathcal{H}_{m+1}$, with unit eigenvector u. Then $\nabla\lambda(A) = u \otimes u^*$.*

Proof. For A' sufficiently close to A, the eigenvalue $\lambda(A')$ and (with proper choice of phase) normalized eigenvector $u(A')$ vary analytically in a neighborhood of λ, u. Take a curve $A(t)u(t) = \lambda(t)u(t)$, and take the inner product with the unit length $u(t)$ to get $\lambda(t) = (A(t)u(t), u(t))$. Differentiate and set $t = 0$, and use $(Au, \dot{u}(0)) + (A\dot{u}(0), u) = \lambda((u, \dot{u}(0)) + (\dot{u}(0), u)) = 0$, resulting in

$$\dot{\lambda}(0) = (\dot{A}(0)u, u) = \mathrm{Tr}(\dot{A}(0)u \otimes u^*),$$

as was to be shown. ∎

We write $E_j(A)$ for the spectral projection onto the λ_j eigenspace of A; the lemma thus states that $\nabla\lambda_j(A) = E_j(A)$, and in particular $\nabla\lambda_{ij}(\mathbf{e}) = E_j(A_i)$.

Proposition 6.2. *For $i = 1, \ldots, n-3$ and $j = 1, \ldots, m+1$, the function $\lambda_{ij} : \mathbf{e} \mapsto \lambda_j(A_i)$ is smooth wherever that eigenvalue is simple. It is the Hamiltonian for the system*

$$\dot{e}_k = \begin{cases} \sqrt{-1}\,[E_j(e_1 + \cdots + e_{i+1}), e_k], & \text{if } 1 \le k \le i+1, \\ 0, & \text{if } i+1 < k \le n. \end{cases} \tag{6.2}$$

The Hamiltonian flow $\phi^t_{ij}(\mathbf{e}) = \mathbf{e}(t)$ is given by

$$e_k(t) = \begin{cases} \big(Ad\exp(\sqrt{-1}\,tE_j(A_i))\big)(e_k), & \text{if } 1 \le k \le i+1, \\ e_k, & \text{if } i+1 < k \le n. \end{cases} \tag{6.3}$$

Proof. To obtain the system (6.2) we wish to apply Proposition 6.1. It is necessary to relate the partial gradients $\nabla_k\lambda_{ij}$ to the full gradient, $\nabla\lambda_{ij} = E_j(A_i)$. According to Lemma 6.1, the former are found by computing

$$\dot{A}_i(t) = (e_1 + \cdots + e_k(t) + \cdots + e_i)^{\cdot} = \dot{e}_k(t),$$

but because $\dot{A}_i(0)$ is tangent to $\mathcal{O}_{r_k}$, this only determines $\nabla\lambda_{ij}$ up to a vector normal to the orbit:

$$\nabla_k\lambda_{ij}(A_i) = E_j(A_i) + Y_k, \quad Y_k \in N_{e_k}\mathcal{O}_{r_k}.$$

Then, since $[Y_k, e_k] = 0$, we have $[\nabla_k\lambda_{ij}, e_k] = [E_j(A_i), e_k]$, and (6.2) follows.

Next, add the equations (6.2) for $1 \le k \le i+1$ to find

$$\dot{A}_i(t) = \sqrt{-1}\,[E_j(A_i(t)), A_i(t)].$$

Since A_i commutes with its own spectral projections, we get $\dot{A}_i(t) = 0$ and $A_i(t) = A_i$. With constant A_i, the solution of (6.2) is immediate. □

Corollary 6.1. *The flows ϕ_{ij} have period 2π in t.*

Proof. If P is a projection, then $P^2 = P$. Consequently, $\exp(\sqrt{-1}\,tP) = \mathbb{I} + (\exp(\sqrt{-1}\,t) - 1)P$, which has period 2π. □

6.2. **Involutivity.** It is not *a priori* clear from the formulas for ϕ_{ij} that these flows commute. This is a short calculation; we again work only with simple eigenvalues of the A_i.

Proposition 6.3. $\{\lambda_{ij}, \lambda_{k\ell}\} = 0$ *for* $1 \le i, k \le n-3$ *and* $1 \le j, \ell \le m+1$.

Proof. By Proposition 6.1 and the proof of Proposition 6.2,

$$\{\lambda_{ij}, \lambda_{k\ell}\}(\mathbf{e}) = \sqrt{-1} \sum_{s=1}^{i+1} \mathrm{Tr}\Big(e_s[E_j(A_i) + Y_s, E_\ell(A_k) + Z_s] \Big),$$

where Y_s, Z_s are normal to $\mathcal{O}_{r_s}$. The ad-invariance of the trace bilinear form shows that both Y_s and Z_s are annihilated by e_s. This leaves

$$\begin{aligned} \{\lambda_{ij}, \lambda_{k\ell}\}(\mathbf{e}) &= \sqrt{-1} \sum_{s=1}^{i+1} \mathrm{Tr}\Big([e_s, E_j(A_i)] E_\ell(A_k) \Big) \\ &= \sqrt{-1}\, \mathrm{Tr}\Big([A_i, E_j(A_i)] E_\ell(A_k) \Big) \\ &= 0. \end{aligned}$$

□

Remark. The proof works more generally, if instead of A_i and A_k one has $\sum_I e_i$ and $\sum_J e_j$, with $I \subset J$. Thus, for example, the eigenvalues of $e_2 + e_3$ and $e_1 + \cdots + e_5$ are in involution. On the other hand, if λ, μ are eigenvalues of $e_1 + e_2$ and $e_2 + e_3$, respectively, then

$$\{\lambda, \mu\}(\mathbf{e}) = \sqrt{-1}\, \mathrm{Tr}\Big(e_2[E_\lambda(e_1 + e_2), E_\mu(e_2 + e_3)] \Big),$$

which need not be zero. See [KM01] for more information.

7. The Weinstein-Aronszajn Formula

The diagonal A_i is a rank-one perturbation of A_{i-1}, and because of this, the eigenvalues λ_{ij} and $\lambda_{i-1,j}$ are related in a special way. This connection is the simplest instance of the Weinstein-Aronszajn formula [Kato, Ch.4, §6]. We describe the formula and two consequences that will be used later.

Let A be an $(m+1) \times (m+1)$ Hermitean matrix with eigenvalues $\lambda_1, \ldots, \lambda_{m+1}$ and let $u_1, \ldots, u_{m+1}$ be corresponding orthonormal eigenvectors. (If an eigenvalue has multiplicity > 1, which is now permitted, the choice of its eigenvectors is irrelevant). Let $w \in \mathbb{C}^{m+1}$ be a unit vector, and let $r \in \mathbb{R}$. Set $L = A + rw \otimes w^*$, and call its eigenvalues $\nu_1, \ldots, \nu_{m+1}$. Finally, define $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{C}$ by $w = \sum_{j=1}^{m+1} \alpha_j u_j$.

Proposition 7.1.

$$\frac{\det(z\mathbb{I} - L)}{\det(z\mathbb{I} - A)} = 1 - r \sum_{j=1}^{m+1} \frac{|\alpha_j|^2}{z - \lambda_j}. \tag{7.1}$$

.

Proof. Write $R_z = (z\mathbb{I} - A)^{-1}$ for the resolvent of A. The left side of (7.1) is

$$\begin{aligned} &= \det\left((z\mathbb{I} - A)^{-1}(z\mathbb{I} - A - rw \otimes w^*)\right) \\ &= \det(\mathbb{I} - R_z(rw \otimes w^*)) \\ &= \det(\mathbb{I} - r(R_z w) \otimes w^*). \end{aligned}$$

Now, $\det(\zeta\mathbb{I} - r(R_z w) \otimes w^*)$ is the characteristic polynomial of a rank-one matrix, and so has an m-fold root at $\zeta = 0$ and a simple root at $\zeta = r(R_z w, w)$. Setting $\zeta = 1$ we get

$$\det(\mathbb{I} - r(R_z w) \otimes w^*) = 1 - r(R_z w, w). \tag{7.2}$$

The lemma now follows by expanding w in (7.2) in the basis u_j. □

It is convenient to write (7.1) more explicitly:

$$\frac{(z-\nu_1)\dots(z-\nu_{m+1})}{(z-\lambda_1)\dots(z-\lambda_{m+1})} = 1 - r\sum_{j=1}^{m+1} \frac{|\alpha_j|^2}{z-\lambda_j}. \tag{7.3}$$

Corollary 7.1. *The $|\alpha_j|^2$ are rational functions of $\nu_k, \lambda_\ell, 1 \le k, \ell \le m+1$.*

Finally, we show that the eigenvalues of A and L interlace. This will play a basic role below.

Proposition 7.2. *If $r > 0$, then $\nu_1 \ge \lambda_1 \ge \nu_2 \dots \ge \nu_{m+1} \ge \lambda_{m+1}$. If $r < 0$, we have $\lambda_1 \ge \nu_1 \dots$ instead.*

Proof. Suppose $r > 0$. It suffices to prove the proposition for a dense set of w, so that we may assume $|\alpha_j|^2 > 0$ for all j. Let $R(z)$ be the rational function on the right side of (7.3). Since $\lim_{z\to\infty} R(z) = 1$ and $\lim_{z\downarrow\lambda_1} = -\infty$, R has a zero in (λ_1, ∞). Likewise, because $\lim_{z\uparrow\lambda_j} = +\infty$ and $\lim_{z\downarrow\lambda_{j+1}} = -\infty$, R has a zero in $(\lambda_{j+1}, \lambda_j)$. This provides $m+1$ zeros of R, which must coincide with the zeros ν_j of the left side of (7.3). □

8. A complete set of bending flows

The eigenvalues $\lambda_{ij}(\mathbf{e})$ have been shown to Poisson commute, and to generate 2π-periodic flows. If there were $\frac{1}{2}\dim M_{\mathbf{r}}$ of them and if they were smooth, they would constitute a set of action variables on $M_{\mathbf{r}}$. Smoothness cannot be achieved, but there are $\frac{1}{2}\dim M_{\mathbf{r}}$ that are smooth and functionally independent on a dense open submanifold of $M_{\mathbf{r}}$. This section presents the proof.

We will arrange the eigenvalues of $A_0 = e_1, A_1, \dots, A_{n-1}$ in a triangle with vertex at the bottom. The eigenvalues of A_k are written in row k

of the triangle, along with some space-filling zeros. For $0 \le k \le m$, the rank of A_k is at most $k+1$, so zero must be at least an $(m-k)$-fold eigenvalue of A_k. Those zeros are not recorded. When $k > m$, there are $m+1$ eigenvalues, potentially nonzero; these are recorded *along with* $k - m$ *zeros*. Figure 2 shows the case $m = 2, n = 6$. Note that entries of successive rows are offset to reflect the interlacing property deduced in Proposition 7.2. This diagram is called a *Gel'fand-Tsetlin pattern*, or *GTs pattern* for short. It is denoted by $\Gamma(\mathbf{e})$. The extra zeros will be explained §11, see Remark 5.

$$\begin{array}{ccccccccccc}
\Lambda & & \Lambda & & \Lambda & & 0 & & 0 & & 0 \\
& d_1 & & d_2 & & d_3 & & 0 & & 0 & \\
& & c_1 & & c_2 & & c_3 & & 0 & & \\
& & & b_1 & & b_2 & & b_3 & & & \\
& & & & a_1 & & a_2 & & & & \\
& & & & & r_1 & & & & &
\end{array}$$

Figure 2

Since $\mathbf{e} \in \widetilde{M_{\mathbf{r}}}$, there are additional restrictions on the entries of $\Gamma(\mathbf{e})$. Row $n-1$ must consist of $m+1$ Λ's (because $e_1 + \cdots + e_n = \Lambda \mathbb{I}$) and $(n-m-1)$ zeros. The interlacing property forces the first m entries of row $n-2$ to be Λ, so in Figure 2, $d_1 = d_2 = \Lambda$. Likewise, $c_1 = \Lambda$. It becomes apparent that the extra zeros remind one that (for example) the eigenvalues $d_3 = \lambda_{4,3}$ and $c_3 = \lambda_{3,3}$ must be non-negative.

Moreover,

$$\text{Tr} A_k = \text{Tr}(e_1 + \cdots + e_{k+1}) = r_1 + \cdots + r_{k+1}, \tag{8.1}$$

which is a linear constraint on the rows of $\Gamma(\mathbf{e})$. In Figure 2, that leaves c_2, b_1, b_2, a_1 as potentially independent commuting Hamiltonians, and indeed $\dim_{\mathbb{R}} M_{\mathbf{r}} = 8$ in this case.

We summarize this discussion.

Definition 8.1. Let $m, n, \mathbf{r}$ be fixed. We write $\mathbf{P}$ for the convex polytope of GTs patterns satisfying the following conditions.

(1) There are n rows numbered $0, \ldots, n-1$ (starting at the bottom);
(2) Row $n-1$ consists of $m+1$ Λ's and $n-m-1$ zeros;
(3) The sum of the entries of row k is $\sum_{i=0}^{k} r_{i+1}$.
(4) The interlacing property $\lambda_{ij} \ge \lambda_{i-1,j} \ge \lambda_{i,j+1}$ holds.

Proposition 8.1. $\dim \mathbf{P} = (n-m-2)m = \frac{1}{2} \dim_{\mathbb{R}} M_{\mathbf{r}}$.

Proof. There are two cases: (1) $n \ge 2(m+1)$ and (2) $n \le 2m+1$. The difference comes from the position of row m, corresponding to the eigenvalues of $A_m = e_1 + \cdots + e_{m+1}$. Generically, this matrix will have full rank. In case (2), some of its eigenvalues are forced, by interlacing, to be Λ. In case (1), all the automatic Λ's have been "exhausted". (Figure 2 falls into the latter category). Let us sketch the counting.

Case (1): Unconstrained λ_{ij} can appear in rows $i = 1, \ldots, n-3$. Break this index set into three parts: $S_1 = \{1, \ldots, m\}$, $S_2 = \{m+1, \ldots, n-m-2\}$, $S_3 = \{n-m-1, \ldots, n-3\}$. If $n = 2(m+1)$ (as in Figure 2), then $S_2 = \emptyset$. The numbers of unconstrained λ_{ij} for the corresponding A_k are

- In S_1, $1, \ldots, m$;
- in S_2, $m, \ldots, m$;
- in S_3, $m-1, \ldots, 1$.

Adding, we obtain

$$\frac{m(m+1)}{2} + (n - (2(m+1))m + \frac{m(m-1)}{2} = (n-m-2)m.$$

Case (2): We set $S_1 = \{1, \ldots, n-m-2\}$, $S_2 = \{n-m-1, \ldots, m\}$, $S_3 = \{m+1, \ldots, n-3\}$ (if $m = 1, 2$, then $S_3 = \emptyset$). The numbers of unconstrained λ_{ij} are:

- In S_1, $1, \ldots, n-m-2$;
- in S_2, $n-m-2, \ldots, n-m-2$;
- in S_3, $n-m-3, \ldots, 1$.

Now add. □

9. Constructing a polygon with given GTs pattern

In the last section, we saw that $\Gamma(M_{\mathbf{r}}) \subset \mathbf{P}$. We now prove the converse.

Theorem 9.1. *(i)* $\Gamma(M_{\mathbf{r}}) = \mathbf{P}$. *(ii) There are* $\frac{1}{2} \dim M_{\mathbf{r}}$ *functionally independent* λ_{ij}*'s.*

Proof. Let $\mathcal{S}_{m+1} \subset \mathcal{H}_{m+1}$ denote the space of real symmetric matrices, and let $\widetilde{M}_{\mathbf{r}}(\mathcal{S}_{m+1})$ be the set of polygons in $\widetilde{M}_{\mathbf{r}}$ with each $e_i \in \mathcal{S}_{m+1}$. The obvious inclusion $\widetilde{M}_{\mathbf{r}}(\mathcal{S}_{m+1}) \hookrightarrow \widetilde{M}_{\mathbf{r}}$ is the analog of the inclusion $\widetilde{M}_{\mathbf{r}}(\mathbb{R}^2) \hookrightarrow \widetilde{M}_{\mathbf{r}}(\mathbb{R}^3)$ used in [KM96]. We will see later that elements of $\mathcal{S}_{m+1}(\widetilde{M}_{\mathbf{r}})$ can be thought of as "unbent" polygons; these will be important in our proof of the involutivity of the angle variables in the next section. We now show that

$$\Gamma \widetilde{M}_{\mathbf{r}}(\mathcal{S}_{m+1}) = \mathbf{P}. \tag{9.1}$$

Since $\Gamma : \widetilde{M}_{\mathbf{r}} \to \mathbf{P}$ is continuous (though not differentiable), the image of Γ is closed, and it suffices to prove that the image of Γ contains the interior $\mathbf{P}^o$ of $\mathbf{P}$. Thus, choose a GTs pattern γ in which all unconstrained inequalities are strict; we are to find $\mathbf{e}$ such that $\Gamma(\mathbf{e}) = \gamma$.

Set $A_0 = r_1 w_1 \otimes w_1^*$, where w_1 is an arbitrary real unit vector. Assuming that a real symmetric A_{k-1} with a given spectrum has been found, we want $w_{k+1} \in \mathbb{R}^{m+1}$ so that

$$A_k = A_{k-1} + r_{k+1} w_{k+1} \otimes w_{k+1}^* \tag{9.2}$$

has the required next spectrum.

We carry out the induction step for Case (1), in the terminology of Proposition 8.1. First, let $k \in S_1$. Thus

$$A_{k-1} = \sum_{j=1}^{k} r_j w_j \otimes w_j^*,$$

it has spectrum $\{\lambda_1, \ldots, \lambda_k, 0, \ldots, 0\}$ with $\lambda_1 > \cdots > \lambda_k > 0$, and $\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} r_i$. We are further given ν_i with

$$\nu_1 > \lambda_1 > \nu_2 > \cdots > \lambda_k > v_{k+1} > 0,$$

and $\sum_{i=1}^{k+1} \nu_i = \sum_{i=1}^{k+1} r_i$.

Let $u_1, \ldots, u_k, u$ be normalized (real) eigenvectors of A_{k-1} corresponding to $\lambda_1, \ldots, \lambda_k, 0$, and seek w_{k+1} in the form

$$w_{k+1} = \sum_{j=1}^{k} \alpha_j u_j + \alpha u.$$

Now solve for $\alpha_j^2, 1 \le j \le k$ and α^2 in equation (7.3), which takes the special form

$$\frac{(z-\nu_1)\ldots(z-\nu_{k+1})z^{m-k}}{(z-\lambda_1)\ldots(z-\lambda_k)z^{m-k+1}} = 1 - r_{k+1}\left(\sum_{j=1}^{k} \frac{|\alpha_j|^2}{z-\lambda_j} + \frac{\alpha^2}{z}\right).$$

Taking traces in equation (9.2), we get

$$\sum_{j=1}^{k+1} r_j = \sum_{j=1}^{k+1} \nu_j = \sum_{j=1}^{k} r_j + r_{k+1}\|w_{k+1}\|^2,$$

whence $\|w_{k+1}\| = 1$.

The same procedure works in the remaining subcases as well; for $k \in S_2$ the eigenvalues λ_j and ν_j are simple, while for $k \in S_3$, account must be taken of the multiplicity of Λ.

□

Remark 2. The proof shows that, if w_{k+1} is not required to be real, each term $\alpha_j u_j$ is determined only up to a multiple $\exp(\sqrt{-1}\,\theta_{k+1,j})$. Thus, the possible polygons $\mathbf{e}$ corresponding to a given pattern γ lie on a torus. The angle coordinates are studied in the next section.

We conclude by making a choice of functionally independent action variables.

Definition 9.1. Let $\mathcal{I}$ be the set of pairs (i,j) satisfying $1 \leq i \leq n-1, 1 \leq j \leq i$ which index eigenvalues λ_{ij} such that λ_{ij} is not forced to be 0 or Λ, with the further property that $\lambda_{i,j+1}$ is not forced to be 0 (this last condition says that in each row we throw away the right-most j such that λ_{ij} is not forced to be 0).

Corollary 9.1. *The set $\mathcal{I}$ indexes a functionally independent set of action variables λ_{ij}.*

Proof. Indeed, these action variables map onto a polyhedron of dimension equal to the cardinality of $\mathcal{I}$. □

10. Angle Variables and Four-Point Functions

In this section, we construct angle variables θ_{ij} conjugate to the action variables λ_{ij} discussed thus far. The angles are implicit in Corollary 6.1 and Remark 2; what we now find is a global description.

10.1. **Four point functions and polygons.** The geometric picture in [KM96] serves as model. For the moment, think of the sides e_j as vectors in $\mathbb{R}^3$. The action variables are the lengths of the diagonals $A_i = e_1 + \ldots + e_{i+1}$ of the polygon. The corresponding conjugate angle is the *oriented* dihedral angle between the two triangles spanned, respectively, by A_{i-1}, e_{i+1}, A_i and A_i, e_{i+2}, A_{i+1}. By this we mean the oriented angle between the two normal vectors to the triangles. These two vectors are elements of the plane orthogonal to A_i. We orient this plane so that a positively oriented basis for the plane followed by A_i is a positively oriented basis for $\mathbb{R}^3$.

Remark 3. In an oriented plane Π equipped with a positive definite inner product $u \cdot v$ we can define the oriented angle $\angle(u,v)$ for a pair of vectors u and v in Π as follows. First we say that two unit vectors make an angle of ninety degrees if $u \cdot v = 0$ and the basis $\{u,v\}$ is positively oriented. We let J be the operation of rotation by ninety degrees. We make Π into a complex vector space by defining $\imath v := Jv$. Then the unit circle in $\mathbb{C}$ acts simply-transitively on the oriented lines in Π. We define $\angle(u,v) = \theta$ if $\exp(\imath\theta)u$ is a positive real multiple of v. If $\theta = \angle(u,v)$ then we have

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$
$$\sin\theta = \frac{Ju \cdot v}{\|u\|\|v\|}$$

For the case at hand, the oriented angle θ_i is given by

$$\cos\theta_i = \frac{(A_i \times e_{i+1}) \cdot (A_i \times e_{i+2})}{\|A_i \times e_{i+1}\| \, \|A_i \times e_{i+2}\|} \tag{10.1}$$

$$\sin\theta_i = \frac{(A_i \times e_{i+1}) \times (A_i \times e_{i+2}) \cdot A_i}{\|A_i \times e_{i+1}\| \, \|A_i \times e_{i+2}\| \, \|A_i\|}. \tag{10.2}$$

Note that $\theta_i = 0$ when the triangles are coplanar, so that the collection of planar polygons forms a reference cross-section for the angle variables.

We now transfer (10.1) and (10.2) back to our Lie algebra $\mathcal{H}_2^0$ of tracefree Hermitean 2×2 matrices. Define $f : \mathbb{R}^3 \to \mathcal{H}_2^0$ by

$$f : \mathbf{x} = (x_1, x_2, x_3) \mapsto \hat{\mathbf{x}} = \frac{1}{2}\begin{pmatrix} x_1 & x_2 + \sqrt{-1}\, x_3 \\ x_2 - \sqrt{-1}\, x_3 & -x_1 \end{pmatrix}. \tag{10.3}$$

Then $\widehat{\mathbf{x} \times \mathbf{y}} = \sqrt{-1}\,[\hat{\mathbf{x}}, \hat{\mathbf{y}}]$, $\mathbf{x} \cdot \mathbf{y} = 2\mathrm{Tr}\hat{\mathbf{x}}\hat{\mathbf{y}}$, and a vector in the $x_3 = 0$ plane corresponds to a real symmetric matrix. (Thus, a planar polygon is represented by a symmetric matrix, cf. Theorem 9.1).

We return to identifying vectors with matrices via (10.3).
Let $\lambda > 0$ and $-\lambda$ be the eigenvalues of A_i, with orthonormal eigenvectors u, v, so that $A_i = \lambda(u \otimes u^* - v \otimes v^*)$. Write, for notational simplicity,

$$e_{i+1} = r_1 w_1 \otimes w_1^* - (r_1/2)\mathbb{I}, e_{i+2} = r_2 w_2 \otimes w_2^* - (r_2/2)\mathbb{I}.$$

Then the numerator of (10.1) becomes (since $\mathbb{I}$ does not contribute)

$$2\mathrm{Tr}\big(\sqrt{-1}\,[A_i, r_1 w_1 \otimes w_1^*]\sqrt{-1}\,[A_i, r_2 w_2 \otimes w_2^*]\big), \tag{10.4}$$

and the numerator of (10.2) becomes

$$2\|A_i\|\mathrm{Tr}\big(\sqrt{-1}\, A_i[r_1 w_1 \otimes w_1^*, r_2 w_2 \otimes w_2^*]\big). \tag{10.5}$$

Definition 10.1. Let $a, b, c, d \in \mathbb{C}^{m+1}$. Define the *four-point function* by

$$F_4(a, b, c, d) = \frac{(a, b)(b, c)(c, d)(d, a)}{\|a\|^2\|b\|^2\|c\|^2\|d\|^2}$$

where $(\cdot, \cdot)$ is the usual Hermitean inner product.

Two properties of F_4 are important:

1. (1) $F_4(a, b, c, d)$ may be thought of as function on $(\mathbb{CP}^m)^4$; in particular, F_4 is independent of the phases of its arguments.
2. (2) $\overline{F_4(a, b, c, d)} = F_4(a, d, c, b)$ (plus other such symmetries).

A longish calculation, using property (2), gives the following.

Proposition 10.1. *Expression (10.4) reduces to*

$$16\lambda^2 r_1 r_2 Re F_4(w_1, u, w_2, v).$$

Expression (10.5) reduces to

$$16\lambda^2 r_1 r_2 Im F_4(w_1, u, w_2, v).$$

The denominator in (10.1) and (10.2) becomes

$$16\lambda^2 r_1 r_2 |F_4(w_1, u, w_2, v)|.$$

Thus, the oriented dihedral angle is $\theta = \arg F_4(w_1, u, w_2, v)$*.*

This formula, suitably adapted, will be shown to define the conjugate angles in the more general case as well.

We mention, as an aside, that the argument of the four-point function has an interesting geometric description.

Theorem 10.1. *Let* $a_j, j = 1, \dots, 4$ *be four points in* $\mathbb{C}^{m+1}$ *defining points* $p_j \in \mathbb{CP}^m$*. Construct a geodesic quadrilateral* π *in* $\mathbb{CP}^m$ *with vertices at the* p_j*. Let* σ *be a two-chain with boundary* π *and let* ω *be the Kähler form on* $\mathbb{CP}^m$*. Then*

$$\arg F_4(a_1, a_2, a_3, a_4) = -\int_\sigma \omega. \tag{10.6}$$

Proof. Draw a geodesic segment (a diagonal of the quadrilateral) from p_1 to p_3. The analogue of (10.6) for triangles was proved in [HM], see also [Go, Ch. 7]. Now choose σ to be the union of two two-chains each of which has as boundary one of the two triangles created by drawing the diagonal $p_1 p_3$. Combining (10.6) for the triangles gives the equation for the quadrilateral. □

10.2. **Construction of angle variables.** We will define the angle variables as in Proposition 10.1, via the four-point function of the w's associated with two consecutive edges and eigenvectors of the diagonal between them. These vectors all involve a choice of phase, and the first goal will be to remove the ambiguity.

Let $M_{\mathbf{r}}^0$ be the open subset of $M_{\mathbf{r}}$ on which the interlacing inequalities $\lambda_{ij} > \lambda_{i-1,j} > \lambda_{i,j+1}$ are strict, and let $\widetilde{M}_{\mathbf{r}}^0$ be its inverse image in $\widetilde{M}_{\mathbf{r}}$. We consider only polygons in $M_{\mathbf{r}}^0$, so that the (unconstrained) eigenvalues and eigenvectors may be taken to be locally smooth functions of $\mathbf{e}$.

Let ϕ^t be one of the λ_{ik}-flows defined in Proposition 6.2. We will follow the transformed n-gon $\phi^t(\mathbf{e})$. Its ℓ-th edge, $r_\ell w_\ell^t \otimes (w_\ell^t)^*$, and the normalized λ_{ij}-eigenvector, u_{ij}^t, of the diagonal $\phi^t(A_i)$, will depend on time t. They may be taken to be locally smooth on $M^0_{\mathbf{r}}$, but will depend on an initial choice, while the n-gon $\phi^t(\mathbf{e})$ itself is well defined.

Definition 10.2. Make smooth local choices of w_ℓ and u_{ij}. Define

$\alpha_{ij} : \widetilde{M}^0_{\mathbf{r}} \to \mathbb{C}, (i,j) \in \mathcal{I},$ by $\alpha_{ij} : \mathbf{e} \mapsto (w_{i+1}(\mathbf{e}), u_{ij}(\mathbf{e}))(u_{ij}(\mathbf{e}), w_{i+2}(\mathbf{e}))$; this depends on the phases of w_{i+1}, w_{i+2}. (We will usually drop the argument $\mathbf{e}$). Set

$$\beta_{ij} = F_4(w_{i+1}, u_{ij}, w_{i+2}, u_{i,j+1}) = \alpha_{ij}\overline{\alpha_{i,j+1}}.$$

The β_{ij} are *independent* of all phase choices. Finally, we define the angle variables $\theta_{ij}, (i,j) \in \mathcal{I}$, by

$$\theta_{ij} = \arg \beta_{ij}.$$

Clearly the number of four-point functions β_{ij} is the same as the number of independent, unconstrained λ_{ij}'s, since for every i there is one more λ_{ij} than β_{ij} and there are no β_{ij}'s corresponding to the eigenvalues 0 and Λ. Thus we obtain the correct formal count of angle variables. We now prove that the angle variables are well-defined on $M^0_{\mathbf{r}}$.

Lemma 10.1.

(1) *All* $|\alpha_{ij}|^2$ *are constant under all bending flows* $\phi_{k\ell}$.
(2) *All* $|\alpha_{ij}|^2$ *are nonzero on* $M^0_{\mathbf{r}}$.

Proof. The first statement follows from Proposition 7.1 and Corollary 7.1. Indeed,

$$A_{i-1} = A_i - r_{i+1} w_{i+1} \otimes w^*_{i+1}.$$

Hence $|(w_{i+1}, u_{ij})|^2$, being a rational function of action variables, is a constant of motion. Likewise,

$$A_{i+1} = A_i + r_{i+2} w_{i+2} \otimes w^*_{i+2}$$

implies that $|(w_{i+2}, u_{ij})|^2$ is a constant of motion.

To prove the second statement we apply the Weinstein-Aronszajn formula to obtain

$$\frac{\det(z\mathbb{I} - A_{i-1})}{\det(z\mathbb{I} - A_i)} = 1 + r_{i+1} \sum_{j=1}^{m+1} \frac{|(w_{i+1}, u_{ij})|^2}{z - \lambda_{ij}}. \tag{10.7}$$

Hence if $|\alpha_{ij}| = |(w_{i+1}, u_{ij})| = 0$, then λ_{ij} is not a pole, so the $(z - \lambda_{ij})$ in the denominator of the left-hand side must cancel with one of the terms in the numerator. Hence one of the interlacing inequalities between the i-th and $(i-1)$-st rows is not strict, contradicting the assumption that $\mathbf{e} \in M^0_{\mathbf{r}}$. Similarly, $(w_{i+2}, u_{ij}) \neq 0$.

□

In the following we will make essential use of

Remark 4. Let $g \in \mathrm{U(m+1)}$ and consider the conjugated polygon $g\mathbf{e}g^{-1}$. Its ℓ-th edge is $r_\ell(gw_\ell)\otimes(gw_\ell)^*$. However, the choice $w_\ell(g\mathbf{e}g^{-1})$ made in Definition 10.2 may not coincide with gw_ℓ. If they differ, it is by a multiple of modulus one. The four-point function β_{ij} is not affected by such a factor. In calculations involving β_{ij}, we may therefore replace $w_\ell(g\mathbf{e}g^{-1})$ by gw_ℓ, and for the same reason, $u_{ij}(g\mathbf{e}g^{-1})$ by gu_{ij}.

We will now compute the Poisson brackets of the action variables with the angle variables.

Lemma 10.2.

$$\{\lambda_{il},\theta_{ij}\} = \begin{cases} 1, \; l=j \\ -1, \; l=j+1 \\ 0, \; l\neq j, j+1 \end{cases}$$

Proof. We will verify, using (6.2), that

$$\beta_{ij}(\phi^t_{il}(\mathbf{e})) = \begin{cases} \beta_{ij}(\mathbf{e}), & l\neq j, j+1, \\ \exp(\sqrt{-1}\,t)\,\beta_{ij}(\mathbf{e}), & l=j, \\ \exp(-\sqrt{-1}\,t)\,\beta_{ij}(\mathbf{e}), & l=j+1. \end{cases}$$

Note from (6.2) that the i-th diagonal A_i of $\mathbf{e}$ and the $(i+2)$-nd edge are fixed under ϕ^t_{il}. Hence the normalized eigenvectors u_{ij} of A_i are also fixed. Now abbreviate $g_t = \exp(\sqrt{-1}\,tE_l(A_i))$, and as explained in Remark 4, make the replacement

$$w_{i+1}(\phi^t_{il}(\mathbf{e})) = w_{i+1}(g_t\mathbf{e}g_t^{-1}) \rightsquigarrow g_t w_{i+1}(\mathbf{e}).$$

We obtain

$$\begin{aligned}\beta_{ij}(\phi^t_{il}(\mathbf{e})) &= (g_t w_{i+1}, u_{ij})(u_{ij}, w_{i+2})(w_{i+2}, u_{i,j+1})(u_{i,j+1}, g_t w_{i+1}) \\ &= (w_{i+1}, g_t^{-1}u_{ij})(u_{ij}, w_{i+2})(w_{i+2}, u_{i,j+1})(g_t^{-1}u_{i,j+1}, w_{i+1}).\end{aligned}$$

Since $E_l(A_i)u_{ij} = \delta_{jl}u_{ij}$ the lemma follows by definition of g_t. □

Lemma 10.3.

$$\{\lambda_{ij},\theta_{kl}\} = 0, i\neq k.$$

Proof. If $i<k$ then the k-th diagonal, the $(k+1)$-st edge, and the $(k+2)$-nd edge are fixed by the bending flow ϕ^t_{ij}, and hence θ_{kl} is unchanged.

If $i>k$, then the k-th diagonal,the $(k+1)$-st edge and the $(k+2)$-nd edge are rigidly moved by the g_t under the bending flow ϕ^t_{ij}, and hence θ_{kl} is unchanged. (Note that Remark 4 is used once more). □

To remove the redundancy in the λ_{ij}, we define new action variables μ_{ij} by the formula

$$\mu_{ij} = \sum_{k=1}^{j} \lambda_{ik}. \tag{10.8}$$

As a consequence of the two preceding lemmas we obtain

Proposition 10.2. *The action variables $\{\mu_{ij}\}$ and the angle variables $\{\theta_{ij}\}$ are conjugate*

$$\{\mu_{ij}, \theta_{kl}\} = \begin{cases} 1, \ i = k, j = l \\ 0, \ otherwise. \end{cases}$$

We deduce two corollaries.

Corollary 10.1. *The angle variables are functionally independent.*

Corollary 10.2. *The Hamiltonian flows of the new action variables $\{\mu_{ij}\}$ permute the simultaneous level sets $\{\theta_{ij} = c_{ij}, (i,j) \in \mathcal{I}\}$ transitively.*

We now begin the proof that

$$\{\theta_{ij}, \theta_{kl}\} = 0.$$

Recall that $\mathcal{S}_{m+1}$ is the space of real symmetric $(m+1) \times (m+1)$ matrices. Let $\sigma : \mathcal{H}_{m+1} \to \mathcal{H}_{m+1}$ be complex conjugation. Then $\mathcal{S}_{m+1}$ is the fixed subspace of σ. The following lemma is immediate from (2.1):

Lemma 10.4. *The involution σ is anti-Poisson (a Poisson isomorphism from $\mathcal{H}_{m+1}$ equipped with the Lie Poisson tensor to $\mathcal{H}_{m+1}$ equipped with the negative of the Lie Poisson tensor).*

We obtain

Corollary 10.3. *If f and g are constant on $\mathcal{S}_{m+1}$, then $\{f, g\}$ vanishes on $\mathcal{S}_{m+1}$.*

Proof. Let $\pi(.,.)$ be the Lie Poisson bivector considered as a skew-symmetric bilinear form on the cotangent bundle of $\mathcal{H}_{m+1}$. For $x \in \mathcal{S}_{m+1}$ and u, v cotangent vectors at x, the Lemma gives $\pi_x(u,v) = -\pi_x(\sigma u, \sigma v)$. If u and v are conormal covectors at x then they are in the -1-eigenspace for σ, and therefore $\pi_x(u,v) = 0$. But if f and g are constant on $\mathcal{S}_{m+1}$, then df_x and dg_x are conormal at x. □

As an immediate consequence we have

Lemma 10.5. *If f and g are constant on $M_{\mathbf{r}}(\mathcal{S}_{m+1})$, then $\{f, g\}$ vanishes on $M_{\mathbf{r}}(\mathcal{S}_{m+1})$.*

Our next goal is to prove that the simultaneous zero level set of the angle variables is $M_{\mathbf{r}}(\mathcal{S}_{m+1})$. In order to obtain this we will need two technical lemmas to handle the regions S_1 and S_3 (in the notation of Proposition 8.1). The first lemma will be used to deal with the region S_3.

Lemma 10.6. *Let $V_i = \ker(A_i - \Lambda \mathbb{I}), n - m - 2 \leq i \leq n - 1$. Then*

$$V_{n-1} \supset V_{n-2} \supset \cdots \supset V_{n-m-2} = \{0\}.$$

Moreover (recalling $A_i = A_{i-1} + r_{i+1} w_{i+1} \otimes w_{i+1}^*$) we have

$$V_{i-1} = \{v \in V_i : (v, w_{i+1}) = 0\}.$$

Proof. Let $v \in V_{i-1}$ and $\|v\| = 1$. Then

$$\Lambda = (A_{i-1}v, v) = (A_i v, v) - r_{i+1}|(w_{i+1}, v)|^2.$$

But Λ is the largest eigenvalue of A_i so $(A_i v, v) \leq \Lambda$. Hence the above equation can hold if and only if

$$(A_i v, v) = \Lambda \quad (\text{so } v \in V_i) \quad \text{and} \quad (w_{i+1}, v) = 0.$$

□

Corollary 10.4. *Let w_{i+1}^{Λ} be the orthogonal projection of w_{i+1} on the Λ-eigenspace of A_{i-1}. Then*

$$w_{i+1}^{\Lambda} = 0.$$

The next lemma will be used to deal with the region S_1.

Lemma 10.7. *Let $U_i = \ker A_i, 1 \leq i \leq m$. Then*

$$U_1 \supset U_2 \supset \cdots \supset U_m = \{0\}.$$

Moreover

$$U_i = \{u \in U_{i-1} : (u, w_{i+1}) = 0\}.$$

Proof. Suppose $A_i u = 0$. Then

$$0 = (A_i u, u) = (A_{i-1}u, u) + r_{i+1}|(w_{i+1}, u)|^2.$$

But A_{i-1} is positive semidefinite and $r_{i+1} > 0$. Hence $u \in \ker A_{i-1}$ and $(u, w_{i+1}) = 0$. □

Corollary 10.5. *Let w_{i+1}^0 be the projection of w_{i+1} on ker A_i. Then*

$$w_{i+1}^0 = 0.$$

Now we can prove the result we need. Let $Z(\Theta)$ be the simultaneous zero level set of the angle variables $\{\theta_{ij}\}$.

Proposition 10.3.

$$Z(\Theta) = M_{\mathbf{r}}(\mathcal{S}_{m+1}).$$

Proof. The inclusion

$$M_{\mathbf{r}}(\mathcal{S}_{m+1}) \subset Z(\Theta)$$

is obvious (all the edges and diagonals are real, so the eigenvectors are real, so the β_{ij} are real). The point is to prove the reverse inclusion. We will assume $n \geq 2(m+1)$ and leave the case $n \leq 2m+1$, which is similar, to the reader.

Given a polygon $\mathbf{e}$ with all $\theta_{ij} = 0$. We wish to show that a sequence of conjugations of $\mathbf{e}$ by elements of U(m+1) will make all sides e_k real symmetric, or equivalently, all the w_k real. The proof is by descending induction, starting with the last diagonal $A_{n-1} = e_1 + \cdots + e_n = \Lambda\mathbb{I}$, which is of course real symmetric. First, conjugate $\mathbf{e}$ by $g \in$ U(m+1) (without changing A_{n-1}) to arrange that A_{n-2} is diagonal, hence real. This moves all the w_k to gw_k, but in the sequel we do not need to keep track of those changes. Now we know that A_{n-3} has the form

$$A_{n-3} = A_{n-2} - r_{n-1} w_{n-1} \otimes w_{n-1}^*,$$

and we want to show that we can move w_{n-1} to a real vector. We have

$$\ker(A_{n-2} - \Lambda\mathbb{I}) = \{\epsilon_1, \ldots, \epsilon_m\},$$

where $\{\epsilon_1, \ldots, \epsilon_{m+1}\}$ is the standard basis for $\mathbb{C}^{m+1}$. Suppose $A_{n-2}\epsilon_{m+1} = \mu\epsilon_{m+1}, \mu = \Lambda - r_n$.

Write w_{n-1} in the form $w_{n-1} = w_{n-1}^{\Lambda} + w_{n-1}^{\perp}$, where w_{n-1}^{Λ} is the orthogonal projection of w_{n-1} onto $\ker(A_{n-2} - \Lambda\mathbb{I})$. Hence there exists $z \in \mathbb{C}$ such that $w_{n-1}^{\perp} = z\epsilon_{m+1}$. Since w_{n-1} is defined only up to a complex multiple of unit length, we may multiply w_{n-1} by an element of S^1 in order to arrange that z be real. Let $c = \|w_{n-1}^{\Lambda}\|$. Now choose $g \in$ U(m+1) such that $g\epsilon_{m+1} = \epsilon_{m+1}$ and $gw_{n-1}^{\Lambda} = c\epsilon_m$.Then $gA_{n-2}g^{-1} = A_{n-2}$ (because $g\epsilon_{m+1} = \epsilon_{m+1}$ and $gw_{n-1} = c\epsilon_m + z\epsilon_{m+1}$). We change $\mathbf{e} = (e_1, \ldots, e_n)$ to $g\mathbf{e}g^{-1} = (ge_1g^{-1}, \ldots, ge_ng^{-1})$.

Next, we show how to find a conjugation $g\mathbf{e}g^{-1}$ that keeps A_{n-2}, A_{n-3} and w_{n-1} real and also makes gw_{n-2} real. This step exhibits the general pattern.

By Lemma 10.6,

$$\begin{aligned}\ker(A_{n-3} - \Lambda\mathbb{I}) &= \{v \in \ker(A_{n-2} - \Lambda\mathbb{I}) : (v, w_{n-1}) = 0\} \\ &= \operatorname{span}\{\epsilon_1, \ldots, \epsilon_{m-1}\}.\end{aligned}$$

The matrix A_{n-3} has two new eigenvalues (in addition to Λ); let their eigenvectors be $u_{n-3,m+1}, u_{n-3,m}$. There is one angle variable

$$\theta_{n-3,m+1} = \arg[(w_{n-2}, u_{n-3,m})(u_{n-3,m}, w_{n-1}) \\ (w_{n-1}, u_{n-3,m+1})(u_{n-3,m+1}, w_{n-2})]$$

We have seen that A_{n-3} is real symmetric, hence $u_{n-3,j}$ can be chosen to be real for all $1 \leq j \leq m+1$. Since w_{n-1} is real, we may normalize $u_{n-3,m}$ and $u_{n-3,m+1}$ so that $(w_{n-1}, u_{n-3,m}) > 0$ and $(w_{n-1}, u_{n-3,m+1}) > 0$. Since, by assumption, $\theta_{n-3,m+1} = 0$, we have

$$\arg(w_{n-2}, u_{n-3,m+1}) = \arg(w_{n-2}, u_{n-3,m}).$$

Hence by multiplying w_{n-2} by an element in S^1 we may assume that $(w_{n-2}, u_{n-3,m+1})$ and $(w_{n-2}, u_{n-3,m})$ are real. Now we may write

$$w_{n-2} = w_{n-2}^{\Lambda} + w_{n-2}^{\perp},$$

where

$$w_{n-2}^{\Lambda} \in ker((A_{n-3} - \Lambda\mathbb{I}) = \text{span}\{\epsilon_1, \ldots, \epsilon_{m-1}\}$$

and

$$w_{n-2}^{\perp} \in \text{span}\{u_{n-3,m}, u_{n-3,m+1}\} = \text{span}\{\epsilon_m, \epsilon_{m+1}\}.$$

We have arranged for $w_{n-2}^{\perp}$ to be real. Choose $g \in$ U(m+1) with $g\epsilon_m = \epsilon_m$ and $g\epsilon_{m+1} = \epsilon_{m+1}$ such that

$$gw_{n-2}^{\Lambda} = c'\epsilon_{m-1},$$

with $c' = \|w_{n-2}^{\Lambda}\|$ as in the preceding step. Now change $\mathbf{e}$ to $g\mathbf{e}g^{-1}$ and proceed to w_{n-3}.

We continue in this way until $\ker(A_k - \Lambda\mathbb{I}) = 0$ and we enter the region S_2. The argument for this region is simpler and is left to the reader. Note that the vanishing of the angle variables says that *all* the coordinates $(w_k, u_{k-1,j})$ in the eigenvector basis of A_{k-1} have a common phase which can be eliminated by multiplication by an element of S^1; no conjugation is needed, so the preceding edges all remain real symmetric. However, the zero eigenvalue, which is unavoidable when we enter region S_1, causes new problems, and Lemma 10.7 is required.

Suppose then we have proved that A_m is real (note that A_m has rank m). We want to prove that A_{m-1} is real. We know that

$$A_m = A_{m-1} + r_{m+1} w_{m+1} \otimes w_{m+1}^*,$$

and since $\ker A_m = \{0\}$, we have enough angle variables to prove that all coordinates of w_{m+1} have a common phase. We clear this phase as before and move on to A_{m-2}. We have $A_{m-1} = A_{m-2} + r_m w_m \otimes w_m^*$, and wish to prove that one can make w_m real without destroying reality of $w_{n-1}, \ldots, w_{m+1}$. Write $w_m = w_m^{\perp} + w_m^0$ with $A_{m-1}w_m^0 = 0$ and $w_m^{\perp}$

orthogonal to $\ker A_{m-2}$ (the latter has dimension 2). By the corollary to Lemma 10.7, we have $w_m^0 = 0$. Also, we have enough angle variables to conclude that the coordinates of $w_m^\perp$ relative to the eigenvectors of A_{m-1} orthogonal to $\ker A_{m-2}$ have a common phase. Thus, no conjugations are required to make w_m real, and all preceding edges remain real symmetric. Now continue. □

We remark that the proof could equally well be done by ascending induction; in that case, region S_1 would be the one requiring conjugations, while an overall scaling would do in S_2, S_3.

We are now ready to prove

Proposition 10.4.

$$\{\theta_{ij}, \theta_{kl}\} = 0.$$

Proof. Let $\mathbf{e} \in M_{\mathbf{r}}$ be given. By Corollary 10.2, the bending deformations flows permute the level sets of the θ_{ij}'s transitively. Hence we may apply a bending ϕ to move $\mathbf{e}$ into $Z(\Theta)$. Since ϕ is symplectic and the Hamiltonian vector fields of the θ_{ij} are invariant under bending, we have

$$\{\theta_{ij}, \theta_{kl}\}(\mathbf{e}) = \{\theta_{ij}, \theta_{kl}\}(\phi\mathbf{e}).$$

But by Proposition 10.3

$$Z(\Theta) = M_{\mathbf{r}}(\mathcal{S}_{m+1}).$$

Hence by Lemma 10.5

$$\{\theta_{ij}, \theta_{kl}\} = 0.$$

□

11. The duality between the bending systems and the Gel'fand-Tsetlin systems on Grassmannians

In this section we use Gel'fand-MacPherson duality, following [HK97] for the case of $m = 1$, to show that the bending system is equivalent to the Gel'fand-Tsetlin integrable system (as defined in [GS83]) on a torus quotient of the Grassmannian $G(m+1, \mathbb{C}^n)$. This equivalence will explain the appearance and form of the Gel'fand-Tsetlin patterns in §8.

Our first goal is to construct a symplectomorphism Φ from $M_{\mathbf{r}}$ to a symplectic quotient of $G(m+1, \mathbb{C}^n)$ by the n-torus T of diagonal matrices in U(n). This is the symplectic version of Gel'fand-MacPherson duality.

Let $\mathcal{M}$ denote the vector space of $(m+1) \times n$ complex matrices. We give $\mathcal{M}$ the Hermitean form (,) defined by $(X, Y) = \mathrm{Tr}(XY^*)$, and thus $\mathcal{M}$ is a symplectic vector space. The product group U(m+1) $\times$

U(n) acts isometrically and symplectically on $\mathcal{M}$. Denote the i-th row (resp. j-th column) of $N \in \mathcal{M}$ by R_i (resp. C_j).

Proposition 11.1. *The action of U(n) has momentum map*

$$\mu_{U(n)} : \mathcal{M} \to \mathcal{H}_n, \quad \mu_{U(n)} : N \mapsto N^*N.$$

In particular, the momentum map for the T-action is

$$\mu_T : N \mapsto (\|C_1\|^2, \dots, \|C_n\|^2).$$

The momentum map for the U(m+1) action is

$$\mu_{U(m+1)} : \mathcal{M} \to \mathcal{H}_{m+1}, \quad \mu_{U(m+1)} : N \mapsto NN^*.$$

Note that

$$\mu_{\mathrm{U(m+1)}}(N) = \sum_{i-1}^{n} C_i \otimes C_i^*. \tag{11.1}$$

This will provide the connection with polygons.

We construct the desired symplectomorphism by computing the symplectic quotient corresponding to the μ_T-level $\mathbf{r}$ and the $\mu_{\mathrm{U(m+1)}}$ level $\Lambda\mathbb{I}$ in two different orders. If we first quotient with respect to T with momentum level $\mathbf{r}$ and then with respect to U(m+1) with momentum level $\Lambda\mathbb{I}$, we get the space $M_{\mathbf{r}}$. In order to see this, we note that the (left) action of $\prod_1^n$ U(m+1) on $\mathcal{M}$ (acting on the columns) commutes with the (right) action of T (in fact one obtains a dual pair in the sense of Howe, see [KKS78]). We first compute the symplectic quotient by T.

Lemma 11.1. *The momentum map $\mu_{(U(m+1))^n}$ induces an embedding of the symplectic quotient $\mu_T^{-1}(\mathbf{r})/T$ into $\Pi_1^n \mathcal{H}_{m+1}$, with image $\prod_1^n \mathcal{O}_{r_i}$.*

Proof. This follows because it is a general feature of dual pairs, see [KKS78], that the momentum map for one action embedds the symplectic quotient of the other as an orbit in (the dual of) the Lie algebra of the first group. This principle, applied to the pair $(\mathrm{U(m+1)})^n \times T$, implies the lemma. □

Thus we have identified the quotient by T with the correct product of rank one orbits in $\mathcal{H}_{m+1}$. Clearly, after taking the symplectic quotient of this product by the diagonal action of U(m+1) (at momentum level $\Lambda\mathbb{I}$), we obtain $M_{\mathbf{r}}$.

Suppose instead we first quotient with respect to U(m+1) and momentum level $\Lambda\mathbb{I}$. We get the Grassmannian $G(m{+}1, \mathbb{C}^n)$ with a certain U(n)-invariant symplectic structure.

Lemma 11.2. *The momentum map $\mu_{U(n)}$ induces an embedding of the symplectic quotient $\mu^{-1}_{U(m+1)}(\Lambda\mathbb{I})/U(m+1)$ into $\mathcal{H}_n$, with image the $U(n)$-orbit $\mathcal{O}_\Lambda$ consisting of those matrices that have eigenvalue Λ with multiplicity $m+1$ and eigenvalue 0 with multiplicity $n-m-1$.*

Proof. The argument is the analogous to the previous case, only this time we use the dual pair U(m+1) $\times$ U(n). □

Denote the torus quotient at momentum level $\mathbf{r}$ of the Grassmannian with the Kostant-Kirillov symplectic structure corresponding to Λ by $\mathcal{M}_\Lambda$. We have now obtained the desired symplectomorphism Φ from $M_{\mathbf{r}}$ to $\mathcal{M}_\Lambda$.

Of course this symplectomorphism gives a Poisson isomorphism between the Poisson algebras of smooth functions of $M_{\mathbf{r}}$ and $\mathcal{M}_\Lambda$. However, we want to make this more explicit and to localize it. Let $\mathcal{M}_{\mathbf{r},\Lambda}$ be the subset of $\mathcal{M}$ consisting of matrices N such that $\|C_j\|^2 = r_j$ and $N^*N = \Lambda\mathbb{I}$. Thus we have U(m+1)$\times T$ quotient mappings $\pi_1 : \mathcal{M}_{\mathbf{r},\Lambda} \to M_{\mathbf{r}}$ (first quotient by T then by U(m+1)) and $\pi_2 : \mathcal{M}_{\mathbf{r},\Lambda} \to \mathcal{M}_\Lambda$ (first quotient by U(m+1) then by T). We use the mappings π_1 and π_2 to realize (and localize) the Poisson isomorphism Φ from above. Let f be a function which is smooth on an open subset of $M_{\mathbf{r}}$. Use π_1 to pull f back to a U(m+1) $\times$ T-saturated open subset of $\mathcal{M}_{\mathbf{r},\Lambda}$. Since π_2 is a quotient map and $\pi_1^* f$ is invariant under U(m+1), we can first descend it to to a T-saturated open subset of the Grassmannian, then to the torus quotient of that open set, which is an open subset of $\mathcal{M}_\Lambda$. We note that Φ is determined by the equation

$$\Phi(\pi_1(N)) = \pi_2(N).$$

We now briefly review the Gel'fand-Tsetlin integrable system - for the details see [GS83]. We recall we have identified the space $\mathcal{H}_n$ of $n\times n$ Hermitean matrices with the (dual of) the Lie algebra of U(n). We now construct $n(n+1)/2$ Poisson commuting functions on $\mathcal{H}_n$ which are smooth on a dense open subset. Let $X \in \mathcal{H}_n$. Let $\beta_i(X)$ be the principal $i \times i$ diagonal block. Define γ_{ij} on $\mathcal{H}_n$ by

$$\gamma_{ij}(X) = \lambda_j(\beta_i(X)),$$

where λ_j is the j-th eigenvalue of the block. As usual, we assume that the eigenvalues of the i-th block are arranged in nonincreasing order. It is proved in [GS83] that the γ_{ij}'s Poisson commute. We note that the γ_{nj} are Casimirs. The restrictions of the remaining Gel'fand-Tsetlin Hamiltonians to generic orbits are functionally independent and give rise to integrable system on such orbits. The eigenvalues of the blocks interlace and can be arranged in a "Gel'fand-Tsetlin" pattern (we take $n = 6$).

$$\begin{array}{ccccccccccc}
\gamma_{61} & & \gamma_{62} & & \gamma_{63} & & \gamma_{64} & & \gamma_{65} & & \gamma_{66} \\
& \gamma_{51} & & \gamma_{52} & & \gamma_{53} & & \gamma_{54} & & \gamma_{55} & \\
& & \gamma_{41} & & \gamma_{42} & & \gamma_{43} & & \gamma_{44} & & \\
& & & \gamma_{31} & & \gamma_{32} & & \gamma_{33} & & & \\
& & & & \gamma_{21} & & \gamma_{22} & & & & \\
& & & & & \gamma_{11} & & & & &
\end{array}$$

Figure 3

Since we are dealing with a degenerate orbit here (the Grassmannian), many of the γ_{ij}'s (at the ends of the rows) will be zero (see Remark 5 below, and Figure 2 above). The next proposition, combined with the earlier sections, shows how to extract a functionally independent set of Gel'fand-Tsetlin Hamiltonians and obtain angle variables for the Gel'fand-Tsetlin system on the Grassmannian.

Proposition 11.2. $\Phi^* \gamma_{ij} = \lambda_{ij}$.

Proof. Let $\mathbb{I}_k$ be the diagonal n by n matrix whose first k eigenvalues are equal to 1 and last $n-k$ eigenvalues are equal to 0. We use $\mathbb{I}_k$ to "truncate" N, N^*N and NN^*. Put $N_k := N\mathbb{I}_k$. Then

$$\mu_{\mathrm{U(n)}}(N_k) = \mathbb{I}_k N^* N \mathbb{I}_k$$
$$\mu_{\mathrm{U(m+1)}}(M_k) = N \mathbb{I}_k \mathbb{I}_k N^*.$$

The matrix on the first line is $\beta_k(N^*N)$, the principal k by k block of the $n \times n$ matrix N^*N, and the matrix on the second line is the diagonal $A_{k-1} = C_1C_1^* + C_2C_2^* + \cdots + C_kC_k^*$. The matrices $\mathbb{I}_k N^* N \mathbb{I}_k$ and $N\mathbb{I}_k\mathbb{I}_k N^*$ have the same nonzero eigenvalues. But the eigenvalues of the second matrix are the bending Hamiltonians λ_{kj}, and the eigenvalues of the first matrix are the Gel'fand-Tsetlin Hamiltonians γ_{kj}. Finally we observe that

$$\gamma_{ij}(\Phi(\pi_1(M))) = \gamma_{ij}(\pi_2(M)) = \lambda_j(\beta_i(\pi_M)) = \lambda_j(A_i(\pi_1(M))) = \lambda_{ij}(\pi_1(M)).$$

□

We conclude this section with three remarks.

Remark 5. Proposition 11.2 explains the appearance of Gel'fand-Tsetlin patterns in connection with the bending Hamiltonians. The appearance of the zeroes at the end of the rows in our patterns is explained because the Gel'fand-Tsetlin system in question is defined on a subset of the Hermitean matrices of rank at most $m+1$. Hence $\gamma_{ij} = 0, j > m+1$.

Remark 6. The reconstruction process in §9 may be interpreted as saying that the class of patterns introduced in §8 is precisely the class corresponding to Hermitean matrices of the form N^*N, where N is as above.

Remark 7. Fixing the row sums in the patterns in §8 to be partial sums of the r_j corresponds to taking the quotient of the Grassmannian by T (at level $\mathbf{r}$).

12. Pieri's formula and the duality at the quantum level

In this section we will assume that the r_i's are (positive) integers. The orbit $\mathcal{O}_{r_i}$ then corresponds under geometric quantization to the irreducible representation $\mathcal{S}^{r_i}(V)$ of U(m+1), where V denotes the standard (or vector) representation of U(m+1) on $\mathbb{C}^{m+1}$ and $\mathcal{S}^{r_i}(V)$ the r_i-th symmetric power.

The (classical) duality result of the last section should have a quantum version. We note that the duality connected an integrable system (bending) on a symplectic quotient of $\prod_1^n \mathcal{O}_{r_i}$ by the diagonal action of U(m+1) and an integrable system (Gel'fand-Tsetlin) on a torus quotient of the Grassmannian $G(m+1, \mathbb{C}^n)$. Thus, according to geometric quantization, at the quantum level we would expect a relation between an n-fold tensor product multiplicity for $GL(m+1)$ and a weight multiplicity for a Cartan power of the the $m+1$-st exterior power of $GL(n)$. The bending system provides a (singular) real polarization of the space $M_{\mathbf{r}}$, the symplectic quotient (at level $\Lambda\mathbb{I}$) of $\prod_i \mathcal{O}_{r_1}$. Thus the number of lattice points in the momentum polyhedron $\mathbf{P}$ for bending should be equal to the multiplicity of the the 1-dimensional representation $(det)^\Lambda$ in $\otimes_1^n \mathcal{S}^{r_i}(V)$. But on the other hand, the Gel'fand-Tsetlin system is a real polarization of the torus quotient of the Grassmannian (at level $\mathbf{r}$) where the Grassmannian is given the symplectic structure which corresponds to the orbit of U(n) through the diagonal matrix with $m+1$ Λ's and $n-m-1$ zeroes. Thus the above number of lattice points should also be the multiplicity of the $\mathbf{r}$-th weight space in $C^\Lambda \bigwedge^{m+1} V$, the Λ-th Cartan power of the $m+1$-st exterior power of the vector representation V of $GL(n)$. (We recall that if W^α is a representation with highest weight ν, then the p-th Cartan power $C^p W^\alpha$ is the irreducible representation with highest weight $p\alpha$). This equality of multiplicities predicted is in fact true, and will be proved below.

Remark 8. It is unfortunate that the theory of geometric quantization using a real polarization is not sufficiently well developed to allow one to deduce theorems in representation theory from equalities of numbers of lattice points in momentum polyhedra. At this time we can only regard such equalities as predictions of theorems in representation theory.

We first note how the interlacing of the spectra of the perturbed matrix and the unperturbed matrix (see §8)from the Weinstein-Aronszajn formula predicts Pieri's formula in representation theory.

12.1. **The Weinstein-Aronszajn and Pieri formulas.** We recall Pieri's formula for tensoring an irreducible polynomial representation of U(m+1) with a symmetric power of the vector representation, [FH, §A.1].

Theorem 12.1 (Pieri's Formula). *Let $\lambda = (\lambda_1, \ldots, \lambda_{m+1})$ be the highest weight of the polynomial representation $V(\lambda_1, \ldots, \lambda_{m+1})$ of U(m+1). Let k be a positive integer. Then*

$$V(\lambda_1, \ldots, \lambda_{m+1}) \otimes \mathcal{S}^k(V) = \oplus V(\nu_1, \ldots, \nu_{m+1})$$

where the sum is taken over all dominant $\nu = (\nu_1, \ldots, \nu_n)$ satisfying

$$\nu_1 \geq \lambda_1 \geq \nu_2 \geq \ldots \geq \nu_{m+1} \geq \lambda_{m+1} \geq 0$$

and

$$\sum_{i=1}^{m+1} \nu_i = \sum_{i=1}^{m+1} \nu_i + k.$$

This is of course Proposition 7.2 restricted to integer eigenvalues. If $A \in \mathcal{O}_\lambda$, then the spectrum of the rank one perturbation $A + k\, w \otimes w^*$ satisfies the interlacing and row sum conditions of the Pieri formula.

12.2. **Duality at the quantum level.** In this subsection we prove the theorem from representation theory that is predicted by the equality (of the numbers of lattice points) of the momentum polyhedra for bending and Gel'fand-Tsetlin. The required facts from representation theory can be found in [FH] and [Ze].

Theorem 12.2. *The multiplicity of the 1-dimensional representation $(det)^{\Lambda}$ in $\otimes_1^n V(r_j)$ is equal to the multiplicity of the weight $\mathbf{r}$ in the irreducible representation $C^{\Lambda} \bigwedge^{m+1} V$ of U(n). This common multiplicity is in fact equal to the number of lattice points in $\mathbf{P}$.*

The theorem will be a consequence of the next three lemmas. We will need

Definition 12.1. Let λ be an l-tuple of positive integers and μ be a partition. Then the Kostka number $K_{\mu\lambda}$ is the number of ways to fill in the Young diagram corresponding to μ with λ_1 1's, λ_2 2's, …, λ_l l's so that the rows are weakly increasing and the columns are strongly increasing.

By applying Pieri's formula iteratively one gets [FH, (A.9)]:

Lemma 12.1.

$$\mathcal{S}^{r_1}(V) \otimes \mathcal{S}^{r_2}(V) \otimes \cdots \otimes \mathcal{S}^{r_n}(V) = \oplus_\mu K_{\mu \mathbf{r}} V(\mu).$$

We obtain

Corollary 12.1. *The multiplicity of the* 1*-dimensional representation* $(det)^\Lambda$ *in* $\otimes_1^n V(r_j)$ *is equal to the Kostka number* $K_{\Lambda(1^{m+1})\mathbf{r}}$.

Here the symbol $\Lambda(1^{m+1})$ means the partition $(\Lambda, \Lambda, \ldots, \Lambda)$ (there are $m+1$ Λ's). The corresponding Young diagram has $m+1$ rows and Λ columns.

In order to compare $K_{\Lambda(1^{m+1})\mathbf{r}}$ with the multiplicity of the weight $\mathbf{r}$ in the irreducible representation $C^\Lambda \bigwedge^{m+1} V$ of U(n) we recall there is a basis for an irreducible representation of $GL(n)$ labelled by semistandard Young tableaux. Suppose the highest weight of the representation is μ. We also use μ to denote the Young diagram associated to μ. A *semistandard filling* of μ is an assignment of the integers between 1 and n to the boxes of μ such that the rows are weakly increasing and the columns are strongly increasing. The associated basis is a weight basis, and the weight of of the basis vector corresponding to a semistandard tableau is $(k_1, \ldots, k_n)$, where k_i is the number of i's in the tableau. Thus we have proved

Lemma 12.2. $K_{\Lambda(1^{m+1})\mathbf{r}}$ *is also the multiplicity of the weight* $\mathbf{r}$ *in* $C^\Lambda \wedge^{m+1} V$ *of* $GL(n)$.

It still remains to prove that the number of lattice points in $\mathbf{P}$ is the common multiplicity.

To see this we recall that there is an orthonormal basis (the Gel'fand-Tsetlin basis) for the irreducible representation $C^\Lambda \wedge^{m+1} V$ indexed by Gel'fand-Tsetlin patterns whose top row consists of $m+1$ Λ's and $n-m-1$ zeroes. Moreover, this basis is a weight basis, and the weight of a basis vector corresponding to a Gel'fand-Tsetlin pattern is given by the differences in the row sums starting with the bottom entry in the pattern. Thus we have

Lemma 12.3. *The number of lattice points in* $\mathbf{P}$ *is equal to the dimension of the* $\mathbf{r}$*-th weight space in* $C^\Lambda \bigwedge^{m+1} V$.

It follows that the count of lattice points in $\mathbf{P}$ gives the correct answer for both multiplicities. We conclude with a remark.

Remark 9. One might ask whether there is a direct combinatorial argument to establish the last lemma above, i.e. that the number of semistandard Young tableaux of weight **r** is equal to the number of Gel'fand-Tsetlin patterns of weight **r**. In fact, there is a one to one weight preserving correspondence between semistandard Young tableaux and Gel'fand-Tsetlin patterns, see [GZ86].

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