

THE NOETHER-LEFSCHETZ CONJECTURE AND GENERALIZATIONS

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ABSTRACT. We prove the Noether-Lefschetz conjecture on the moduli space of quasi-polarized K3 surfaces. This is deduced as a particular case of a general theorem that states that low degree cohomology classes of arithmetic manifolds of orthogonal type are coming from special cycles, i.e. sub-arithmetic manifolds of the same type. For compact manifolds this was proved in [2], here we extend the latter to non-compact manifolds. This allows to apply our result to some moduli spaces. As in [2] our results are conditional on the stabilization of the trace formula for the (disconnected) groups. At present the stabilization of the trace formula has been published only for the case of *connected* groups. However, a proof of the extension by Mœglin and Waldspurger, [47], is now on the Archive, we refer to [2] for more details.

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INTRODUCTION

0.1. From moduli theory to Shimura varieties of orthogonal type. The study of Picard groups of moduli problems has been started by Mumford [50] in 1960's. For moduli space \mathcal{M}_g of genus g curves, Mumford and Harer (cf. [51, 24]) showed that the Picard group $\text{Pic}(\mathcal{M}_g)$ of \mathcal{M}_g is isomorphic to its second cohomology group $H^2(\mathcal{M}_g, \mathbb{Z})$, which is a finitely generated abelian group of rank one for $g \geq 3$. Moreover, the generator of $H^2(\mathcal{M}_g, \mathbb{Q})$ is the first chern class of the Hodge bundle on \mathcal{M}_g .

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In higher dimensional moduli theory, the *quasi-polarized* K3 surface of genus g is a two dimensional analogue of the genus g smooth projective curve. Here, a quasi-polarized K3 surface of genus $g \geq 2$ is defined by a pair (S, L) where S is a K3 surface and L is line bundle on S with primitive Chern class $c_1(L) \in H^2(S, \mathbb{Z})$ satisfying

$$\int_S L^2 = 2g - 2 \quad \text{and} \quad \int_S L \cdot [C] \geq 0$$

for every curve $C \subset S$. Let \mathcal{K}_g be the coarse moduli space of quasi-polarized K3 surfaces of genus g . Unlike the case of $\text{Pic}(\mathcal{M}_g)$, O'Grady has [52] shown that the rank of $\text{Pic}(\mathcal{K}_g)$ is increasing to infinity as g increases. Besides the Hodge line bundle, there are actually many other natural divisors on \mathcal{K}_g coming from Noether-Lefschetz theory developed by Griffiths and Harris in [21] (see also [41]). More precisely, the Noether-Lefschetz locus in \mathcal{K}_g parametrizes K3 surfaces in \mathcal{K}_g with Picard number greater than 2; it is a countable union of divisors. Each of them parametrizes the K3 surfaces whose Picard lattice contains a special curve class; these divisors are called *Noether-Lefschetz (NL) divisors* on \mathcal{K}_g . The following theorem implies that any curve on \mathcal{K}_g will meet some NL-divisor on \mathcal{K}_g (see also [7, Theorem 1.1]).

Theorem 0.1.1 (Oguiso). [53, Main Theorem] *Let $\mathfrak{X} \rightarrow B$ be a connected family of quasi-polarized K3 surfaces in \mathcal{K}_g with constant Picard number. Then \mathfrak{X} is isotrivial.*

So it is natural to ask whether the Picard group $\text{Pic}_{\mathbb{Q}}(\mathcal{K}_g)$ of \mathcal{K}_g with rational coefficients is spanned by NL-divisors. This is conjectured to be true by Maulik and Pandharipande, see [41, Conjecture 3]. More generally, one can extend this question to higher NL-loci on \mathcal{K}_g , which parametrizes K3 surfaces in \mathcal{K}_g with higher Picard number, see [31]. Call the irreducible components of higher NL-loci the NL-cycles on \mathcal{K}_g . Each of them parametrizes K3 surfaces in \mathcal{K}_g whose Picard lattice contains a special primitive lattice. In this paper, we prove that

Theorem 0.1.2. *For all $g \geq 2$ and all $n \leq 4$, the cohomology group $H^{2n}(\mathcal{K}_g, \mathbb{Q})$ is spanned by NL-cycles of codimension n . In particular (taking $n = 1$), $\text{Pic}_{\mathbb{Q}}(\mathcal{K}_g) \cong H^2(\mathcal{K}_g, \mathbb{Q})$ and the Noether-Lefschetz conjecture holds on \mathcal{K}_g for all $g \geq 2$.*

Remark 0.1.3. There is a purely geometric approach (cf. [20]) for low genus case ($g \leq 12$), but it can not be applied for large genera. It remains interesting to give a geometric proof for this conjecture.

Combined with works of Borchers and Bruinier in [6] and [12], we get the following:

Corollary 0.1.4. *We have*

$$(0.1) \quad \text{rank}(\text{Pic}(\mathcal{K}_g)) = \frac{31g + 24}{24} - \frac{1}{4}\alpha_g - \frac{1}{6}\beta_g \\ - \sum_{k=0}^{g-1} \left\{ \frac{k^2}{4g-4} \right\} - \#\left\{k \mid \frac{k^2}{4g-4} \in \mathbb{Z}, 0 \leq k \leq g-1\right\}$$

where

$$\alpha_g = \begin{cases} 0, & \text{if } g \text{ is even;} \\ \binom{2g-2}{2g-3} & \text{otherwise.} \end{cases}, \quad \beta_g = \begin{cases} \binom{\frac{g-1}{4g-5}}{\frac{g-1}{4g-5}} - 1, & \text{if } g \equiv 1 \pmod{3}, \\ \binom{\frac{g-1}{4g-5}}{\frac{g-1}{4g-5}} + \binom{\frac{g-1}{3}}{\frac{g-1}{3}} & \text{otherwise.} \end{cases}$$

and $\left(\frac{a}{b}\right)$ is the Jacobi symbol.

Theorem 0.1.2 is deduced from a general theorem on arithmetic manifolds. Indeed: let (S, L) be a K3 surface in \mathcal{K}_g , then the middle cohomology $H^2(S, \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$ under the intersection form \langle, \rangle :

$$H^2(S, \mathbb{Z}) = U^{\oplus 3} \oplus (-E_8)^{\oplus 2},$$

where U is the hyperbolic lattice of rank two and E_8 is the positive definite lattice associated to the Lie group of the same name; see [41]. The primitive part $H^2(S, \mathbb{Z})_{\text{prim}} := c_1(L)^\perp$ is isometric to the even lattice

$$\Lambda_g := \langle 2 - 2g \rangle \oplus U^{\oplus 2} \oplus (-E_8)^{\oplus 2}.$$

There is a Hodge structure on $H^2(S, \mathbb{Z})_{\text{prim}}$ given by the Hodge decomposition

$$H^2(S, \mathbb{Z})_{\text{prim}} \otimes_{\mathbb{Z}} \mathbb{C} = H^{2,0}(S, \mathbb{C}) \oplus H^{1,1}(S, \mathbb{C})_{\text{prim}} \oplus H^{0,2}(S, \mathbb{C})$$

with Hodge number $(1, 19, 1)$. Let $V = \Lambda_g \otimes \mathbb{Q}$, which is unique up to isomorphism. The period domain associated to Λ_g :

$$D = D(V) = \{\omega \in V \otimes_{\mathbb{Q}} \mathbb{C} \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\} / \mathbb{C}^\times \\ \cong \{\text{oriented positive 2-planes in } V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}\} \\ \cong \text{SO}(V_{\mathbb{R}}) / K_{\mathbb{R}}$$

parametrizes the Hodge structures of primitive middle cohomology group of K3 surfaces in \mathcal{K}_g , where $K_{\mathbb{R}} \cong \text{SO}(2) \times \text{SO}(19)$ is the stabilizer of an oriented positive 2-plane in $\text{SO}(V_{\mathbb{R}}) \cong \text{SO}(2, 19)$. It comes with a natural action of the monodromy group

$$\Gamma_g = \{\gamma \in \text{SO}(\Lambda_g) \mid \gamma \text{ acts trivially on } \Lambda_g^\vee / \Lambda_g\},$$

and the action is properly discontinuous. The global Torelli theorem for K3 surfaces (cf. [54, 16]) says that the coarse moduli space \mathcal{K}_g is isomorphic to the arithmetic quotient $\Gamma_g \backslash D$ via the period map.

Note that the quotient $\Gamma_g \backslash D$ is a connected component of a Shimura variety associated to the group $\text{SO}(2, 19)$ and NL-cycles on \mathcal{K}_g can be interpreted as special cycles on $\Gamma_g \backslash D$: vectors in Λ_g correspond to additional elements

of $\text{Pic}(S) \cong H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z})$. Kudla [31, Proposition 3.2] shows that the codimension r special cycle associated to a quadratic sub-lattice $N \subset \Lambda_g$ of signature $(0, r)$ is precisely the locus in \mathcal{K}_g of quasi-polarized $K3$ surfaces (S, L) such that there exists a quadratic embedding $N \hookrightarrow \text{Pic}(S)$ whose image is orthogonal to $c_1(L)$. E.g. for $r = 1$, we are imposing a single additional class in $\text{Pic}(S)$ and the corresponding special cycles are the NL-divisors of [41]. Our main result will be more generally concerned with special cycles in (non-compact) Shimura varieties associated to orthogonal groups.

Remark 0.1.5. General (non-compact) Shimura varieties associated to Λ_g correspond to moduli space of primitively quasi-polarized $K3$ surfaces with level structures; see [56, §2]. Recently, these moduli spaces play more and more important role in the study of $K3$ surfaces (cf. [29, 40]). The Hodge-type result above can be naturally extended to these moduli spaces.

0.2. Shimura varieties of orthogonal type. Let Y be a connected smooth Shimura variety of orthogonal type, that is, a congruence locally Hermitian symmetric variety associated to the orthogonal group $\text{SO}(p, 2)$. One attractive feature of these Shimura varieties is that they have many algebraic cycles coming from sub-Shimura varieties of the same type in all codimensions. These are so called special cycles on Y and play the central role in Kudla's program. A natural question arising from geometry and also arithmetic group cohomology theory is:

Question. *Do the classes of special cycles of codimension n exhaust all the cohomology classes in $H^{2n}(Y, \mathbb{Q})$ for sufficiently small n ?*

For $2n < \frac{p}{2}$, the cohomology group $H^{2n}(Y, \mathbb{Q})$ has a pure Hodge structure of weight $2n$ with only Hodge classes (cf. §6.2), i.e. $H^{2n}(Y, \mathbb{C}) = H^{n,n}(Y)$. So when Y is compact, this can be viewed as a strong resemblance of the Hodge conjecture on Y . And indeed in the compact case, the main result of [2] provides a positive answer to both Question 0.2 and the Hodge conjecture as long as $n < \frac{p+1}{3}$. The proof is automorphic frame, based on Kudla-Millson theory and Arthur's endoscopic classification of automorphic representations of orthogonal groups.

When Y is non-compact, one expect a similar surjective theorem holding for certain low degree Hodge classes of Y . Indeed, before [2], Hoffman and He considered the case $p = 3$ in [26]. In their situation, Y is a smooth Siegel modular threefold and they prove that the $\text{Pic}(Y) \otimes \mathbb{C} \cong H^{1,1}(Y)$ is generated by Humbert surfaces. In the present paper, one of our goals is to extend [2] to all non-compact Shimura varieties of orthogonal type:

Theorem 0.2.1. *Assume that Y is a connected Shimura variety associated to $\text{SO}(p, 2)$. If $n < \frac{p+1}{3}$, any cohomology class in $H^{2n}(Y, \mathbb{Q}) \cap H^{n,n}(Y, \mathbb{C})$ is a linear combination (with rational coefficients) of classes of special cycles.*

As in the case of [2], our proof also relies on the Arthur's theory [1] which depends on the stabilization of the trace formula for disconnected

groups. The ordinary trace formula has been established by Arthur, while the stabilized twisted trace formula was unavailable until recently. In a series of recent work (cf. [64, 65, 66]), Mœglin and Waldspurger have announced a proof of the stabilized twisted trace formula for disconnected groups. This means that our results are now unconditional.

0.3. General results for arithmetic manifolds. As we will see later, Theorem 0.2.1 actually follows from a more general result. Let \mathbb{A} be the ring of adèles of \mathbb{Q} . Let $G = \mathrm{SO}(V)$ be a special orthogonal group associated to a quadratic space V over F . Assume that V has signature (p, q) over $\mathbb{R} = \mathbb{Q}_\infty$. Then $G(\mathbb{R}) = \mathrm{SO}(p, q)$ and the quotient

$$D = \mathrm{SO}_0(p, q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$$

is a symmetric space, where $\mathrm{SO}_0(p, q)$ is the connected component of $G(\mathbb{R})$ containing the identity. Fix a neat compact open subgroup $K \subseteq G(\mathbb{A}_f)$, then $Y_K = \Gamma_K \backslash D$ is a smooth congruence arithmetic manifold associated to G , where $\Gamma_K = G(\mathbb{Q}) \cap K$ is a congruence arithmetic subgroup of $G(\mathbb{Q})$. The special cycles on Y_K can be considered as the linear combinations of arithmetic submanifolds of the same type.

Let us consider the space $\tilde{H}^{nq}(Y_K, \mathbb{C})$ of square integrable harmonic nq -forms in $H^{nq}(Y_K, \mathbb{C})$. We recall from [2] that the decomposition of exterior powers of the cotangent bundle of D under the action of the holonomy group, i.e. the maximal compact subgroup of G , yields a natural notion of *refined Hodge decomposition* of the cohomology groups of the associated locally symmetric spaces. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the (complexified) Cartan decomposition of $G(\mathbb{R})$ associated to some base-point in D . As a representation of $\mathrm{SO}(p, \mathbb{C}) \times \mathrm{SO}(q, \mathbb{C})$ the space \mathfrak{p} is isomorphic to $V_+ \otimes V_-^*$ where $V_+ = \mathbb{C}^p$ (resp. $V_- = \mathbb{C}^q$) is the standard representation of $\mathrm{SO}(p, \mathbb{C})$ (resp. $\mathrm{SO}(q, \mathbb{C})$). The refined Hodge types therefore correspond to irreducible summands in the decomposition of $\wedge^\bullet \mathfrak{p}^*$ as a $(\mathrm{SO}(p, \mathbb{C}) \times \mathrm{SO}(q, \mathbb{C}))$ -module. In the case of the group $\mathrm{SU}(n, 1)$ (then D is the complex hyperbolic space) it is an exercise to check that one recovers the usual Hodge-Lefschetz decomposition. But in general the decomposition is much finer and in our orthogonal case it is hard to write down the full decomposition of $\wedge^\bullet \mathfrak{p}$ into irreducible modules. Note that, as a $\mathrm{GL}(V_+) \times \mathrm{GL}(V_-)$ -module, the decomposition is already quite complicated. We have (see [17, Equation (19), p. 121]):

$$(0.2) \quad \wedge^R (V_+ \otimes V_-^*) \cong \bigoplus_{\mu \vdash R} S_\mu(V_+) \otimes S_{\mu^*}(V_-)^*.$$

Here we sum over all partition of R (equivalently Young diagram of size $|\mu| = R$) and μ^* is the conjugate partition (or transposed Young diagram).

Note that since $\wedge^\bullet \mathfrak{p} = \wedge^\bullet (V_+ \otimes V_-^*)$ the group $\mathrm{SL}(q) = \mathrm{SL}(V_-)$ acts on $\wedge^\bullet \mathfrak{p}^*$. We will be mainly concerned with elements of $(\wedge^\bullet \mathfrak{p}^*)^{\mathrm{SL}(q)}$ — that is elements that are trivial on the V_- -side. Note that in general $(\wedge^\bullet \mathfrak{p}^*)^{\mathrm{SL}(q)}$ is *strictly* contained in $(\wedge^\bullet \mathfrak{p}^*)^{\mathrm{SO}(q)}$. Recall that if q is even there exists an

invariant element

$$e_q \in (\wedge^q \mathfrak{p}^*)^{\mathrm{SO}(p) \times \mathrm{SL}(q)},$$

the *Euler class/form*. We define $e_q = 0$ if q is odd.

The subalgebra $\wedge^\bullet(\mathfrak{p}^*)^{\mathrm{SL}(q)}$ of $\wedge^\bullet(\mathfrak{p}^*)$ is invariant under $K_\infty = \mathrm{SO}(p) \times \mathrm{SO}(q)$. Hence, we may form the associated subbundle

$$F = D \times_{K_\infty} (\wedge^\bullet \mathfrak{p}^*)^{\mathrm{SL}(q)}$$

of the bundle

$$D \times_{K_\infty} (\wedge^\bullet \mathfrak{p}^*)$$

of exterior powers of the cotangent bundle of D . The space of sections of F is invariant under the Laplacian and hence under harmonic projection, compare [15, bottom of p. 105]. The space of sections of F is a subalgebra of the algebra of differential forms.

We denote by $\bar{H}^\bullet(Y)^{\mathrm{SC}}$ the corresponding subspace of $\bar{H}^\bullet(Y)$. Note that when $q = 1$ we have $\bar{H}^\bullet(Y)^{\mathrm{SC}} = \bar{H}^\bullet(Y)$ and when $q = 2$ we have

$$\bar{H}^\bullet(Y)^{\mathrm{SC}} = \bigoplus_{n=1}^p \bar{H}^{n,n}(Y).$$

As above we may associate to n -dimensional totally positive sub-quadratic spaces of V special cycles of (real) codimension nq in Y . They yield classes in $\bar{H}^{nq}(Y)$; in fact these classes belong to the subspace $\bar{H}^{nq}(Y)^{\mathrm{SC}}$ and we have:

$$(0.3) \quad \bar{H}^\bullet(Y)^{\mathrm{SC}} = \bigoplus_{r=0}^{\lfloor p/2 \rfloor} \bigoplus_{k=0}^{p-2r} e_q^k \bar{H}_{\mathrm{cusp}}^{r \times q}(Y, E(\lambda)),$$

(see [2]). Compared with the usual Hodge-Lefschetz decomposition, we call $\bar{H}^{n \times q}(Y)$ the *primitive part* of $\bar{H}^{nq}(Y)^{\mathrm{SC}}$. We see then that if q is odd the above special classes have *pure refined Hodge type* and if q is even each such class is the sum of at most $n + 1$ refined Hodge types. In what follows we will consider the primitive part of the special cycles i.e. their projections into the subspace associated to the refined Hodge type $n \times q$. Then our main result is:

Theorem 0.3.1. *Let Y_K be a connected arithmetic manifold associated to $\mathrm{SO}(V)$. Then the subspace $\bar{H}^{n \times q}(Y_K, \mathbb{C})$ is spanned by the Poincaré dual of special cycles with rational coefficients for $n < \min\{\frac{p+q-1}{3}, \frac{p}{2}\}$.*

Remark 0.3.2. The bound $\frac{p+q-1}{3}$ is conjectured to be the sharp bound; see [2] for some evidences.

In small degree it follows from works of Zucker that $\bar{H}^\bullet(Y) = H^\bullet(Y)$, we review these results in §6. If $q = 1$, Y_K is a congruence hyperbolic manifold and the special cycles on Y_K are linear combinations of totally geodesic submanifolds. We obtain

Corollary 0.3.3. *Let Y be a smooth hyperbolic manifold of dimension p . Then for all $n < \frac{p}{3}$ the groupe $H^n(Y, \mathbb{Q}) = \bar{H}^n(Y, \mathbb{Q})$ is spanned by the classes of totally geodesic submanifolds of codimension n .*

Similar as in [2], our proof of Theorem 0.3.1 is also valid for cohomology groups with non-trivial coefficients (cf. [2] Theorem 1.13). The argument is basically the same but would require much more notations. For the ease of readers, we restrict ourselves to the case of trivial coefficients in this paper.

0.4. Outline. The Sections 2-4 are devoted to generalizing the result in [2] to all discrete square integrable automorphic representation of orthogonal groups with non-zero low degree cohomology at infinity. We prove our main result in Section 6 and 7 using Kudla-Millson's special theta lifting theory.

1. NOTATION AND CONVENTIONS

1.1. General notations. Throughout this paper, let \mathbb{A} be the adèle ring of \mathbb{Q} and $\Gamma_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We write \mathbb{A}_{∞} and \mathbb{A}_f for its infinite and finite component respectively. We denote by $|\cdot|_p$ the absolute value on local fields \mathbb{Q}_p and $|\cdot|_{\mathbb{A}} = \prod_p |\cdot|_p$ the absolute value of adelic numbers in \mathbb{A} .

Let G be a classical group over \mathbb{Q} and let $G(\mathbb{A})$ be the group of adelic numbers. Fix a minimal parabolic \mathbb{Q} -subgroup P_0 with Levi decomposition $P_0 = M_0N_0$ and let A_0 be the maximal \mathbb{Q} -split torus in the center of M_0 . The Weyl group of A_0 in G is denote by $W(G)$. For a standard parabolic subgroup $P = MN$ containing P_0 , let A be the maximal split \mathbb{Q} -torus of M and denote by $\Delta(P, A)$ the set of simple roots. Write $\mathfrak{a}_M = \text{Hom}_{\mathbb{R}}(X(M)_{\mathbb{Q}}, \mathbb{R})$ and $\mathfrak{a}_M^* = X(M)_{\mathbb{Q}} \otimes \mathbb{R}$, with natural pairing $\langle \cdot, \cdot \rangle$, where $X(M)_{\mathbb{Q}}$ is the group of characters of M which are defined over \mathbb{Q} . The Harish-Chandra map $H_M : M(\mathbb{A}) \rightarrow \mathfrak{a}_M$ is defined by

$$(1.1) \quad \exp(\langle H_M(m), \chi \rangle) = |\chi(m)|_{\mathbb{A}}, \text{ for all } \chi \in \mathfrak{a}_M^*.$$

We extend H_M to $H_P : P \rightarrow \mathfrak{a}_M$ by extending it trivially on $N(\mathbb{A})$.

1.2. Automorphic forms. Let $G(\mathbb{A})^1 = \{x \in G(\mathbb{A}) : |\chi(x)| = 1, \chi \in X(G)_{\mathbb{Q}}\}$ be the subgroup of $G(\mathbb{A})$. The quotient $X_G := G(\mathbb{Q}) \backslash G(\mathbb{A})^1$ has finite volume with respect to the natural Haar measure. Let $L^2(X_G)$ be the Hilbert space of square integrable functions on X_G . According to Langlands' spectrum decomposition theory, the discrete part $L_{\text{dis}}^2(X_G)$ of $L^2(X_G)$

$$L_{\text{dis}}^2(X_G) = \widehat{\bigoplus} m(\pi)\pi$$

is the Hilbert sum of all irreducible $G(\mathbb{A})$ -submodules π in $L^2(X_G)$ with finite multiplicity $m(\pi)$.

1.3. Orthogonal groups. Here we fix some notations for orthogonal groups. Let V be a non-degenerate quadratic space of dim m over \mathbb{Q} and let $G = \text{SO}(V)$ be the corresponding special orthogonal group. Set $N = m$ if m is even and $N = m - 1$ if m is odd. Throughout this paper, we denote by

$\mathrm{SO}(n)$ the special orthogonal group associated to

$$J = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}.$$

The group $G = \mathrm{SO}(V)$ is an inner form of a quasi-split form G^* , where G^* is the odd orthogonal group $\mathrm{SO}(m)$ when m is odd or the outer twist $\mathrm{SO}(m, \eta)$ of the split group $\mathrm{SO}(m)$ when m is even. Let θ_N be the automorphism of $\mathrm{GL}(N)$:

$$\theta_N(g) = J \cdot {}^t g^{-1} \cdot J^{-1}, \quad g \in \mathrm{GL}(N)$$

and let $\widetilde{\mathrm{GL}}(N) = \mathrm{GL}(N) \rtimes \langle \theta_N \rangle$. The complex dual of G^* is $G^\vee = \mathrm{Sp}(N, \mathbb{C})$ if m is odd and $G^\vee = \mathrm{SO}(N, \mathbb{C})$ if m is even. In the first case the L -group of G is ${}^L G = G^\vee \times \Gamma_{\mathbb{Q}}$ whereas, in the second case, we have ${}^L G = G^\vee \rtimes \Gamma_{\mathbb{Q}}$, where $\Gamma_{\mathbb{Q}}$ acts on G^\vee by an order 2 automorphism — trivial on the kernel of η — and fixes a splitting, see [4, p. 79] for an explicit description.

We finally denote by $\Pi \mapsto \Pi^\theta$ the action of the automorphism $\theta = \theta_N$ on representations of $\mathrm{GL}(N)$.

2. ARTHUR'S CLASSIFICATION THEORY

In this section, we review Arthur's theory of classification of square integrable automorphic representations of special orthogonal groups.

2.1. Preliminaries. Let G be an algebraic group over \mathbb{Q} . Any $\pi \in \mathcal{A}_2(G)$ has a unique decomposition

$$\pi = \otimes_v \pi_v,$$

where π_v is an admissible representation of $G(\mathbb{Q}_v)$ over all places. Moreover, π_v is unramified for all v outside a finite set S of places. Langlands and Arthur theories investigate how the discrete automorphic representation are assembled from the local representations.

We first recall that the local Langlands classification says that the set of admissible representations of $G(\mathbb{Q}_v)$ are parametrized by equivalence classes of morphisms,

$$(2.1) \quad \phi_v : L_{\mathbb{R}} \longrightarrow {}^L G.$$

where $L_{\mathbb{Q}_\infty} = W_{\mathbb{R}}$, if $v = \infty$, and $L_{\mathbb{Q}_p} = W_{\mathbb{Q}_p} \times \mathrm{SU}(2)$, if v is a (finite) prime, in any case $W_{\mathbb{Q}_v}$ is the local Weil group of \mathbb{Q}_v and ${}^L G$ the complex dual group. We call ϕ_v a Langlands parameter and denote by $\prod(\phi_v)$ the packet of admissible representations associated to ϕ_v . When $G = \mathrm{GL}(N)$, the local Langlands packet $\prod(\phi_v)$ contains exactly one element.

2.2. Formal parameters of orthogonal groups. Arthur [1] parametrizes discrete automorphic representations of G by some set of formal sums of formal tensor products (called global Arthur parameter)

$$(2.2) \quad \Psi = \boxplus(\mu_j \boxtimes R_{a_j}),$$

where each μ_j is an irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}(d_j)$, R_{a_j} denotes the irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ of dimension a_j , $\sum a_j d_j = N$, and we have:

- (1) the $\mu_j \boxtimes R_{a_j}$ are all distinct, and
- (2) for each j , we have $\mu_j^\theta = \mu_j$.

To each such global Arthur parameter Ψ , we associate an irreducible representation Π_Ψ of the general linear group $\mathrm{GL}(N, \mathbb{A})$: the representation Π_Ψ is the induced representation

$$(2.3) \quad \mathrm{ind}(\Pi_{\mu_1, a_1} \otimes \Pi_{\mu_2, a_2} \cdots \otimes \Pi_{\mu_\ell, a_\ell}),$$

where the inducing parabolic subgroup of $\mathrm{GL}(N)$ has Levi factor $\mathrm{GL}(d_1 a_1) \times \cdots \times \mathrm{GL}(d_\ell a_\ell)$. Here Π_{μ_j, a_j} is the unique Langlands quotient of the normalized parabolically induced representation

$$(2.4) \quad \mathrm{ind}(\mu_j | \det |_{\mathbb{A}}^{\frac{a_j-1}{2}} \otimes \mu_j | \det |_{\mathbb{A}}^{\frac{a_j-3}{2}} \otimes \cdots \otimes \mu_j | \det |_{\mathbb{A}}^{\frac{1-a_j}{2}}).$$

from a parabolic subgroup of $\mathrm{GL}(d_j a_j)$ whose Levi factor is the group of block diagonal matrices $\mathrm{GL}(d_j) \times \cdots \times \mathrm{GL}(d_j)$ (a_j -copies).

2.3. Let $k = \mathbb{Q}_v$ with v finite or infinite. Arthur has duplicated the construction from the global case to define the local parameters of admissible representations of $G(k)$. More precisely, one can define the local Arthur parameter of $G(k)$ to be the formal symbol

$$(2.5) \quad \Psi_k = \boxplus(\mu_j \boxtimes R_{a_i})$$

where μ_i is a tempered, θ -stable, irreducible representation of $\mathrm{GL}(d_i, k)$ that is square integrable modulo the center and R_{a_i} is the a_i -dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$. Similar as the global construction (2.4), it determines a unique irreducible representation Π_{Ψ_k} of $\mathrm{GL}(N, k)$.

By local Langlands correspondence, the local Arthur parameter Ψ_k can be represented by a continuous homomorphism

$$(2.6) \quad \Psi_k : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G,$$

which has bounded image on its restriction to L_k and is algebraic on $\mathrm{SL}(2, \mathbb{C})$. By abuse of notation, we still use Ψ_k to denote this homomorphism. One attach an L -map $\phi_{\Psi_k} : L_k \rightarrow {}^L G$ to Ψ_k by composition, i.e.

$$(2.7) \quad \phi_{\Psi_k}(w) = \Psi_k \left(w, \left(\begin{array}{c} |w|^{1/2} \\ |w|^{-1/2} \end{array} \right) \right).$$

Note that if G is not quasi-split, the map ϕ_{Ψ_k} might not be relevant (in Langlands' sense) and therefore does not define an L -parameter in general.

However outside a finite set of places ϕ_{Ψ_k} indeed defines a (relevant) L -parameter.

Remark 2.3.1. As in [2], here we ignore the complication of lack of generalized Ramanujan conjecture just for simplicity of notations. In general setting, we need to introduce a larger set of homomorphisms without the boundedness condition. Then the approximation to Ramanujan's conjecture in [39] is enough for our purpose (see also [5] §3.1).

2.4. Arthur's packets. Given a local Arthur parameter Ψ_k , Arthur has associated a finite packet $\prod(\Psi_k)$ of representations of $G(k)$ with multiplicities to each Ψ_k using the twisted Langlands-Shelstad transfer. The details of transfer are not important for us, so we omit the description and only recall the relevant properties of Arthur local packets. We will first describe these properties when G is quasi-split.

2.5. Suppose that $G = G^*$ is quasi-split. Let π be a unitary representation of $G = G(k)$. By the local Langlands' classification [10, Theorem XI.2.10] there exist 'Langlands data' P, σ, λ , with $\lambda \in (\mathfrak{a}_P^*)^+$, such that π is the unique quotient of the corresponding standard induced module. The parabolic subgroup P comes with a canonical embedding of its dual chamber $(\mathfrak{a}_P^*)^+$ into the closure $\overline{(\mathfrak{a}_B^*)^+}$ of the dual chamber of the standard Borel subgroup B of $\mathrm{GL}(N)$. We shall write Λ_π for the corresponding image of the point λ ; and call it the *exponent* of the representation π . One writes

$$\Lambda' \leq \Lambda, \quad \Lambda', \Lambda \in \overline{(\mathfrak{a}_B^*)^+},$$

if $\Lambda - \Lambda'$ is a nonnegative integral combination of simple roots of (B, A_B) . This determines a partial order.

The key property of local Arthur packets that we will use is the following proposition that follows from [1, Theorem 2.2.1 and Eq. (2.2.12)]. By local Langlands correspondence, each unitary representation π of $G(k)$ belongs to a unique L -packet $\prod(\phi)$ associated to an L -parameter $\phi : L_k \rightarrow {}^L G$. Moreover: if π and π' both belong to $\prod(\phi)$ then $\Lambda_\pi = \Lambda_{\pi'}$, we may therefore define Λ_ϕ as their common value.

Proposition 2.5.1. *Let Ψ_k be a local Arthur parameter. Then the associated L -packet $\prod(\phi_{\Psi_k})$ is included in the Arthur packet $\prod(\Psi_k)$. Moreover: if $\pi \in \prod(\Psi_k)$ then*

$$\Lambda_\pi \leq \Lambda_{\phi_{\Psi_k}}.$$

Note that a global Arthur parameter Ψ gives rise to a local Arthur parameter Ψ_v at each place v . The global part of Arthur's theory (see [1, Theorem 1.5.2]) then implies:

Proposition 2.5.2. *Let π be an irreducible automorphic representation of $G(\mathbb{A})$. Then there exists a global Arthur parameter Ψ such that $\pi_v \in \prod(\Psi_v)$ at each place v . Moreover, there exists a finite set S of places of \mathbb{Q} containing all infinite places such that for all $v \notin S$, the representation π_v is unramified and its L -parameter is ϕ_{Ψ_v} , in particular π_v belongs to the L -packet $\prod(\phi_{\Psi_v})$.*

2.6. Suppose now that G is **not quasi-split**. Recall the map ϕ_{Ψ_k} associated to a local Arthur parameter Ψ_k might not be relevant (in Langlands' sense) now and therefore might not define a local L -packet. In particular Proposition 2.5.1 does not make sense. Arthur stable trace formula however allows to compare the discrete automorphic spectrum of G with that of G^* . This is the subject of work in progress of Arthur, the results being announced in [1, Chapter 9]. We will only need weak version of them that can be directly deduced from what Arthur has already written in the quasi-split case. It first follows from [1, Proposition 9.5.2] that:

- (1) it corresponds to any irreducible automorphic representation π of $G(\mathbb{A})$ a global Arthur parameter Ψ , and
- (2) at each place, one can still define a local packet $\prod(\Psi_v)$ of irreducible unitary representations of $G(\mathbb{Q}_v)$ (via the stable trace transfer),

in such a way that Proposition 2.5.2 still holds when S is chosen so that for all $v \notin S$, the group $G(\mathbb{Q}_v) = G^*(\mathbb{Q}_v)$ is quasi-split. We subsume this in the following

Proposition 2.6.1. *Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. Then there exists a global Arthur parameter Ψ and a finite set S of places of \mathbb{Q} containing all Archimedean ones such that for all $v \notin S$, the group $G(\mathbb{Q}_v) = G^*(\mathbb{Q}_v)$ is quasi-split, the representation π_v is unramified and the L -parameter of π_v is ϕ_{Ψ_v} .*

We now explain how to replace Proposition 2.5.1. Let v be a place of \mathbb{Q} and let $k = \mathbb{Q}_v$. By composition (2.7) the local Arthur parameter Ψ_k still defines an L -parameter for $\mathrm{GL}(N)$. Let Λ_{Ψ_k} be its exponent. Note that if $G(k) = G^*(k)$ is quasi-split case we have:

$$\Lambda_{\Psi_k} = \Lambda_{\phi_{\Psi_k}}.$$

Now recall that if π is a unitary representation of $G = G(k)$ then π is the unique quotient of some standard module $\mathrm{St}(\pi)$ associated to a Langlands datum. Here again the inducing parabolic subgroup comes with a canonical embedding of its dual chamber into the closure of the dual chamber of the standard Borel subgroup B of $\mathrm{GL}(N)$. Therefore the exponent Λ_π still makes sense as an element of $(\mathfrak{a}_B^*)^+$.

Now following [1, pp. 66–72] we may replace Proposition 2.5.1 by the following proposition (see also [3, Appendix A] and [2, Appendix] for similar results and more details).

Proposition 2.6.2. *Let $k = \mathbb{Q}_v$ with v finite or not. For every irreducible unitary representation π of $G(k)$ that belongs to a local Arthur packet $\prod(\Psi_k)$ we have:*

$$\Lambda_\pi \leq \Lambda_{\Psi_k}.$$

Remark 2.6.3. (1) Due to our particular choice of groups here

$$\mathrm{St}(\pi) = \mathrm{ind} \left(\otimes_{(\rho, a, x)} \mathrm{St}(\rho, a) \mid \cdot \mid^x \otimes \sigma \right)$$

where each triplet (ρ, a, x) consists of a unitary cuspidal representation ρ of a linear group, a integer a and positive real number x , $\text{St}(\rho, a)$ is the corresponding Steinberg representation — the unique irreducible sub-representation of the representation induced from $|\rho| \cdot |^{(a-1)/2} \otimes \dots \otimes \rho| \cdot |^{(1-a)/2}$ — and σ is a tempered representation of an orthogonal group of the same type as $G(k)$ but of smaller dimension. The set of x 's with multiplicities is the same as the real part of the exponent of π .

- (2) Let ϕ be the L -parameter of π . By embedding ${}^L G$ into ${}^L \text{GL}(N)$, we may associate to ϕ an irreducible unitary representation of $\text{GL}(N, k)$ -module. Arthur's proof of [1, Proposition 3.4.3] in fact shows that the latter occurs as an (irreducible) sub-quotient of the local standard representation associated to the local Arthur parameter Ψ_k .

2.7. Automorphic L -functions. Let $\pi \in \mathcal{A}_2(G)$ and let $\Psi = \boxplus(\mu_i \boxtimes R_{a_i})$ be the associated global Arthur parameter. Let $L(s, \Pi_\Psi) = \prod_v L_v(s, \Pi_\Psi)$ be the L -function of Π_Ψ . At a unramified place v of π , we have

$$L(s, \pi_v) = L_v(s, \Pi_\Psi) = \prod_j L\left(s - \frac{a_j - 1}{2}, \mu_{j,v}\right) \times \dots \times L\left(s - \frac{1 - a_j}{2}, \mu_{j,v}\right),$$

where $L(s, \pi_v)$ is the local L -function of π_v . Let S be a finite set of places that contains the finite set of Proposition 2.6.1 and the set of places where some $(\mu_i)_v$ is ramified. Then, for $v \notin S$, the partial L -function $L^S(s, \pi)$ is defined as the formal Euler product $\prod_{v \notin S} L(s, \pi_v)$ and we have

$$L^S(s, \pi) = L^S(s, \Pi_\Psi),$$

by Proposition 2.6.1. For a self-dual automorphic character η , we can simply write $L^S(s, \eta \times \pi)$ as the partial L -function associated to the parameter $\boxplus((\eta \otimes \mu_i) \boxtimes R_{a_i})$.

Similarly, we can define the L -function $L(s, \Pi_\Psi, r_G)$ attached to Ψ , where r_G is the finite dimensional representation Sym^2 if m is odd and \wedge^2 if m is even.

2.8. As in [2], we say an automorphic representation $\pi \in \mathcal{A}_2(\text{SO}(V))$ is *highly non-tempered* if its global Arthur parameter contains a factor $\eta \boxtimes R_a$, where η is a quadratic character and $3a > m - 1$. Assuming further that π has a regular infinitesimal character, the poles of the partial L -function of a highly non-tempered representations can be easily determined using the following lemma.

Lemma 2.8.1. *Suppose $\pi \in \mathcal{A}_2(\text{SO}(V))$ is highly non-tempered with corresponding factor $\eta \boxtimes R_a$ in its global Arthur parameter Ψ . If π has a regular infinitesimal character, then Ψ must have the form:*

$$(2.8) \quad \Psi = (\boxplus_{(\tau,b)} \tau \boxtimes R_b) \boxplus \eta \boxtimes R_a,$$

where each pair (τ, R_b) satisfies the condition either $b < a$ or $b = a$ and $\tau \neq \eta$, and $\sum_{(\tau, b)} bd_\tau + a = N$. Moreover, the partial L -function $L^S(s, \Pi_\Psi)$ is holomorphic for $\operatorname{Re}(s) > \frac{a+1}{2}$ and has a simple pole at $s = \frac{a+1}{2}$.

Proof. See [2] Lemma 4.3. ♣

3. COHOMOLOGICAL UNITARY REPRESENTATIONS FOR ORTHOGONAL GROUPS

In this section, we review the cohomological representations of classical groups and determine the (local) Arthur parameter of square integrable automorphic representations of $\operatorname{SO}(V)(\mathbb{A})$ with cohomology at infinity. The local parameters have been computed in [2]. We refer the readers to [2, Section 4 and Appendix] for more details.

3.1. Cohomological $(\mathfrak{g}, K_{\mathbb{R}})$ -module. Let $G_{\mathbb{R}}$ be a real connected reductive group with finite center. Let $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ be a maximal compact subgroup and denote by θ the associated Cartan involution. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the corresponding Cartan decomposition of $\mathfrak{g}_0 = \operatorname{Lie}(G_{\mathbb{R}})$. In this paper, we write $\mathfrak{a} = (\mathfrak{a}_0)_{\mathbb{C}}$ for the complexification of a real Lie algebra \mathfrak{a}_0 .

A unitary representation $(\pi_{\mathbb{R}}, V_{\pi_{\mathbb{R}}})$ is *cohomological* if the associated $(\mathfrak{g}, K_{\mathbb{R}})$ -module $(\pi_{\mathbb{R}}^{\infty}, V_{\pi_{\mathbb{R}}}^{\infty})$ has nonzero relative Lie algebra cohomology, i.e.

$$H^{\bullet}(\mathfrak{g}, K_{\mathbb{R}}; V_{\pi_{\mathbb{R}}}^{\infty}) \neq 0.$$

3.2. The unitary cohomological $(\mathfrak{g}, K_{\mathbb{R}})$ -modules have been classified by Vogan and Zuckerman [62] as following: let $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ be a Cartan subalgebra of \mathfrak{k}_0 . For $x \in \mathfrak{t}_0$, let \mathfrak{q} be the sum of nonnegative eigenspaces of $\operatorname{ad}(x)$. Then \mathfrak{q} is a θ -stable parabolic subalgebra of \mathfrak{g} with a θ -stable Levi decomposition

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}.$$

The normalizer of \mathfrak{q} in G is the connected Levi subgroup $L \subseteq G$ with Lie algebra $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$. Via cohomological induction \mathfrak{q} determines a unique irreducible unitary representation $A_{\mathfrak{q}}$ of G . More precisely, let $\mathfrak{h} \supseteq \mathfrak{t}$ to a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and we fix a positive system $\Delta^+(\mathfrak{h}, \mathfrak{g})$ of roots of \mathfrak{h} in \mathfrak{g} . Let ρ be the half sum of roots in $\Delta^+(\mathfrak{h}, \mathfrak{g})$ and $\rho(\mathfrak{u} \cap \mathfrak{p})$ the half sum of roots of \mathfrak{t} in $\mathfrak{u} \cap \mathfrak{p}$. Denote by $\mu(\mathfrak{q})$ the irreducible representation of $K_{\mathbb{R}}$ of highest weight $2\rho(\mathfrak{u} \cap \mathfrak{p})$. Then we have

Proposition 3.2.1. *The module $A_{\mathfrak{q}}$ is the unique irreducible unitary $(\mathfrak{g}, K_{\mathbb{R}})$ -module such that:*

- (1) $A_{\mathfrak{q}}$ contains the $K_{\mathbb{R}}$ -type $\mu(\mathfrak{q})$ occurring with multiplicity one.
- (2) $A_{\mathfrak{q}}$ has the infinitesimal character ρ .

Moreover, let $(\pi_{\mathbb{R}}, V_{\pi_{\mathbb{R}}})$ be an irreducible unitary $(\mathfrak{g}, K_{\mathbb{R}})$ -module such that

$$H^*(\mathfrak{g}, K_{\mathbb{R}}; V_{\pi_{\mathbb{R}}}^{\infty}) \neq 0.$$

Then there is a θ -stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of \mathfrak{g} so that $\pi_{\mathbb{R}} \cong A_{\mathfrak{q}}$ and

$$(3.1) \quad H^i(\mathfrak{g}, K; V_{\pi}) \cong \text{Hom}_{\mathfrak{l} \cap \mathfrak{t}}(\wedge^{i-\dim(\mathfrak{u} \cap \mathfrak{p})}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}).$$

Note that the isomorphism class of $A_{\mathfrak{q}}$ only depends on the intersection $\mathfrak{u} \cap \mathfrak{p}$, we may furthermore choose \mathfrak{q} such that the Levi subgroup L associated to \mathfrak{l} has no non-compact (non-abelian) factor.

3.3. Now let $G_{\mathbb{R}} = \text{SO}_0(p, q)$ and fix a maximal compact subgroup $K_{\mathbb{R}} \cong \text{SO}(p) \times \text{SO}(q)$. The unitary $(\mathfrak{g}, K_{\mathbb{R}})$ -modules with nonzero low degree cohomology have been classified as below:

Proposition 3.3.1. [2] *Let $A_{\mathfrak{q}}$ be a cohomological $(\mathfrak{g}, K_{\mathbb{R}})$ -module. Suppose that $R = \dim(\mathfrak{u} \cap \mathfrak{p})$ is strictly less than $p + q - 3$ and $pq/4$. Then either $L = C \times \text{SO}(p - 2n, q)$ with $C \subseteq K_{\mathbb{R}}$ and $R = nq$ or $L = C \times \text{SO}(p, q - 2n)$ with $C \subseteq K_{\mathbb{R}}$ and $R = np$.*

By Proposition 3.2.1, the irreducible summands of the decomposition of $\wedge^* \mathfrak{p}$ give a refined Hodge structure of $H^*(\mathfrak{g}, K_{\mathbb{R}}, V_{\pi})$. Recall that the space $\mathfrak{p} = \mathbb{C}^p \otimes (\mathbb{C}^q)^*$, where \mathbb{C}^p (resp. \mathbb{C}^q) is the standard representation of $\text{SO}(p, \mathbb{C})$ (resp. $\text{SO}(q, \mathbb{C})$). Then there is a decomposition

$$(3.2) \quad \wedge^R \mathfrak{p} = \bigoplus_{\mu \vdash R} \mathbb{S}_{\mu}(\mathbb{C}^p) \otimes \mathbb{S}_{\mu^*}(\mathbb{C}^q)^*$$

where μ is a partition of R , μ^* is the conjugation partition and $\mathbb{S}_{\mu}(\mathbb{C}^p)$ is the Schur functor associated to μ . Then the cohomological module $A_{\mathfrak{q}}$ with Levi subgroup $C \times \text{SO}(p - 2n, q)$ in Proposition 3.3.1 corresponds to the irreducible representation $\mathbb{S}_{[n \times q]}(\mathbb{C}^p) \otimes (\wedge^q \mathbb{C}^q)^n$ of $\text{SO}(p, \mathbb{C}) \times \text{SO}(q, \mathbb{C})$, where $\mathbb{S}_{[n \times q]}(\mathbb{C}^p)$ is the harmonic Schur functor associated to the partition (q, q, \dots, q) of $R = nq$ (see [2, §5.11]).

Moreover, let $e_q \in \Omega^q(D)$ be the Euler form defined in [2], which is a $K_{\mathbb{R}}$ invariant q -form on D . It is nonzero only when q is even. Wedging with the (qr) -form e_q^r induces a morphism:

$$(3.3) \quad e_q^r : H^i(\mathfrak{g}, K_{\mathbb{R}}; A_{\mathfrak{q}}) \rightarrow H^{i+qr}(\mathfrak{g}, K_{\mathbb{R}}; A_{\mathfrak{q}})$$

which is an isomorphism when q is even. We refer the readers to [2, Proposition 5.15] for the description of $H^i(\mathfrak{g}, K_{\mathbb{R}}; A_{\mathfrak{q}})$.

In case $G_{\mathbb{R}} = \text{SO}(p, 2)$, we write $A_{n,n}$ for the cohomological $(\mathfrak{g}, K_{\mathbb{R}})$ -module with corresponding Levi subgroup $U(1)^n \times \text{SO}(p - 2n, 2)$ for $n < \frac{p}{2}$. Then $H^{2k}(\mathfrak{g}, K_{\mathbb{R}}, A_{n,n})$ has a pure Hodge structure of type (k, k) . In small degree, this refined Hodge structure and the isomorphism (3.3) will recover the Hodge-Lefschetz structure on the cohomology group of connected Shimura varieties via Matsushima formula (see §6).

3.4. Local parameters of cohomological representations. For the special orthogonal group $G_{\mathbb{R}} = \mathrm{SO}_0(p, q)$, let $\pi_{\mathbb{R}}$ be a unitary representation of $G_{\mathbb{R}}$ whose underlying $(\mathfrak{g}, K_{\mathbb{R}})$ -module is equivalent to $A_{\mathfrak{q}}$ associated to the Levi subgroup $L = \mathrm{U}(1)^r \times \mathrm{SO}(p - 2r, q)$. A conjectural description of the local Arthur packets that can contain $\pi_{\mathbb{R}}$ is given in [2, §6.6]. It would follow that if $\pi_{\mathbb{R}}$ belongs to a local Arthur packet $\prod(\Psi)$ then the real part of the exponent Λ_{Ψ} satisfies

$$(3.4) \quad \mathrm{Re}(\Lambda_{\Psi}) \geq (m - 2r - 2, m - 2r - 4, \dots, 0, \dots, 0, \dots, 2r + 2 - m).$$

To our knowledge the proof of (3.4) is still open.

However letting P_0 be the minimal standard parabolic subgroup of $G_{\mathbb{R}}$ with Langlands decomposition $P_0 = {}^0MA_0N$, where $M \cong \mathrm{SO}(p - q, \mathbb{R}) \times (\mathbb{R}^{\times})^q$ and A is the maximal \mathbb{R} -split torus of P_0 and using the standard labeling of simple roots:

$$\Delta(P_0, A_0) = \{\epsilon_1, \epsilon_2, \dots, \epsilon_q\},$$

we may write $\pi_{\mathbb{R}}$ as the unique Langlands quotient of the normalized induced representation

$$(3.5) \quad \mathrm{ind}_{P_0}^G(e^{\nu} \otimes \sigma)$$

(see [62, §7]) or, equivalently, $\pi_{\mathbb{R}}$ is the unique irreducible submodule of the normalized induced representation $\mathrm{ind}_{P_0}^G(e^{-\nu} \otimes \sigma)$ where

$$(3.6) \quad \nu = \frac{m - 2r - 2}{2}\epsilon_1 + \frac{m - 2r - 4}{2}\epsilon_2 + \dots + \frac{m - 2r - 2q}{2}\epsilon_q \in \mathfrak{a}_0^*,$$

called the exponent of $\pi_{\mathbb{R}}$, is in the positive chamber of the Weyl group and σ is an irreducible representation on the compact group $\mathrm{SO}(p - q)$.¹ Using Langlands' notation we say that $\pi_{\mathbb{R}}$ is a Langlands' quotient of a standard representation whose non-tempered "exponents" are

$$(z\bar{z})^{m-2r-2}, (z\bar{z})^{m-2r-4}, \dots, (z\bar{z})^{m-2r-2q}.$$

It follows in particular from Proposition 2.6.2 that if Ψ is a local Arthur parameter such that $\pi_{\mathbb{R}}$ belongs to $\prod(\Psi)$ then the real part of the exponent Λ_{Ψ} satisfies:

$$(3.7) \quad \mathrm{Re}(\Lambda_{\Psi}) \geq (m - 2r - 2, m - 2r - 4, \dots, m - 2r - 2q, 0, \dots, \dots, 0, 2r + 2q - m, \dots, 2r + 2 - m).$$

It is slightly weaker than the expected bound (3.4) but it will be enough for our purpose.

¹The Harish-Chandra module of σ is determined by $\rho + \rho_u$ (cf. See [10] Chapter VI when $q = 1$).

3.5. Cohomological automorphic representation for $\mathrm{SO}(V)$. Coming back to the global case, we consider the Arthur parameters of the automorphic representations $\pi = \otimes_v \pi_v \in \mathcal{A}_2(\mathrm{SO}(V))$ with π_∞ cohomological. Let Ψ be the global Arthur parameter of π . By Proposition 3.2.1, we know that the infinitesimal character of π_∞ is regular (equal to that of the trivial representation) and so is the infinitesimal character of π^{GL} . This implies that for any two factors $\mu \boxtimes R_a$ and $\mu' \boxtimes R_{a'}$ in Ψ , we have $\mu \neq \mu'$ unless a or $a' = 1$ (see [2, Appendix]).

Furthermore: if π_∞ is the cohomological representation of $\mathrm{SO}(p, q)$ associated to some Levi subgroup $L = \mathrm{SO}(p - 2r, q) \times \mathrm{U}(1)^r$, the inequality (3.7) on exponents at infinity forces Ψ_∞ to contain a “big” $\mathrm{SL}(2, \mathbb{C})$ -factor R_a with $a \geq m - 2r - 1$. The following theorem is proved in [2], see Proposition 6.9. Here we quickly review the main steps of the proof.

Theorem 3.5.1. *Let $\pi = \otimes_v \pi_v$ be an irreducible square integrable automorphic representation of $G(\mathbb{A})$. If π_∞ is a cohomological representation of $\mathrm{SO}(p, q)$ associated to a Levi subgroup $L = \mathrm{SO}(p - 2r, q) \times \mathrm{U}(1)^r$ with $p > 2r$ and $m - 1 > 3r$, then π is highly non-tempered.*

Proof. Since the $\mathrm{SL}(2, \mathbb{C})$ -factors of the Arthur parameter is global, the global Arthur parameter Ψ of π contains some factor $\mu \boxtimes R_a$ with $a \geq m - 2r - 1 > \frac{m-1}{3}$. Then either μ is a character or μ is 2-dimensional by dimension consideration. We have to exclude the latter case. If $a > (m - 1)/2$, this is automatic. In general, we first observe that the explicit description of π_∞ as a Langlands’ quotient not only shows π_∞ has a big exponent but that this exponent is a *character exponent*. Then the proof [2, Proposition 6.9] (postponed to the Appendix there) shows that Ψ_∞ also has a big character exponent and this excludes the possibility that μ is 2-dimensional. ♣

4. RESIDUAL REPRESENTATIONS

In this section, we study the residual spectrum of $G = \mathrm{SO}(V)$ via the theory of Eisenstein series developed by Langlands (cf. [34, 45]). Our main purpose is to show that non-cuspidal representations with cohomology at infinity are residues of Eisenstein series from the rank one maximal parabolic subgroup, see Theorem 4.4.2.

4.1. Residual spectrum. The right regular representation of $G(\mathbb{A})$ acting on the Hilbert space $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ has continuous spectrum and discrete spectrum:

$$L^2(G(\mathbb{Q}) \backslash G(\mathbb{A})) = L_{\mathrm{dis}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A})) \oplus L_{\mathrm{cont}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A})).$$

We are interested in the discrete spectrum. Using his theory of Eisenstein series, Langlands decomposes this space as:

$$L_{\mathrm{dis}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A})) = \bigoplus_{(M, \sigma)} L_{\mathrm{dis}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))_{(M, \sigma)},$$

where (M, σ) is a Levi subgroup with a cuspidal automorphic representation σ (taken up to conjugacy and whose central character restricts trivially to the center of G) and $L_{\text{dis}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))_{(M, \sigma)}$ is a space of residues of Eisenstein series associated to (M, σ) .

Note that the subspace

$$L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A})) := \oplus_{(G, \pi)} L_{\text{dis}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))_{(G, \pi)}$$

is the space of cuspidal automorphic representations of G . Its orthogonal complement in $L_{\text{dis}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ is called the residual spectrum and we denote it by $L_{\text{res}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. In order to understand it we have to introduce the relevant Eisenstein series.

4.2. Eisenstein series. Let $P = MN$ be a maximal standard parabolic subgroup of G and let (σ, V_σ) be an irreducible automorphic representation of $M(\mathbb{A})$. Choosing a section f_s of the parabolically induced representation

$$I(s, \sigma) = \text{ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma | \det |^s),$$

where $|\cdot|$ is the adelic absolute value, and the induction is normalized,² we define the Eisenstein series by the analytic continuation from the domain of convergence of the series

$$(4.1) \quad E(f_s, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} f_s(\gamma g).$$

When σ is cuspidal, Langlands (see [45, Sect. IV.1]) shows that the Eisenstein series (4.1) is a meromorphic function of s with finitely many poles, all simple, in the region $\text{Re}(s) > 0$. Then $L_{\text{dis}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))_{(M, \sigma)}$ is the space spanned by the residues $((s - s_0)E(f_s, g))_{s=s_0}$ of the Eisenstein series (4.1) at a simple pole $s = s_0$ when varying the sections f_s in V_σ .

4.3. Parabolic subgroups. The maximal parabolic subgroups of $G = \text{SO}(V)$ can be described as below: let

$$V = V_t + U_{t,t},$$

be the Witt decomposition of V , where V_t is anisotropic and $U_{t,t}$ is t -copies of hyperbolic planes spanned by u_i, u'_i for $i = 1, 2, \dots, t$ with $(u_i, u'_j) = \delta_{ij}$. The integer t is the Witt index of V . Fixing an anisotropic quadratic space V_t , for any $0 \leq d \leq t$, we write

$$V = U_d + V_d + U'_d,$$

where U_d, U'_d are subspaces spanned by u_1, \dots, u_d and u'_1, \dots, u'_d respectively, and $V_t \subset V_d$. The subgroup $P \subseteq \text{SO}(V)$ which stabilizes U'_d is a standard parabolic subgroup over \mathbb{Q} with Levi decomposition $P = MN$, where

$$(4.2) \quad M \cong \text{GL}(U_d) \times \text{SO}(V_d).$$

²Note that, since P is maximal, the group of characters of M is one dimensional generated by \det .

Let $W(M)$ be the set of elements w in the Weyl group of G , of minimal length in their left coset modulo the Weyl group of M , such that wMw^{-1} is the Levi factor of a standard parabolic subgroup of G . Then $W(M) = \{1, w_0\}$, where w_0 is a Weyl group element that can be lifted to $G(\mathbb{Q})$ so that $w_0Nw_0^{-1} = N^-$ is the opposite of N . Recall that P is said to be *self-associate* if $w_0Mw_0^{-1} = M$. One can prove (see [60]) that P as above is self-associate, unless t is odd and $d = t$. Since we will only be concerned with low rank parabolic subgroup, we may always assume that P is self-associate for the sake of simplicity.

4.4. Eisenstein series associated to non-cuspidal representations.

The paper [42] deals with Eisenstein series where the hypothesis of the cuspidality of $\sigma \in \mathcal{A}_2(M)$ is removed. There the following theorem is proved.

Let $\pi = \otimes_v \pi_v$ be an irreducible sub-representation of $G(\mathbb{A})$ in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ and let Ψ be the global Arthur parameter of π .

Theorem 4.4.1. *Suppose that π_∞ is a unitary cohomological representation. Then: either π is cuspidal or Ψ contains a factor $\rho \boxtimes R_a$ for some $a > 1$ and $\rho \in \mathcal{A}_c(\mathrm{GL}(d))$ and there exists $\pi' \in \mathcal{A}_2(\mathrm{SO}(V_d))$ such that*

- (1) *the global Arthur parameter of π' is obtained from that of π by replacing the factor $\rho \boxtimes R_a$ by $\rho \boxtimes R_{a-2}$,*
- (2) *the Eisenstein series $E(f_s, g)$ associated to $\rho \otimes \sigma$ have at most a simple pole at $s_0 = \frac{1}{2}(a-1)$, and*
- (3) *the representation π is realized in the space spanned by the residues at $s_0 = \frac{1}{2}(a-1)$ of the Eisenstein series associated to $\sigma = \rho \otimes \pi'$.*

We shall abbreviate the last item by:

$$(4.3) \quad \pi \hookrightarrow \left((s - (a-1)/2) E(s, \rho \times \pi') \right)_{s=(a-1)/2}.$$

Here we precise the preceding theorem assuming strong conditions on π_∞ .

Proposition 4.4.2. *Suppose that π_∞ is isomorphic to the cohomological representation of $\mathrm{SO}(p, q)$ associated to the Levi subgroup $L = \mathrm{SO}(p-2r, q) \times \mathrm{U}(1)^r$. Then in Theorem 4.4.1, if π is not cuspidal, we have:*

- (1) *the representation ρ is a quadratic character (so that $d = 1$) and $a = m - 2r - 1$, and*
- (2) *the inclusion (4.3) is an equality.*

Remark 4.4.3. 1. Since $a \geq 2$ recall from §3.5 that being a character uniquely determines ρ .

2. Since π is irreducible, if (4.3) is an equality then the space spanned by the residues at $s_0 = \frac{1}{2}(a-1)$ of the Eisenstein series associated to $\sigma = \eta \otimes \pi'$ is irreducible.

Proof of Proposition 4.4.2(1). Let π be as in Theorem 4.4.1 and not cuspidal. Then the Arthur parameter Ψ associated to π contains a factor $\rho \boxtimes R_a$ for some $a > 1$ and $\rho \in \mathcal{A}_c(\mathrm{GL}(d))$ and there exists $\pi' \in \mathcal{A}_2(\mathrm{SO}(V_d))$ such that (4.3) holds.

Denote by ρ_∞ the local Archimedean component of ρ . To prove Proposition 4.4.2(1), we shall prove that ρ_∞ is a character.

Because π_∞ is cohomological, the infinitesimal character of ρ_∞ is integral and this implies that ρ_∞ is an induced representation

$$(4.4) \quad \text{ind}(\delta_1, \dots, \delta_t) \quad \text{or} \quad \text{ind}(\delta_1, \dots, \delta_t, \varepsilon)$$

(normalized induction from the standard parabolic subgroup of type $(2, \dots, 2)$ or $(2, \dots, 2, 1)$ depending on the parity of d), where the δ_j 's are discrete series of $\text{GL}(2, \mathbb{R})$ and ε is a quadratic character of \mathbb{R}^\times .³ Now by localizing (4.3) we conclude (see [42] for more details) that

$$(4.5) \quad \pi_\infty \hookrightarrow \text{I} \left(-\frac{a-1}{2}, \rho_\infty \otimes \pi'_\infty \right),$$

where π'_∞ is the local component of π' at infinity. Since π' is square integrable, the representation π'_∞ is unitary. Moreover, after Salamanca-Riba (see [57, Theorem 1.8]), we also know that the representation π'_∞ is cohomological and has a regular, integral infinitesimal character. Write π'_∞ as the Langlands submodule of a standard module I' .

We claim that the induced representation $\text{I} \left(-\frac{a-1}{2}, \rho_\infty \otimes I' \right)$ contains a unique irreducible sub-module, which is the Langlands sub-module of this induced representation. Write I' as induced of twists of discrete series and character, see e.g. [2, Appendix, §16.12]. In this way $\text{I} \left(-\frac{a-1}{2}, \rho_\infty \otimes I' \right)$ is a standard module. If it is in Langlands' position, then the claim immediately follows (see e.g. [10, Proposition 2.6]). To prove the claim in general, we will reduce to this case. To do that we look at the exponents of I' (see again [2, Appendix, §16.12] for more about exponents). Fix δ' a representation that occurs in the induction. Then δ' is either a discrete series of parameter ℓ' or a quadratic character corresponding to $\ell' = 0$.

Now let x the biggest possible exponent for such a representation. Then the infinitesimal character of π' "contains" the segment $\ell' + [x, -x]$ and is symmetric around 0. Denote by ℓ the parameter of one of the δ_j above. To put the induced representation in the positive Weyl chamber (i.e. in Langlands' position) we have to exchange $\delta_j |\det|^{(a-1)/2}$ and $\delta' |\det|^x$ if $(a-1)/2 < x$. This is done using the following intertwining operator (between two $\text{GL}(4, \mathbb{R})$ -representations):

$$\text{ind}(\delta_j |\det|^{s_1} \otimes \delta' |\det|^{s_2}) \mapsto \text{ind}(\delta' |\det|^{s_2} \otimes \delta_j |\det|^{s_1})$$

(normalized induction from the standard parabolic of type $(2, 2)$.) To prove the claim it suffices to show that the right-hand side induced representation (which is in Langlands' position) is irreducible. We distinguish between the two following cases:

1. Let us first assume that $\ell' \neq 0$ (i.e. that δ' is a discrete series and not a character). Then it follows from [46, Lemme I.7] (and in fact, in this case

³Note that only one quadratic character can occur because of the regularity of the infinitesimal character.

it is due to Speh) that the induced representation

$$\text{ind}(\delta'|\det|^x \otimes \delta_j|\det|^{(a-1)/2})$$

can be reducible only when we have both $\ell' + (a-1)/2 < \ell + x$ and $-\ell' + (a-1)/2 < -\ell + x$. The second inequality is equivalent to $\ell + (a-1)/2 < \ell' + x$. But, by the regularity of the infinitesimal character of π_∞ , this implies $\ell + (a-1)/2 < \ell' - x$. From which we conclude that

$$x + \ell < \ell' - (a-1)/2 < \ell' + (a-1)/2.$$

This proves that our parameters avoid the bad cases (of possible reducibility).

2. Let us now assume that $\ell' = 0$. In that case, the infinitesimal character of π'_∞ contains the segment $[x, -x]$ which is symmetric around 0, so we certainly have $\ell + (a-1)/2 > x$ and irreducibility follows.

This concludes the proof of the claim.

Proposition 4.4.2(1) follows: since π_∞ is not a Langlands sub-module of a standard representation containing a non trivial twist of a discrete series, we have $t = 0$ in (4.4) and ρ is a character and $d = 1$. Finally, computing the L-parameter of π_∞ and using (3.6) we conclude that $(a-1)/2 = m/2 - r - 1$, i.e. $a = m - 2r - 1$.



Remark 4.4.4. It follows from the proof above, and inspection of the Langlands parameter of π_∞ , that the representation π'_∞ is isomorphic to the cohomological representation of $\text{SO}(p-1, q-1) = \text{SO}(V_1)$ associated to the Levi subgroup $L = \text{SO}(p-1-2r, q-1) \times \text{U}(1)^r$.

Before entering the proof of Proposition 4.4.2(2) we shall recall some basic facts on the relation between residues of Eisenstein series and residues of intertwining operators.

4.5. Use of intertwining operators. If P is self-associate and σ is cuspidal then Langlands [34], see also [45, Sect. IV.1], has shown that the poles of $E(f_s, g)$ coincide with the poles of its constant term

$$E_P(f_s, g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} E(f_s, ng) dn$$

along the parabolic subgroup P . On the other hand, by [45, Sect. II.1.7], the constant term equals

$$(4.6) \quad E_P(f_s, g) = f_s(g) + M(s, w_0, \sigma) f_s(g),$$

where

$$M(s, w_0, \sigma) : \text{I}(s, \sigma) \rightarrow \text{I}(-s, \sigma)$$

given — whenever the integral below is convergent — by

$$(4.7) \quad (M(s, w_0, \sigma) f)(g) = \int_{N(\mathbb{Q}) \cap N^-(\mathbb{Q}) \backslash N(\mathbb{A})} f(w_0^{-1} ng) dn \quad (f \in \text{I}(s, \sigma))$$

is the standard intertwining operator defined in [45, Sect. II.1.6]. Note that we have identified σ with its conjugate by w_0 and furthermore note that $w_0(s) = -s$. Away from its poles $M(s, w_0, \sigma)$ intertwines $I(s, \sigma)$ and $I(-s, \sigma)$.

In general, if σ is non-cuspidal (but discrete), an Eisenstein series associated to σ is holomorphic at $s = s_0$ if and only if all its cuspidal constant terms are holomorphic, see e.g. [45, Sect. I.4.10]. If $\sigma = \rho \otimes \pi'$, as in Theorem 4.4.1, the cuspidal constant terms are obtained as linear combinations of intertwining operators defined on representations induced from $\rho|\cdot|^s \otimes \pi'_Q$ where π'_Q is a cuspidal constant term along the unipotent radical of some standard parabolic subgroup Q .

In order to understand the singularities of the Eisenstein series, one is therefore reduced to study the singularities of the standard intertwining operators.

4.6. Normalization of intertwining operators. Intertwining operators are meromorphic and can be decomposed as Eulerian products using local intertwining operators (that are also meromorphic); see [45, §II.1.9]. Consider a square integrable representation $\sigma = \rho \otimes \pi' \in \mathcal{A}_2(M)$, as in Theorem 4.4.1, with $\rho \in \mathcal{A}_c(\mathrm{GL}(d))$ and $\pi' \in \mathcal{A}_2(\mathrm{SO}(V_d))$. If $v = p$ is a finite place where everything is unramified the value of a local (at p) intertwining operator is — up to an invertible holomorphic function — an explicit expression of local L-functions:

$$\frac{L_p(2s, \rho_p \times (\pi'_p)^{\mathrm{GL}})}{L_p(2s+1, \rho_p \times (\pi'_p)^{\mathrm{GL}})} \cdot \frac{L_p(2s, r_G, \rho_p)}{L_p(2s+1, r_G, \rho_p)},$$

where $(\pi'_p)^{\mathrm{GL}}$ is the local functorial lift of π_p to a representation of $\mathrm{GL}_{N_1}(\mathbb{Q}_p)$ (with $N_1 = N - 2d$), see e.g. [60].

Note that if Ψ' is the global Arthur parameter of π' , it follows from Proposition 2.5.2 (and the fact that being unramified we have $p \notin S$) that $(\pi'_p)^{\mathrm{GL}}$ is associated to the L -parameter $\phi_{\Psi'_p}$. This suggests to normalize the global intertwining operators using a product of L -functions (and there inverses) analogue as the one above but using the global Arthur parameter Ψ' : let $\Pi_{\Psi'}$ be the representation of $\mathrm{GL}(N_1)$ defined in (2.3). In [42], the rank one global standard intertwining operator $M(s, w_0, \rho \otimes \pi')$ are normalized in terms of the Arthur parameter Ψ' ,

$$(4.8) \quad N_{\Psi'}(s, w_0, \rho \otimes \pi') := r(s, \Psi')^{-1} M(s, w_0, \rho \otimes \pi'),$$

where the normalizing factor

$$(4.9) \quad r(s, \Psi') = \frac{L(2s, \rho \times \Pi_{\Psi'})}{L(2s+1, \rho \times \Pi_{\Psi'})} \cdot \frac{L(2s, r_G, \rho)}{L(2s+1, r_G, \rho)}$$

is the quotient of L -functions defined in §2.7. Write $N_{\Psi'}(s, w_0, \rho \otimes \pi') = \otimes_v N_{\Psi'_v}(s, w_0, \rho_v \times \pi'_v)$. Then if $v = p$ is unramified, the local factor of

$N_{\Psi'}(s, w_0, \rho \times \pi')$ at p is the identity on spherical vectors and at any place v

$$N_{\Psi'_v}(s, w_0, \rho_v \otimes \pi'_v) = r(s, \Psi'_v)^{-1} M(s, w_0, \rho_v \otimes \pi'_v)$$

intertwines the induced representations $I(s, \rho_v \otimes \pi'_v)$ and $I(-s, \rho_v \otimes \pi'_v)$. If one can show that these normalized intertwining operators are holomorphic, one gets that the poles of the global intertwining operator $M(s, w_0, \eta \otimes \pi')$ are precisely the poles of the global normalizing factor $r(s, \Psi')$. The latter being expressed in terms of standard L -functions, we can easily determine its poles.

4.7. Proof of Proposition 4.4.2(2). Let π be as in Theorem 4.4.1 and not cuspidal. Since Proposition 4.4.2(1) is proved, the Arthur parameter Ψ associated to π contains a factor $\eta \boxtimes R_a$ for some $a > 1$ and η a quadratic character and there exists $\pi' \in \mathcal{A}_2(\mathrm{SO}(V_1))$ such that (4.3) holds. Moreover: the global Arthur parameter of π' is obtained from that of π by replacing the factor $\eta \boxtimes R_a$ by $\eta \boxtimes R_{a-2}$.

Recall the following notation introduced in [42]: if $\tilde{\pi}$ is any automorphic representation (not necessarily irreducible), $P \subset P_d$ is a standard maximal parabolic subgroup and $\rho \in \mathcal{A}_c(\mathrm{GL}(d))$, we denote by $\tilde{\pi}_P[\rho]$ the projection on the ρ -cuspidal part of the constant term of $\tilde{\pi}$ along P .

Now let $\tilde{\pi}$ be the (automorphic) representation in the space spanned by the residues at $s_0 = \frac{1}{2}(a-1)$ of the Eisenstein series associated to $\sigma = \eta \otimes \pi'$.

Lemma 4.7.1. *Suppose that π_∞ is isomorphic to the cohomological representation of $\mathrm{SO}(p, q)$ associated to the Levi subgroup $L = \mathrm{SO}(p-2r, q) \times \mathrm{U}(1)^r$. Then $\tilde{\pi}_P[\rho] = \{0\}$ unless $P = P_1$ (with Levi $\mathrm{GL}(1) \times \mathrm{SO}(V_1)$) and $\rho = \eta$. Furthermore: in this latter case $\tilde{\pi}_P[\rho]$ is precisely the image of the induced representation $I(s_0, \eta \otimes \pi')$ by the intertwining operator $((s-s_0)M(s, w_0, \eta \otimes \pi'))_{s=s_0}$.⁴*

Proof of Lemma 4.7.1. We first remark that if there is only $\tilde{\pi}_{P_1}[\eta] \neq 0$, then the last statement follows from [45, §II.1.7]. So it suffices to show that only $\tilde{\pi}_{P_1}[\eta]$ is nonzero. According to Remark 4.4.4, we know that π'_∞ is isomorphic to the cohomological representation of $\mathrm{SO}(p-1, q-1) = \mathrm{SO}(V_1)$ associated to the Levi subgroup $L = \mathrm{SO}(p-1-2r, q-1) \times \mathrm{U}(1)^r$. By induction (on $p+q$), we may therefore assume that Lemma 4.7.1 holds for π' . (Note that the induction start with $\dim V = 2$ or 3 where the Lemma is obvious.)

Now if $\tilde{\pi}_P[\rho] \neq \{0\}$ for some $P \neq P_1$ (and some ρ), then π' is not cuspidal. (Recall that Langlands has proved that if $\sigma = \eta \otimes \pi'$ is cuspidal the associated Eisenstein series has only one non-trivial constant term, along P_1 .) And since constant terms of Eisenstein series are sums of Eisenstein series of constant terms (see [45, §II.1.7]), we have $\pi'_{P'}[\rho'] \neq \{0\}$ for some standard parabolic subgroup $P' \subset \mathrm{SO}(V_1)$ with Levi isomorphic to $\mathrm{GL}(d) \times \mathrm{SO}(V_{d+1})$

⁴Note that this operator is well defined since, as a pole of the Eisenstein series associated to $\eta \otimes \pi'$, s_0 is at most simple (see Theorem 4.4.1).

and some $\rho' \in \mathcal{A}_c(\mathrm{GL}(d))$. Now since Lemma 4.7.1 holds for π' , we conclude that $d = 1$, $\rho' = \eta$ and π' must be a quotient of an induced representation $\mathrm{I}(s_0, \eta \otimes \pi'')$ for some square integrable automorphic representation π'' on $\mathrm{SO}(V_2)$ with the *same* s_0 . But this is not possible because of the form of the Arthur parameter of π' . Similarly, one shows that $\tilde{\pi}_{P_1}[\rho] = 0$ unless $\rho = \eta$. \clubsuit

Let us now prove that Lemma 4.7.1 implies Proposition 4.4.2(2). The representation $\tilde{\pi}$ injects into the sum of its constant terms along standard maximal parabolic subgroups. But this sum reduces to the image of the induced representation $\mathrm{I}(s_0, \eta \otimes \pi')$ by the intertwining operator $((s - s_0)M(s, w_0, \eta \otimes \pi'))_{s=s_0}$. We claim that the image of this intertwining operator is irreducible. This implies that $\tilde{\pi}$ is irreducible and the proposition follows.

To prove the claim, let us apply $((s - s_0)M(s, w_0, \eta \otimes \pi'))_{s=s_0}$ to some function f which is the characteristic function of a hyperspecial maximal compact subgroup outside a finite set of places. Let S be a finite set of places that contains all these bad places and all the places of ramification. Then for any $v \notin S$, the normalized local intertwining operator

$$(4.10) \quad N_{\Psi'_v}(s, w_0, \eta_v \otimes \pi'_v) = r(s, \Psi'_v)^{-1} M(s, w_0, \eta_v \otimes \pi'_v)$$

is the identity. It is proved in [44, §5.3.1] that for any finite $v \in S$ the image of the local intertwining operator (4.10) is either 0 or irreducible. We shall now prove that this is still true if $v = \infty$. First of all we remark that the standard intertwining operator $M(s, w_0, \eta_\infty \otimes \pi'_\infty)$ is holomorphic at $s = s_0$ with irreducible image. Indeed we have:

$$M(s, w_0, \eta_\infty \otimes \pi'_\infty) : \mathrm{I}(s, \eta_\infty \otimes \pi'_\infty) \rightarrow \mathrm{I}(-s, \eta_\infty \otimes \pi'_\infty).$$

But the exponents of π'_∞ are strictly less than s_0 (see the proof of Proposition 4.4.2(1)). So the induced modules are in Langlands position for $s = s_0$ and it follows from [10, Proposition 2.6] that $M(s, w_0, \eta_\infty \otimes \pi'_\infty)$ is holomorphic in $s = s_0$ with irreducible image. It remains to prove that the local normalization factor does not add a pole, i.e. that $r(s, \Psi'_\infty)$ does not vanish in $s = s_0$. But this follows from the explicit formula for $r(s, \Psi'_\infty)$, see [46] for details.

Since the global normalization factor has a simple pole at $s = s_0$, we know that the image of the intertwining operator $((s - s_0)M(s, w_0, \eta \otimes \pi'))_{s=s_0}$ is either 0 or irreducible. As it contains the space of π which is not 0, this concludes the proof of our claim and therefore of Proposition 4.4.2(2). \clubsuit

5. THETA CORRESPONDENCE FOR ORTHOGONAL GROUPS

The theory of Howe's theta correspondence between reductive dual pairs has been applied to construct many automorphic representations with non-zero cohomology at infinity (cf. [36]). Conversely, the results of [2] show that small degree cohomological *cuspidal* representations of $\mathrm{SO}(V)$ come

from the theta correspondence. In this section, we generalize the latter to non-cuspidal automorphic representations.

5.1. Theta correspondence. In this section, we will consider the reductive dual pair $(\mathrm{O}(V), \mathrm{Sp}(W))$, where W is a symplectic space over \mathbb{Q} of dimension $2n$ and $\mathrm{Sp}(W)$ is the associated symplectic group. Let $\widetilde{\mathrm{Sp}}(\cdot)$ be the metaplectic double cover of the symplectic group $\mathrm{Sp}(\cdot)$. Then we denote by $\mathrm{Mp}(W)$ the symplectic group $\mathrm{Sp}(W)$ if m is odd and the metaplectic group $\widetilde{\mathrm{Sp}}(W)$ if m is even. There is a natural morphism

$$(5.1) \quad \mathrm{O}(V) \times \mathrm{Mp}(W) \rightarrow \widetilde{\mathrm{Sp}}(V \otimes W).$$

We denote by $\mathrm{Mp}_{2n}(\mathbb{A})$, $\mathrm{O}_m(\mathbb{A})$ and $\widetilde{\mathrm{Sp}}_{2mn}(\mathbb{A})$ the adelic points of the groups $\mathrm{Mp}(W)$, $\mathrm{O}(V)$ and $\widetilde{\mathrm{Sp}}(V \otimes W)$ respectively.

Fix a nontrivial additive character ψ of \mathbb{A}/\mathbb{Q} , let ω_ψ be the (automorphic) Weil representation of $\mathrm{Mp}_{2mn}(\mathbb{A})$ realized in the Schrödinger model $\mathcal{S}(V^n(\mathbb{A}))$, where $\mathcal{S}(V^n(\mathbb{A}))$ is the space of Schwartz-Bruhat functions on $V^n(\mathbb{A})$. By restricting the Weil representation ω_ψ to $\mathrm{O}(V) \times \mathrm{Mp}(W)$, one can define the theta function on $\mathrm{O}(V) \times \mathrm{Mp}(W)$:

$$(5.2) \quad \theta_{\psi, \phi}(g, g') = \sum_{\xi \in V(\mathbb{Q})^n} \omega_\psi(g, g')(\phi)(\xi),$$

for each $\phi \in \mathcal{S}(V^n(\mathbb{A}))$. Given a cuspidal representation (τ, H_τ) of $\mathrm{Mp}_{2n}(\mathbb{A})$ and $f \in H_\tau$, the integral

$$(5.3) \quad \theta_{\psi, \phi}^f(g) = \int_{\mathrm{Mp}(\mathbb{Q}) \backslash \mathrm{Mp}(\mathbb{A})} \theta_{\psi, \phi}(g, g') f(g') dg',$$

defines an automorphic function on $\mathrm{O}_m(\mathbb{A})$, called the *global theta lift* of f . Let $\theta_{\psi, V}(\tau)$ be the space of all the theta liftings $\theta_{\psi, \phi}^f$ as f and ϕ vary. This is called the global ψ -theta lifting of τ to $\mathrm{O}_m(\mathbb{A})$.

Remark 5.1.1. When the space $\theta_{\psi, V}(\tau)$ contains a nonzero cuspidal automorphic function, it is known that $\tilde{\pi} \cong \theta_{\psi, V}(\tau)$ is irreducible and cuspidal (cf. [48, 28]). Moreover, it will be the first occurrence of the global ψ -theta lifting of τ in the Witt tower of quadratic spaces (cf. [55]).

5.2. For the special orthogonal group $\mathrm{SO}(V)$, we say that $\pi \in \mathcal{A}_2(\mathrm{SO}(V))$ is in the image of ψ -cuspidal theta correspondence from a smaller symplectic group if a lift $\tilde{\pi}$ of π to $\mathrm{O}(V)$ is contained in a global ψ -theta lifting from symplectic groups, i.e. there exists $\tau \in \mathcal{A}^c(\mathrm{Mp}(W))$ such that $\tilde{\pi} \hookrightarrow \theta_{\psi, V}(\tau)$. The restriction of $\tilde{\pi}$ to $\mathrm{SO}(V)$ is isomorphic to π and lift $\tilde{\pi}$ should be uniquely determined. Then one of the key results in [2] states as

Proposition 5.2.1. [2, Theorem 4.2] *Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with regular infinitesimal character. If π is highly non-tempered, then there exists a cuspidal representation τ of $\mathrm{Mp}_{2r}(\mathbb{A})$ such that*

π (up to a twist by a quadratic character) is in the image of ψ -cuspidal theta correspondence from a smaller symplectic group.

5.3. We also recall the local theta correspondence which will be used later. For temporary notations, we let $G = \mathrm{O}(V)(\mathbb{Q}_v)$ and $G' = \mathrm{Mp}(W)(\mathbb{Q}_v)$. Similarly as the global case, the local theta correspondence is obtained from the restriction of the local Weil representation on $G \times G'$. By abuse of notations, we denote by $\omega_{\psi,v}$ the pullback of the Weil representation to $G \times G'$. Then for an irreducible admissible representation τ of G' , we can define

$$S(\tau) = \omega_{\psi,v} / \bigcap_{\phi \in \mathrm{Hom}_G(\omega_{\psi,v}, \tau)} \ker \phi$$

to be the maximal quotient of $\omega_{\psi,v}$ on which G acts as a multiple of τ . By [49, Lemma III.4] if v is finite there is a smooth representation $\Theta(\tau)$ of G such that

$$S(\tau) \simeq \tau \otimes \Theta(\tau)$$

and $\Theta(\tau)$ is unique up to isomorphism. The Howe duality principle (cf. [27, 63, 22]) asserts further that if $\Theta(\tau) \neq 0$, then it has a unique irreducible quotient $\theta_{\psi,v}(\tau)$ and two representations τ_1, τ_2 are isomorphic if $\theta_{\psi,v}(\tau_1) \cong \theta_{\psi,v}(\tau_2)$. If $v = \infty$, Howe [27] proves that if π and π' are two irreducible (\mathfrak{g}, K) -modules for G and G' then, in the category of (\mathfrak{g}, K) -modules, we have:

$$\dim \mathrm{Hom}_{G \times G'}(\omega_{\psi,v}, \pi \otimes \pi') \leq 1.$$

This will be enough for our purpose: there is at most one irreducible representation $\theta_{\psi,v}(\tau)$ such that $\mathrm{Hom}_{G \times G'}(\omega_{\psi,v}, \tau \otimes \theta_{\psi,v}(\tau)) \neq \{0\}$. The irreducible representation $\theta_{\psi,v}(\tau)$ is called the *local ψ -theta lifting* of τ . As indicated by Rallis in [55], there is a natural relation between global and local theta liftings:

Proposition 5.3.1. *For $\tau = \otimes_v \tau_v$ of $\mathrm{Mp}_{2n}(\mathbb{A})$, the local theta lifting $\theta_{\psi,v}(\tau_v)$ occurs as the irreducible constituent of the local component of the global theta lifting $\theta_{\psi,V}(\tau)$. In particular, if $\theta_{\psi,V}(\tau)$ is cuspidal, then $\theta_{\psi,V}(\tau) \cong \otimes_v \theta_{\psi,v}(\tau_v)$.*

Remark 5.3.2. At the Archimedean place $v = \infty$, Li [36] has shown that the cohomological (\mathfrak{g}, K) -module $A_{\mathfrak{q}}(\lambda)$ occurs as the (\mathfrak{g}, K) -module of representations in Howe's theta correspondence of the holomorphic unitary discrete series representation of $G'(\mathbb{R}) = \mathrm{Mp}_{2n}(\mathbb{R})$.

5.4. **Constant terms of theta lifts.** Next, we recall the constant term of global theta lifts computed by Rallis. With the notation as before, let \tilde{P}_i be the standard maximal parabolic group of $\mathrm{O}(V)$ with the Levi decomposition $\tilde{P}_i = \tilde{M}_i \tilde{N}_i$ and

$$\tilde{M}_i = \mathrm{GL}_i \times \mathrm{O}(V_i), \quad i \leq t.$$

For a global theta lift $\theta_{\psi,\phi}^f$ defined in (5.3), Rallis [55] has shown that the constant term

$$(5.4) \quad \theta_{\psi,\phi}^f(g)_{(\tilde{P}_i)} = \int_{\tilde{N}_i(\mathbb{Q}) \backslash \tilde{N}_i(\mathbb{A})} \theta_{\psi,\phi}^f(ng) dn$$

is the lift associated to a theta series in a smaller number of variables. More precisely, using the decomposition $V = U_i + V_i + U'_i$, one can write a vector $v \in V^n$ as

$$v = \begin{bmatrix} u_i \\ v_i \\ u'_i \end{bmatrix}, \quad u_i \in U_i^n, \quad v_i \in V_i^n, \quad u'_i \in (U'_i)^n.$$

Then we have

Lemma 5.4.1. *Let (τ, H_τ) be an irreducible cuspidal representation of $\mathrm{Mp}(W)$. For a cusp form $f \in H_\tau$, the constant term of $\theta_{\psi,V}^f$ along the unipotent radical N_i is*

$$(5.5) \quad \theta_{\psi,\phi}^f(g)_{(\tilde{P}_i)} = \int_{\mathrm{Mp}_{2n}(\mathbb{Q}) \backslash \mathrm{Mp}_{2n}(\mathbb{A})} \theta'(g', g) f(g') dg'$$

where

$$\theta'(g', g) = \sum_{\xi_i \in V_i^n(\mathbb{Q})} \int_{u_i \in U_i^n(\mathbb{A})} \omega_\psi(g', g)(\phi) \begin{bmatrix} u_i \\ \xi_i \\ 0 \end{bmatrix} du_i.$$

Proof. The computation (5.5) can be found in [55] Theorem I.1.1 (2) when m is even. The proof is similar when m is odd. \clubsuit

From Lemma 5.4.1, we see that the constant term of $\theta_{\psi,V}(\tau)$ along P_i lies exactly in the theta lifting of τ to the smaller orthogonal group $\mathrm{O}(V_i)$ where GL_i acts by a character. So the only nonzero cuspidal constant term of $\theta_{\psi,V}(\tau)$ is along the parabolic subgroup contained in some P_s ($s \leq d$) and stabilizing the flag consisting of s isotropic spaces. Moreover: this constant term is isomorphic to

$$(5.6) \quad \eta| \cdot |_{\mathbb{A}}^{-m/2+n+1} \otimes \dots \otimes \eta| \cdot |_{\mathbb{A}}^{-m/2+n+s} \otimes \theta_{\psi,V_s}(\tau),$$

where the irreducible representation $\theta_{\psi,V_s}(\tau)$ is the first occurrence (and cuspidal by [55]) of the theta correspondence of τ , and η is a character that only depends on the choices made for the Weil representation (see [32, Remark 2.3] where Kudla computes the Jacquet modules of local theta lift, see also [49, p. 69]). By the square integrability criterion in [45, I.4.11], each automorphic representation $\theta_{\psi,V_i}(\tau)$ is square integrable for $i \leq s$. Specializing Lemma 5.4.1 we can compute its constant term:

Corollary 5.4.2. *The constant term of $\theta_{\psi,V_i}(\tau)$ along the unipotent radical of a parabolic stabilizing an isotropic line of V_i is equal to*

$$\eta| \cdot |_{\mathbb{A}}^{-m/2+n+i+1} \otimes \theta_{\psi,V_{i+1}}(\tau).$$

5.5. A surjectivity theorem. The following result shows that, under some conditions, representations with cohomology come from the theta correspondence.

Theorem 5.5.1. *Let $\pi \in \mathcal{A}_2(\mathrm{SO}(V))$ be a square integrable automorphic representation of $\mathrm{SO}(V)$. Suppose that the local Archimedean component of π is isomorphic to the cohomological representation of $\mathrm{SO}(p, q)$ associated to the Levi subgroup $L = \mathrm{SO}(p - 2r, q) \times \mathrm{U}(1)^r$. Then, if $3r < m - 1$, there exists a cuspidal representation τ of $\mathrm{Mp}_{2n}(\mathbb{A})$ such that π (up to a twist by a quadratic character) is in the image of theta lift of τ .*

Proof. By Proposition 5.2.1, it suffices to consider the case when π is non-cuspidal. It then follows from Proposition 4.4.2 that the Arthur parameter of π contains a factor $\eta \boxtimes R_a$, where η is a quadratic character and $a = m - 2r - 1 > 1$.

Now note that the Arthur parameter of $\pi_1 = \pi'$ in Proposition 4.4.2 is obtained by replacing the factor $\eta \boxtimes R_a$ in the parameter of π by the factor $\eta \boxtimes R_{a-2}$. So if π_1 is non-cuspidal and $a - 2 > 1$, it will again be the residue of Eisenstein series for some $\pi_2 \in \mathcal{A}_2(\mathrm{SO}(V_2))$ by applying Proposition 4.4.2 one more time.

If π_1 is cuspidal, we shall put $t = 1$ and otherwise inductively define π_2, \dots, π_t until we arrive at a cuspidal representation π_t . Note that the processus stops at finite t less or equal to the Witt index of V and even $t \leq \min(p - 2r, q)$. We furthermore remark that $m/2 - r - i + 1 = (p - 2r + q - 2i)/2 + 1 > 0$. We therefore obtain a sequence of irreducible representations $\pi_i \in \mathcal{A}_2(\mathrm{SO}(V_i))$, $i = 0, 1, 2 \dots t$, such that

(1) $\pi_0 = \pi$, we have

$$\pi_{i-1} = ((s_i - m/2 + r + i)E(s_i, \eta \times \pi_i))_{s_i = m/2 - r - i},$$

for each $i \leq t$, and $\pi_t \in \mathcal{A}_c(\mathrm{SO}(V_t))$ is cuspidal;

(2) the Arthur parameter of π_{i+1} is the same as the Arthur parameter of π_i except that the factor $\eta \boxtimes R_{m-2r-2i-1}$ is replaced by $\eta \boxtimes R_{m-2r-2i-3}$;

(3) at the Archimedean place $v = \infty$, the underlying (\mathfrak{g}, K_∞) -module $(\pi_i)_\infty^\infty \cong A_{\mathfrak{q}_i}$, where $A_{\mathfrak{q}_i}$ is associated to the Levi subgroup $\mathrm{SO}(p - 2r - i, q - i) \times \mathrm{U}(1)^r \subseteq \mathrm{SO}(p - i, q - i)$.

It follows from Theorem 3.5.1 that π_t is the image of a cuspidal representation τ of $\mathrm{Sp}(2r)$ using the θ -lifting: if $t = 0$, we use the hypothesis $r < (m - 1)/3$ and the assertion follows from Proposition 5.2.1. If $t > 0$, the wanted property follows from [48, Theorem on p. 203] if m is even, and [19, Theorem 5.1] if m is odd.⁵

We now want to prove that π itself is in the image of the theta correspondence. Let us take $\pi'_t = \pi_t$. According to Corollary 5.4.2 we can define

⁵It is not explicitly stated there that τ is cuspidal, but this follows from the Rallis theta tower property [55], since τ is the first occurrence.

inductively a sequence π'_i ($0 \leq i \leq t$) of square integrable automorphic representation of $\mathrm{SO}(V_i)$ as follows: let π'_i be the irreducible constituent in the θ -lift of τ on $\mathrm{SO}(V_i)$ whose constant term along the unipotent radical of a parabolic stabilizing an isotropic line of V_i contains $\eta| \cdot |_{\mathbb{A}}^{-m/2+r+i+1} \otimes \pi'_{i+1}$.

At infinity, the Archimedean component $(\pi'_i)_{\infty}$ is in the image of local theta correspondence of τ_{∞} . The local correspondence is known in this case, see [35], and its underlying (\mathfrak{g}, K) -module $(\pi'_i)_{\infty}^{\infty} \cong A_{\mathfrak{q}_i}$. We prove by induction that $\pi'_i = \pi_i$. This is true by construction of π'_t for $i = t$. Assume that it is true until i and let us prove it for $i - 1$. We know that π'_{i-1} is not cuspidal and that it satisfies the condition Proposition 4.4.2. We may therefore realize π'_{i-1} in the space of residues of the Eisenstein series constructed from some character η' and some irreducible representation $\pi' \in \mathcal{A}_2(\mathrm{SO}(V_i))$. Computing the constant term as explained above, we obtain that $\eta' = \eta$ and $\pi' = \pi'_i$. Now using the induction equality $\pi'_i = \pi_i$ and the irreducibility of the space of residues already proved, we conclude that $\pi'_i = \pi_i$, as wanted. This proves that $\pi = \pi'_0$ is in the image of the theta correspondence. \clubsuit

6. COHOMOLOGY OF ARITHMETIC MANIFOLDS

6.1. **Notations.** Let us take

$$\widehat{D} = G(\mathbb{R})/(\mathrm{SO}(p) \times \mathrm{SO}(q)),$$

and let D be a connected component of \widehat{D} . Let \widetilde{G} be the general spin group $\mathrm{Gspin}(V)$ associated to V . For any compact open subgroup $K \subseteq G(\mathbb{A}_f)$, we set \widetilde{K} to be its preimage in $\widetilde{G}(\mathbb{A}_f)$. Then we denote by X_K the double coset

$$\widetilde{G}(\mathbb{Q}) \backslash (\mathrm{SO}(p, q) \times \widetilde{G}(\mathbb{A}_f)) / (\mathrm{SO}(p) \times \mathrm{SO}(q)) \widetilde{K}.$$

Let $\widetilde{G}(\mathbb{Q})_+ \subseteq \widetilde{G}(\mathbb{Q})$ be the subgroup consisting of elements with totally positive spinor norm, which can be viewed as the subgroup of $\widetilde{G}(\mathbb{Q})$ lying in the identity component of the adjoint group of $\widetilde{G}(\mathbb{R})$. Write

$$\widetilde{G}(\mathbb{A}_f) = \coprod_j \widetilde{G}(\mathbb{Q})_+ g_j \widetilde{K},$$

one has that the decomposition of X_K into connected components is

$$X_K = \coprod_{g_j} \Gamma_{g_j} \backslash D,$$

where Γ_{g_j} is the image of $\widetilde{G}(\mathbb{Q})_+ \cap g_j \widetilde{K} g_j^{-1}$ in $\mathrm{SO}_0(p, q)$. When $g_j = 1$, we denote by Γ_K the arithmetic group $\Gamma_1 = K \cap G(\mathbb{Q})$ and $Y_K = \Gamma_K \backslash D$ the connected component of X_K . Throughout this section, we assume that Γ_1 is torsion free. The arithmetic manifold Y_K inherits a natural Riemannian metric from the Killing form on the Lie algebra of $G(\mathbb{R})$, making it a complete manifold of finite volume.

6.2. L^2 -cohomology on arithmetic manifolds. Let $\Omega_{(2)}^i(Y_K, \mathbb{C})$ be the space of \mathbb{C} -valued smooth square integrable i -forms on Y_K whose exterior derivatives are still square integrable. It forms a complex $\Omega_{(2)}^\bullet(Y_K, \mathbb{C})$ under the natural exterior differential operator

$$d : \Omega_{(2)}^i(Y_K, \mathbb{C}) \rightarrow \Omega_{(2)}^{i+1}(Y_K, \mathbb{C}).$$

The L^2 -cohomology $H_{(2)}^*(Y_K, \mathbb{C})$ of Y_K is defined as the cohomology of the complex $\Omega_{(2)}^\bullet(Y_K, \mathbb{C})$. With the distribution exterior derivative \bar{d} , one can work with the full L^2 -spaces $L_{(2)}^i(Y_K, \mathbb{C})$ instead of just smooth forms, i.e. $\omega \in L_{(2)}^i(Y_K, \mathbb{C})$ is a square integrable i -form and $\bar{d}\omega$ remains square integrable, then we can define the reduced L^2 -cohomology group to be

$$\bar{H}_{(2)}^i(Y_K, \mathbb{C}) = \{\omega \in L_{(2)}^i(Y_K, \mathbb{C}) : \bar{d}\omega = 0\} / \overline{\{\bar{d}L_{(2)}^{i-1}(Y_K, \mathbb{C})\}},$$

where $\overline{\{\bar{d}L_{(2)}^{i-1}(Y_K, \mathbb{C})\}}$ denotes the closure of the image $\text{Im}\bar{d}$ in $L_{(2)}^i(Y_K, \mathbb{C})$.

By Hodge theory, the group $\bar{H}_{(2)}^i(Y_K, \mathbb{C})$ is isomorphic to the space of L^2 -harmonic i -forms, which is a finite dimensional vector space with a natural Hodge structure (cf. [9]). As Y_K is complete, there is an inclusion

$$(6.1) \quad \bar{H}_{(2)}^i(Y_K, \mathbb{C}) \hookrightarrow H_{(2)}^i(Y_K, \mathbb{C}),$$

and it is an isomorphism when $H_{(2)}^i(Y_K, \mathbb{C})$ is finite dimensional.

Let $\Omega^i(Y_K, \mathbb{C})$ be the space of smooth i -forms on Y_K . The inclusion

$$\Omega_{(2)}^i(Y_K, \mathbb{C}) \hookrightarrow \Omega^i(Y_K, \mathbb{C})$$

induces a homomorphism

$$(6.2) \quad H_{(2)}^i(Y_K, \mathbb{C}) \rightarrow H^i(Y_K, \mathbb{C}),$$

between the L^2 -cohomology group and ordinary de-Rham cohomology group. We denote by $\bar{H}^i(Y_K, \mathbb{C})$ the image of $\bar{H}_{(2)}^i(Y_K, \mathbb{C})$ in $H^i(Y_K, \mathbb{C})$. In general, the mapping (6.2) is neither injective nor surjective, but as we will see later, (6.1) and (6.2) become isomorphisms when i is sufficiently small.

6.3. Zucker's Result. Now we review Zucker and Borel's results on comparing L^2 -cohomology groups with ordinary de-Rham cohomology groups. Let P_0 be a minimal parabolic subgroup of $G(\mathbb{R})$ and \mathfrak{q}_0 the associated Lie algebra with Levi decomposition $\mathfrak{q}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$. Let $A \subseteq P_0$ be the maximal \mathbb{Q} -split torus and \mathfrak{a}_0 the associated Lie algebra. We consider the Lie algebra cohomology $H^*(\mathfrak{u}, \mathbb{C})$ as a \mathfrak{l}_0 -module. Then Zucker (see also [11]) shows that

Theorem 6.3.1. [67, Theorem 3.20] *The mapping (6.2) is an isomorphism for $i \leq c_G$, where the constant*

$$c_G = \max\{k : \beta + \rho > 0 \text{ for all weights } \beta \text{ of } H^k(\mathfrak{u}, \mathbb{C})\},$$

where ρ be the half sum of positive roots of \mathfrak{a}_0 in \mathfrak{u} . In particular, c_G is at least the greatest number of $\{k : \beta + \rho > 0 \text{ for all weights } \beta \text{ of } \wedge^k \mathfrak{u}^*\}$ which is greater than $\lfloor \frac{m}{4} \rfloor$.

Remark 6.3.2. The result in [67] is actually much more general. Zucker has shown the existence of such a constant not only for cohomology groups with trivial coefficients, but also for cohomology groups with non-trivial coefficients.

Example 6.3.3. When $G(\mathbb{R}) = \mathrm{SO}(p, 1)$ and Y_K is a hyperbolic manifold, the constant c_G is equal to $\lfloor \frac{p}{2} \rfloor - 1$ (cf. [67, Theorem 6.2]) and thus we have the isomorphisms

$$(6.3) \quad \bar{H}_{(2)}^i(Y_K, \mathbb{C}) \xrightarrow{\sim} H_{(2)}^i(Y_K, \mathbb{C}) \xrightarrow{\sim} H^i(Y_K, \mathbb{C})$$

for $i \leq \lfloor \frac{p}{2} \rfloor - 1$.

Example 6.3.4. In case that $G(\mathbb{R}) = \mathrm{SO}(p, 2)$ and Y_K is locally Hermitian symmetric, we can have a better bound for i from Zucker's conjecture to ensure (6.2) being an isomorphism. Remember that the quotient Y_K is a quasi-projective variety with the Baily-Borel-Satake compactification \bar{Y}_K^{bb} , then Zucker's conjecture (cf. [38, 58]) asserts that there is an isomorphism

$$(6.4) \quad H_{(2)}^i(Y_K, \mathbb{C}) \cong IH^i(Y_K, \mathbb{C}),$$

where $IH^i(Y_K, \mathbb{C})$ is the intersection cohomology on \bar{Y}_K^{bb} . Since the boundary of \bar{Y}_K^{bb} has dimension at most one, we have an isomorphism

$$(6.5) \quad H_{(2)}^i(Y_K, \mathbb{C}) \cong IH^i(Y_K, \mathbb{C}) \cong H^i(Y_K, \mathbb{C}) = \bar{H}^i(Y_K, \mathbb{C}),$$

for $i < p - 1$. Moreover, a result of Harris and Zucker (cf. [25, Theorem 5.4]) shows that the map (6.5) is also a Hodge structure morphism. Therefore, $H^i(Y_K, \mathbb{C})$ has a pure Hodge structure when $i < p - 1$.

6.4. Spectrum decomposition of cohomology. Let $L^2(\Gamma_K \backslash G(\mathbb{R}))$ be the space of square integrable functions on $\Gamma_K \backslash G(\mathbb{R})$ and $L^2(\Gamma_K \backslash G(\mathbb{R}))^\infty$ the subspace of smooth vectors in $L^2(\Gamma_K \backslash G(\mathbb{R}))$. It is a (\mathfrak{g}, K_∞) -module and Borel shows that

$$H_{(2)}^*(Y_K, \mathbb{C}) \cong H^*(\mathfrak{g}, K_\infty; L^2(\Gamma_K \backslash G(\mathbb{R}))^\infty).$$

We can exploit Langlands' spectral decomposition of $L^2(\Gamma_K \backslash G(\mathbb{R}))$ to obtain the corresponding decomposition of the cohomology group.

Let $L_{\mathrm{dis}}^2(\Gamma_K \backslash G(\mathbb{R}))^\infty$ be the discrete spectrum of $L^2(\Gamma_K \backslash G(\mathbb{R}))^\infty$. Borel and Casselman [8] have shown that the reduced L^2 -cohomology group is isomorphic to the discrete part

$$H^*(\mathfrak{g}, K_\infty; L_{\mathrm{dis}}^2(\Gamma_K \backslash G(\mathbb{R}))^\infty)$$

of $H^*(Y_K, \mathbb{C})$. Note that the discrete spectrum of $L^2(\Gamma_K \backslash G(\mathbb{R}))$ decomposes as a Hilbert sum of irreducible $G(\mathbb{R})$ -modules with finite multiplicity, we

have

$$(6.6) \quad \bar{H}_{(2)}^*(Y_K, \mathbb{C}) \cong \bigoplus_{\pi_{\mathbb{R}}} m(\pi_{\mathbb{R}}) H^*(\mathfrak{g}, K_{\infty}; V_{\pi_{\mathbb{R}}}^{\infty}).$$

where $(\pi_{\mathbb{R}}, V_{\pi_{\mathbb{R}}})$ runs over all the unitary representation of $G(\mathbb{R})$ occurring in the discrete spectrum of $L^2(\Gamma_K \backslash G(\mathbb{R}))$ with multiplicity $m(\pi_{\mathbb{R}})$. The isomorphism (6.6) also yields a decomposition of $\bar{H}^i(Y_K, \mathbb{C})$ and we can denote by $\bar{H}^*(\mathfrak{g}, K_{\infty}; V_{\pi_{\mathbb{R}}}^{\infty})$ the corresponding image of $H^*(\mathfrak{g}, K_{\infty}; V_{\pi_{\mathbb{R}}}^{\infty})$ in $\bar{H}^i(Y_K, \mathbb{C})$.

Remark 6.4.1. Combined with Zucker's result and classification of cohomological $(\mathfrak{g}, K_{\infty})$ -modules, we can obtain a series of vanishing results of cohomology groups on arithmetic manifolds of orthogonal type. For instance, when $G(\mathbb{R}) = \mathrm{SO}(p, 2)$ and $p \geq 3$, we have $\bar{H}^1(Y_K, \mathbb{C}) \cong H^1(Y_K, \mathbb{C}) = 0$ because there is no $(\mathfrak{g}, K_{\infty})$ -module with non-zero first relative Lie algebra cohomology (cf. [2, §5.10]). This implies that the Albanese variety of a connected Shimura variety of orthogonal type is trivial (cf. [30]).

Similar as in [2], we can define the group $\bar{H}^k(X_K, \mathbb{C})$ as X_K is a finite disjoint union of arithmetic manifolds and set

- $\bar{H}^i(\mathrm{Sh}(G), \mathbb{C}) = \varinjlim_K \bar{H}^i(X_K, \mathbb{C})$.
- $\bar{H}^i(\mathrm{Sh}^0(G), \mathbb{C}) = \varinjlim_K \bar{H}^i(Y_K, \mathbb{C})$.

For a global representation

$$\pi = \pi_{\infty} \otimes \pi_f \in \mathcal{A}_2(G),$$

we write π_f^K as the finite dimensional subspaces of K -invariant vectors in π_f . Then we obtain the decomposition

$$(6.7) \quad \bar{H}^i(X_K, \mathbb{C}) \cong \bigoplus m_{\mathrm{disc}}(\pi) \bar{H}^i(\mathfrak{g}, K; V_{\pi_{\infty}}^{\infty}) \otimes \pi_f^K.$$

A global representation $\pi \in \mathcal{A}_2(G)$ has nonzero contribution to $\bar{H}_{(2)}^k(X_K, \mathbb{C})$ via (6.7) and hence to $\bar{H}_{(2)}^k(Y_K, \mathbb{C})$ only when it has cohomology at infinity, i.e. $V_{\pi_{\infty}}^{\infty} \cong A_{\mathfrak{q}}$ for some θ -stable parabolic subalgebra \mathfrak{q} in §3. Then one can define subspaces of $\bar{H}^i(Y_K, \mathbb{C})$ coming from special automorphic representations.

Definition 6.4.2.

(1) Let

$$\bar{H}^i(\mathrm{Sh}(G), \mathbb{C})_{A_{\mathfrak{q}}} \subseteq \bar{H}^i(\mathrm{Sh}(G), \mathbb{C})$$

be the subspace consisting of cohomology classes contributed from representations $\pi \in \mathcal{A}_2(G)$, whose infinite component π_{∞} has underlying $(\mathfrak{g}, K_{\infty})$ -module $A_{\mathfrak{q}}$. Similarly, we can define $\bar{H}^i(\mathrm{Sh}^0(G), \mathbb{C})_{A_{\mathfrak{q}}}$, $\bar{H}^i(X_K, \mathbb{C})_{A_{\mathfrak{q}}}$ and $\bar{H}^i(Y_K, \mathbb{C})_{A_{\mathfrak{q}}}$.

(2) Let

$$\bar{H}_\theta^i(\mathrm{Sh}(G), \mathbb{C}) \subseteq \bar{H}^i(\mathrm{Sh}(G), \mathbb{C})$$

be the subspace generated by the image of $\bar{H}^i(\mathfrak{g}, K_{\mathbb{R}}; \pi)$ where π varies among the irreducible representations in $\mathcal{A}_2(G)$ which are in the image of ψ -cuspidal theta correspondence from a symplectic group.

Theorem 5.5.1 implies the following

Theorem 6.4.3. *With notations as above, suppose that the Levi subgroup associated to \mathfrak{q} is $U(1)^r \times \mathrm{SO}(p-2r, q)$ with $2r < p$ and $3r < m-1$. Then the natural map*

$$(6.8) \quad \bar{H}_\theta^i(\mathrm{Sh}(G), \mathbb{C}) \cap \bar{H}^i(\mathrm{Sh}(G), \mathbb{C})_{A_{\mathfrak{q}}} \rightarrow \bar{H}^i(\mathrm{Sh}^0(G), \mathbb{C})_{A_{\mathfrak{q}}}$$

is surjective.

Proof. The surjectivity of (6.8) is proved in [2, Theorem 8.10] for the cuspidal part of the reduced L^2 -cohomology groups, i.e. the subspaces generated by the contribution from cuspidal automorphic representations in $\mathcal{A}_2(G)$. Since Theorem 5.5.1 now holds for all square integrable representations, the proof of [2, Theorem 8.10] extends to the whole reduced L^2 -cohomology group. We sketch the proof as below.

By Theorem 5.5.1 the space $\bar{H}^i(\mathrm{Sh}(G), \mathbb{C})$ is generated by the image of $\bar{H}^i(\mathfrak{g}, K_\infty; \sigma \otimes \chi)$, where σ is in the image of ψ -cuspidal theta correspondence from a symplectic group and χ an automorphic quadratic character. Then for any $\omega \in H^i(\mathfrak{g}, K_\infty; \sigma \otimes \chi)$, one can construct a twisting cohomology class $\omega \otimes \chi^{-1}$ (cf. [2, §8.9]) in $\bar{H}^i(\mathfrak{g}, K_\infty; \sigma)$ such that ω and $\omega \otimes \chi^{-1}$ have the same image in $\bar{H}^i(\mathrm{Sh}^0(G), \mathbb{C})$. This immediately yields the assertion. ♣

7. SPECIAL CYCLES ON ARITHMETIC MANIFOLDS OF ORTHOGONAL TYPE

In this section, we briefly review the Kudla-Millson's *special theta lifting* theory and its connection to special cycles on arithmetic manifolds of orthogonal type. In particular, we will apply the recent surjectivity results of special theta liftings in [26] and [2] to prove the main theorem.

7.1. Special cycles on arithmetic manifolds. We keep notations as in §6.1. Given a vector $\mathbf{v} \in V^n$, we let $U = U(\mathbf{v})$ be the \mathbb{Q} -subspace of V spanned by the components of \mathbf{v} . Let $D_{\mathbf{v}} \subset D$ be the subset consisting of q -planes which lie in U^\perp . The codimension k natural cycle $c(U, g_j, K)$ on $\Gamma_{g_j} \backslash D$ is defined to be the image of

$$(7.1) \quad \Gamma_{g_j, U} \backslash D_{\mathbf{v}} \rightarrow \Gamma_{g_j} \backslash D.$$

where $\Gamma_{g_j, U}$ is the stabilizer of U in Γ_{g_j} . When K is small enough, (7.1) is an embedding and hence the natural cycles on $\Gamma_{g_j} \backslash D$ are just arithmetic submanifold of the same type.

For any $\beta \in \text{Sym}^k(\mathbb{Q})$, we set

$$\Omega_\beta = \{\mathbf{v} \in V^n : \frac{1}{2}(\mathbf{v}, \mathbf{v}) = \beta, \dim U(\mathbf{v}) = \text{rank}\beta\}.$$

The special cycle on X_K with trivial coefficients is a linear combination:

$$(7.2) \quad Z(\beta, \varphi, K) = \sum_j \sum_{\substack{\mathbf{v} \in \Omega_\beta(\mathbb{Q}) \\ \text{mod } \Gamma'_{g_j}}} \varphi(g_j^{-1}\mathbf{v})c(U(\mathbf{v}), g_j, K),$$

for some K -invariant Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_f)^n)$. Let t be rank of β . Kudla and Millson [33] have associated a Poincaré dual cohomology class $\{Z[\beta, \varphi, K]\}$ in $H^{tq}(X_K, \mathbb{C})$ and we define

$$[\beta, \varphi] = \{Z[\beta, \varphi, K]\} \wedge e_q^{n-t} \in \bar{H}^{nq}(X_K, \mathbb{C}).$$

Remark 7.1.1. We should remark that the Euler form e_q is spanned by the submanifolds $c(U, g_j, K)$ when q is even (see Corollary 7.2.2). Therefore, $[\beta, \varphi]$ can be viewed as the class of a linear combination of arithmetic manifolds of the same type.

Definition 7.1.2. Let

$$SC^{nq}(\text{Sh}(G)) \subseteq H^{nq}(\text{Sh}(G), \mathbb{C}),$$

be the subspace spanned by $[\beta, \varphi]$ and set

$$(7.3) \quad SC^{nq}(X_K) := SC^{nq}(\text{Sh}(G))^K,$$

to be the K -invariant subspace. Then we define the space of special cycles on Y_K to be the projection of $SC^{nq}(X_K)$ to $H^{nq}(Y_K, \mathbb{C})$, which is denoted by $SC^{nq}(Y_K)$.

According to Remark 7.1.1, the subspace $SC^{nq}(Y_K)$ is spanned by arithmetic submanifolds in Y_K of the same type and codimension nq .

7.2. Kudla-Millson class and special theta lifting. By fixing the infinite component of the global Schwartz function ϕ in (7.4), one can obtain a correspondence between the space of Siegel cusp forms tensored with Schwartz space of the finite adèles to automorphic forms on $\text{SO}(V)$ via theta correspondence.

Let $\Omega^{nq}(\hat{D}, \mathbb{C})$ be the space of smooth nq -forms on \hat{D} . In [33], Kudla and Millson have constructed a Schwartz form:

$$\varphi_{nq} \in [\mathcal{S}(V^n(\mathbb{R})) \otimes \Omega^{nq}(\hat{D}, \mathbb{C})]^{G(\mathbb{R})},$$

(see also [2, Chapter 9]). Fix a level $K \subseteq G(\mathbb{A}_f)$ and a K -invariant Schwartz function $\varphi \in \mathcal{S}(V^n(\mathbb{A}_f))$, by abuse of notation, we define a global Schwartz form:

$$(7.4) \quad \phi = \varphi_{nq} \otimes \varphi \in [\mathcal{S}(V^n(\mathbb{A})) \otimes \Omega^{nq}(\hat{D}, \mathbb{C})]^{G(\mathbb{R})}.$$

Using the Weil representation, we can form a theta form

$$(7.5) \quad \theta_{\psi, \phi}(g, g') = \sum_{\xi \in V^n(\mathbb{Q})} \omega_{\psi}(g, g')(\phi)(\xi)$$

as in (5.2). As a function of $g \in G(\mathbb{R})$, it defines a closed nq -form on X_K , denoted by $\theta_{nq}(g', \varphi)$. Let $[\theta_{nq}(g', \varphi)]$ be the associated class in $\bar{H}^{nq}(X_K, \mathbb{C})$.

Note that there is perfect pairing from Poincaré duality

$$(7.6) \quad \langle \cdot, \cdot \rangle : H^{nq}(X_K, \mathbb{C}) \times H_c^{(p-n)q}(X_K, \mathbb{C}) \rightarrow \mathbb{C},$$

coming from Poincaré duality, where $H_c^*(-)$ denotes the de-Rham cohomology group with compact support. We now recall the main result of [33] (see also [18]):

Proposition 7.2.1. [33, Theorem 2] *As a function of $g' \in \mathrm{Mp}_{2n}(\mathbb{A})$, the cohomology class $[\theta_{nq}(g', \varphi)]$ is a holomorphic Siegel modular form of weight $\frac{m}{2}$ for some congruence subgroup with coefficients in $\bar{H}^{nq}(X_K, \mathbb{C})$. Moreover, for any closed $(p-n)q$ -form η with compact support on X_K , the Fourier expansion of $\langle [\theta_{nq}(g', \varphi)], \eta \rangle$ is*

$$(7.7) \quad \langle [\theta_{nq}(g', \varphi)], \eta \rangle = \sum_{\beta \geq 0} \langle [\beta, \varphi], \eta \rangle W_{\beta}(g'),$$

where

$$W_{\beta}(g') = \omega_{\infty}(g'_{\infty}) \exp(-\pi \mathrm{tr}(x_{\infty}, x_{\infty}))$$

and ω_{∞} is the Weil representation on $\mathrm{Mp}_{2n}(\mathbb{R})$.

Corollary 7.2.2. *When q is even, the Euler form e_q is a linear combination of the cohomology class of special cycles on X_K of codimension q .*

Proof. We prove it by contradiction. Suppose e_q is not in the span of $\{Z(\beta, \varphi, K)\}$. Then there exists an element $\eta \in H^q(X_K, \mathbb{C})^*$ such that $\eta(e_q) \neq 0$ and η vanishes on the span of special cycles. One can identify η as an element in $H_c^{(p-n)q}(X_K, \mathbb{C})$ via the perfect pairing (7.6). Then by Proposition 7.2.1, we know that $\langle [\theta_q(g', \varphi)], \eta \rangle$ is a holomorphic modular form of weight $\frac{m}{2}$ for some congruence subgroup. Moreover, the constant term of $\langle [\theta_q(g', \varphi)], \eta \rangle$ is $\eta(e_q) \neq 0$ (up to a nonzero scalar) while all the other Fourier coefficients are zero by assumption and (7.7). This is impossible and hence the corollary is proved. \clubsuit

Remark 7.2.3. When $q = 2$ and hence D is Hermitian, we know that D is a domain in $\mathbb{P}(V \otimes \mathbb{C})$. There is an ample line bundle \mathbb{L} on Y_K , which is descent of $\mathcal{O}_{\mathbb{P}(V \otimes \mathbb{C})}(1)$ (cf. [41, §4.3]). The Euler form e_2 is just the first chern class $c_1(\mathbb{L})$ of \mathbb{L} (up to a scalar), which is the Kähler class. Corollary 7.2.2 then implies that \mathbb{L} is spanned by connected Shimura subvarieties in Y_K of codimension one associated to $\mathrm{SO}(p-1, 2)$.

7.3. Proof of Theorem 0.3.1. It suffices to show that the natural projection

$$(7.8) \quad SC^{nq}(Y_K) \rightarrow H^{nq}(Y_K, \mathbb{C})_{A_q}$$

is surjective. Let us define

$$(7.9) \quad \bar{H}_{\theta_{nq}}^{nq}(\mathrm{Sh}(G), \mathbb{C})_{A_q} \subseteq \bar{H}_{\theta}^{nq}(\mathrm{Sh}(G), \mathbb{C})_{A_q},$$

to be the subspace generated by the image of $[\theta_{nq}(g', \varphi)]$ for $\varphi \in \mathcal{S}(V(\mathbb{A}_f)^n)$ where the infinite component of the global Schwartz form φ is given by $\varphi_{\infty} = \varphi_{nq}$; which is called the space of Kudla-Millson's special theta lifting classes in $\bar{H}^{nq}(\mathrm{Sh}(G), \mathbb{C})$. By definition, the subspace $\bar{H}_{\theta_{nq}}^{nq}(\mathrm{Sh}(G), \mathbb{C})$ is spanned by the image of the class

$$[\theta_{nq}^f(\varphi)] := \int_{\mathrm{Mp}(X) \backslash \mathrm{Mp}_{2n}(\mathbb{A})} [\theta_{nq}(g', \varphi)] f(g') dg',$$

for some $f \in L_{cusp}^2(\mathrm{Mp}(X) \backslash \mathrm{Mp}_{2n}(\mathbb{A}))$. Then we claim that:

(I) $\bar{H}_{\theta_{nq}}^{nq}(\mathrm{Sh}(G), \mathbb{C})_{A_q} = \bar{H}_{\theta}^{nq}(\mathrm{Sh}(G), \mathbb{C})_{A_q}$.

(II) The projection

$$(7.10) \quad SC^{nq}(\mathrm{Sh}(G)) \rightarrow \bar{H}_{\theta}^{nq}(\mathrm{Sh}(G), \mathbb{C})_{A_q},$$

is surjective.

Combining (I), (II) with Theorem 6.4.3, one can deduce that

$$(7.11) \quad SC^{nq}(Y_K) \rightarrow \bar{H}_{\theta_{nq}}^{nq}(Y_K, \mathbb{C})_{A_q} = \bar{H}^{nq}(Y_K, \mathbb{C})_{A_q}$$

is surjective as long as $2n < p$ and $3n < m - 1$ by taking K -invariant classes on both sides of (7.10) and projecting to the cohomology groups of Y_K .

Proof of the claims. Claims (I) and (II) are actually extensions of the results in [2, §10] for subspaces of cuspidal classes to the entire reduced L^2 -cohomology group. More precisely, the identity in part (I) is already proved in [2, Theorem 10.5] for cuspidal classes, but the proof does not depend on cuspidality at all. So one can directly apply their argument to obtain (I) and we omit the details here.

For part (II), the proof of the inclusion (5.5.1) is analogous to [2, Theorem 10.7]. Using the perfect pairing \langle, \rangle in (7.6), one can define

$$SC^{nq}(\mathrm{Sh}(G))^{\perp}, H_{\theta_{nq}}^{nq}(\mathrm{Sh}(G), \mathbb{C})_{A_q}^{\perp},$$

to be the annihilator of the corresponding spaces in $H_c^{(p-n)q}(\mathrm{Sh}(G), \mathbb{C})$. For any $\eta \in SC^{nq}(\mathrm{Sh}(G))^{\perp}$, assume that η is K' -invariant for some level K' . According to Proposition 7.2.1, we know that the Siegel modular form

$$(7.12) \quad \begin{aligned} \theta_{\varphi}(\eta) &:= \langle [\theta_{nq}(g', \varphi)], \eta \rangle = \int_{X_{K'}} \theta_{nq}(g', \varphi) \wedge \eta \\ &= \sum_{\beta \geq 0} \langle [\beta, \varphi], \eta \rangle W_{\beta}(g) \end{aligned}$$

is zero by our assumption. Then we have

$$\left\langle [\theta_{nq}^f(\varphi)], \eta \right\rangle = \int_{\mathrm{Mp}(W) \backslash \mathrm{Mp}_{2n}(\mathbb{A})} \theta_\varphi(\eta) f(g') dg' = 0$$

which implies that η is orthogonal to the all $[\theta_{nq}^f(\varphi)] \in \bar{H}_{\theta_{nq}}^{nq}(\mathrm{Sh}(G), \mathbb{C})$. It follows that

$$SC^{nq}(\mathrm{Sh}(G))^\perp \subseteq \bar{H}_\theta^{nq}(\mathrm{Sh}(G))_{A_q}^\perp$$

which implies the surjectivity of (7.10). ♣

7.4. Non-compact hyperbolic manifolds and Shimura varieties. For cohomology classes on hyperbolic manifolds and locally Hermitian symmetric varieties, we have the following results.

Theorem 7.4.1. *Let Y be an arithmetic hyperbolic manifold associated to the group $\mathrm{SO}(p, 1)$ as above. Then, for all $n < p/3$, the group $H^n(Y, \mathbb{Q})$ is spanned by the classes of special cycles.*

Proof. We may assume that $p > 3$ (otherwise the statement is trivial). Then $n < p/3$ implies $n < [p/2] - 1$, so that $H^n(Y, \mathbb{C}) \cong \bar{H}^n(Y, \mathbb{C})$. Furthermore, there is a unique $(\mathfrak{g}, K_{\mathbb{R}})$ -module A_n for $\mathrm{SO}(p, 1)$ such that $H^n(Y, \mathbb{C})_{A_n} \neq 0$. Hence we have

$$H^n(Y, \mathbb{C}) = \bar{H}^n(Y, \mathbb{C})_{A_n} = \bar{H}^{n \times 1}(Y, \mathbb{C}).$$

Then the assertion follows easily from Theorem 0.3.1. ♣

Theorem 7.4.2. *Let Y be a smooth Shimura variety associated to the group $\mathrm{SO}(p, 2)$ as above. Then, for all $n < (m - 1)/3$ the group $H^{n,n}(Y, \mathbb{C})$ is defined over \mathbb{Q} and spanned by the classes of special cycles. In particular, when $n < \frac{p}{4}$, $H^{2n}(Y, \mathbb{Q}) \cong H^{n,n}(Y, \mathbb{C})$ is spanned by the classes of special cycles.*

Proof. We may assume that $p \geq 3$ for otherwise the statement is trivial. Then if $n < (p + 1)/3$ we have $2n < p - 1$ and the cohomology group $H^{2n}(Y, \mathbb{C})$ is isomorphic to $\bar{H}^{2n}(Y, \mathbb{C})$ by Example 6.3.4. Moreover, the Killing form provides a canonical element in $H^{1,1}(\mathfrak{g}, K_\infty; \mathbb{C})$, whose image in $H^2(Y, \mathbb{C})$ is just the Kähler class (See also Remark 7.2.3). The cup product with the Kähler form induces a Lefschetz structure on $H^*(Y, \mathbb{C})$,

$$L^k : H^i(Y, \mathbb{C}) \rightarrow H^{i+2k}(Y, \mathbb{C}).$$

For $n < \frac{p-1}{2}$, the pure Hodge structure

$$H^{2n}(Y, \mathbb{C}) = \bigoplus_{i+j=2n} H^{i,j}(Y)$$

is compatible with the Lefschetz structure. It also coincides with the Hodge structure on the relative Lie algebra cohomology group $H^*(\mathfrak{g}, K_\infty; A_{r,r})$. So

$H^{n \times 2}(Y, \mathbb{C}) = H^{n,n}(Y)$ decomposes as

$$H^{n,n}(Y) = \bigoplus_{r=0}^n L^{2n-2r} H^{2r}(Y, \mathbb{C})_{A_{r,r}},$$

(see [2, §14]).

By Theorem 0.3.1, we know that the subspace $H^{2r}(Y, \mathbb{C})_{A_{r,r}}$ is spanned by the classes of special cycles. Note that the cup product with the Kähler form is actually to take the intersection with hyperplane class e_2 , which is a linear combination of special cycles by Corollary 7.2.2. Since the intersections of special cycles remain in the span of special cycles, Theorem 7.4.2 follows.

When $4n < p$, the Hodge structure $H^{2n}(Y, \mathbb{C})$ is of pure weight (n, n) (but is not (in general) if $4n \geq p$, see [2, §5.10]). This immediately yields the last assertion. ♣

Remark 7.4.3. When $(p, q) = (3, 2)$, Y is a Siegel modular threefold. In this case, this is exactly the surjectivity result in [26].

Now suppose $Y = \Gamma \backslash D$ is a connected Shimura variety and not necessarily smooth. Then Y is a smooth quasi-projective orbifold (cf. [23, Section 14]), as we can take a neat subgroup $\Gamma' \subseteq \Gamma$ so that $Y' = \Gamma' \backslash D$ is smooth. In this case, we get

Corollary 7.4.4. *The cohomology group $H^{2n}(Y, \mathbb{Q})$ is spanned by Poincaré dual of special cycles of codimension one for $n \leq \frac{p}{4}$. Moreover, $\text{Pic}_{\mathbb{Q}}(Y) \cong H^2(Y, \mathbb{Q})$ is spanned by special cycles of codimension one. In particular, the Noether-Lefschetz conjecture holds on \mathcal{K}_g for all $g \geq 2$.*

Proof. With notations as above, by Theorem 7.4.2, $H^{2n}(\Gamma' \backslash D, \mathbb{Q}) = SC^{2n}(Y')$ is spanned by the special cycles of codimension n when $n < \frac{p}{4}$. Note that

$$H^{2n}(Y, \mathbb{Q}) = H^{2n}(Y', \mathbb{Q})^{\Gamma/\Gamma'} \quad \text{and} \quad SC^{2n}(Y) = SC^{2n}(Y')^{\Gamma/\Gamma'},$$

it follows that

$$(7.13) \quad SC^{2n}(Y) = H^{2n}(Y, \mathbb{Q}).$$

Next, since Y is a smooth quasi-projective orbifold and $H^1(Y, \mathbb{Q}) = 0$ (cf. Remark 6.4.1), the first chern class map

$$c_1 : \text{Pic}_{\mathbb{Q}}(Y) \rightarrow H^2(Y, \mathbb{Q}),$$

is an injection by [23, Proposition 14.2] and hence has to be an isomorphism by (7.13). ♣

7.5. More applications. Combined with Borchers' theta lifting theory, one may follow Bruinier's work to give an explicit computation of the Picard number for locally Hermitian symmetric varieties associated to an even lattice. This also completes the proof of Theorem 0.1.2.

Let $M \subset V$ be an even lattice of level N and write M^\vee for the dual lattice. Let $\Gamma_M \subset \text{SO}(M)$ be the subgroup consisting of elements in $\text{SO}(M)$ acting trivially on the discriminant group M^\vee/M . Then the arithmetic

manifold Y_M associated to M is defined to be the quotient $\Gamma_M \backslash D$. In this case, Bruinier has shown that there is a natural relation between the space of vector-valued cusp forms of certain type and $SC^q(\Gamma_M \backslash D)$.

For the ease of readers, let us recall the vector-valued modular forms with respect to M . The metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ consists of pairs $(A, \phi(\tau))$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \phi(\tau) = \pm \sqrt{c\tau + d}.$$

Borcherds has defined a Weil representation ρ_M of $\mathrm{Mp}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[M^\vee/M]$ in [6, §4]. Let \mathbb{H} be the complex upper half-plane. For any $k \in \frac{1}{2}\mathbb{Z}$, a vector-valued modular form $f(\tau)$ of weight k and type ρ_M is a holomorphic function on \mathbb{H} , such that

$$f(A\tau) = \phi(\tau)^{2k} \cdot \rho_M(g)(f(\tau)), \quad \text{for all } g = (A, \phi(\tau)) \in \mathrm{Mp}_2(\mathbb{Z}).$$

Let $S_{k,M}$ be the space of $\mathbb{C}[M^\vee/M]$ -valued cusp forms of weight k and type ρ_M . Then we have

Corollary 7.5.1. *Let Y_M be the locally Hermitian symmetric variety associated to an even lattice M of signature $(p, 2)$. Then*

$$\dim_{\mathbb{Q}} \mathrm{Pic}_{\mathbb{Q}}(Y_M) = 1 + \dim_{\mathbb{C}} S_{m/2, M}$$

if $M = U \oplus U(N) \oplus E$ for some even lattice E , where $U(N)$ denotes the rank two lattice $\begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$.

Proof. By [13, Theorem 1.2], there is an isomorphism

$$(7.14) \quad S_{m/2, M} \xrightarrow{\sim} SC^2(Y_M) / \langle \mathbb{L} \rangle$$

via the Borcherd's theta lifting, where \mathbb{L} is the line bundle defined in Remark 7.2.3. Note that $SC^2(Y_M) \cong \mathrm{Pic}_{\mathbb{C}}(Y_M)$ by Corollary 7.4.4. It follows that $\dim_{\mathbb{Q}} \mathrm{Pic}_{\mathbb{Q}}(Y_M) = 1 + \dim_{\mathbb{C}} S_{m/2, M}$. \clubsuit

The dimension of $S_{k,M}$ has been explicitly computed in [12]. In particular, when $M = \Lambda_g$, one can get a simplified formula (0.1) of $\dim S_{k, \Lambda_g}$ in [37]. As Λ_g contains two hyperbolic planes, then Theorem 0.1.2 follows from Theorem 7.4.2 and Corollary 7.5.1.

Remark 7.5.2. One can use the similar idea to compute the q -th Betti number of arithmetic manifolds associated to a unimodular even lattice of signature (p, q) . Note that there is a similar map of (7.14) given in [14, Corollary 1.2].

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