CYCLES WITH LOCAL COEFFICIENTS FOR ORTHOGONAL GROUPS AND VECTOR-VALUED SIEGEL MODULAR FORMS

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Abstract. The purpose of this paper is to generalize the relation between intersection numbers of cycles in locally symmetric spaces of orthogonal type and Fourier coefficients of Siegel modular forms to the case where the cycles have local coefficients. Now the correspondence will involve vector-valued Siegel modular forms.

1. Introduction. Let $\underline{V}$ be a nondegenerate quadratic space of dimension m and signature (p, q) over $\mathbb{Q}$, for simplicity. The general case of a totally real number field is treated in the main body of the paper. We write $V = \underline{V}(\mathbb{R})$ for the real points of $\underline{V}$ and let $G = SO_0(V)$. Let G' denote the nontrivial 2-fold covering group of the symplectic group $Sp(n, \mathbb{R})$ (the metaplectic group) and let K' be the 2-fold covering inherited by $U(n)$. Let $D = G/K$ resp. $D' = G'/K'$ be the symmetric space of G resp. G'. Note that $D' = \mathbb{H}_n$, the Siegel upper half space. In what follows we will choose appropriate (related) arithmetic subgroups $\Gamma \subset G$ and $\Gamma' \subset G'$. We let $M = \Gamma\backslash D$ and $M' = \Gamma'\backslash D'$ be the associated locally symmetric spaces.

We let $\mathbb{E}_n$ denote the holomorphic vector bundle over $\mathbb{H}_n$ associated to the standard representation of $U(n)$, i.e., $\mathbb{E}_n = Sp(n, \mathbb{R}) \times_{U(n)} \mathbb{C}^n$. For each dominant weight λ' of $U(n)$, we have the corresponding irreducible representation space $S_{\lambda'}(\mathbb{C}^n)$ of $U(n)$ and the associated holomorphic vector bundle $S_{\lambda'}(\mathbb{E}_n)$ over M' (see §3, for the meaning of the Schur functor $S_{\lambda'}(\cdot)$). For each half integer $k/2$ we have a character $det^{k/2}$ of K'. Let $\mathbb{L}_{k/2}$ be the associated G'-homogeneous line bundle over the Siegel space. For each dominant weight λ of G, we have the corresponding irreducible representation $S_{[\lambda]}(V)$ of G with highest weight λ and the flat vector bundle $S_{[\lambda]}(\mathcal{V})$ over M with typical fiber $S_{[\lambda]}(V)$ (see §3, for the meaning of the harmonic Schur functor $S_{[\lambda]}(\cdot)$).

Let λ be a dominant weight for G. Let $i(\lambda)$ be the number of nonzero entries in λ when λ is expressed in the coordinates relative to the standard basis $\{\epsilon_i\}$ of [Bou], Planche II and IV. Hence we have $i(\lambda) \leq [m/2]$. We will assume (because of the choice of X below in the construction of our cycles C_X, see Remark 4.7)

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that $i(\lambda) \leq p$. Now we choose n as in the paragraph above to be any integer satisfying $i(\lambda) \leq n \leq p$ and choose for our highest weight of $U(n)$ corresponding to λ the unique dominant weight λ' such that λ' and λ *have the same nonzero entries*, We note that both weights correspond to the same Young diagram and consequently the Schur functors $S_{\lambda'}(\cdot)$ and $S_\lambda(\cdot)$ are the same, and we will not distinguish between them.

The main point of this paper is to use the theta correspondence for the dual pair (G, G') to construct for a pair of dominant weights λ' and λ as above an element

$$\theta_{nq,[\lambda]}(\tau, z) \in C^\infty(M', S_\lambda \mathbb{E}_n^* \otimes \mathbb{L}_{-\frac{m}{2}}) \widehat{\otimes} \mathcal{A}^{nq}(M, S_{[\lambda]}\mathcal{V}),$$

($\tau \in \mathbb{H}_n$, $z \in D$) which is closed as a differential form on M:

$$d\theta_{nq,[\lambda]}(\tau, z) = 0.$$

Here $\mathcal{A}^{nq}(M, S_{[\lambda]}\mathcal{V})$ denotes the space of $S_{[\lambda]}\mathcal{V}$-valued differential nq-forms on M. Note that our notation is justified since n and λ determine λ'. Hence we obtain an induced element $[\theta_{nq,[\lambda]}] \in C^\infty(M', S_\lambda \mathbb{E}_n^* \otimes \mathbb{L}_{-\frac{m}{2}}) \otimes H^{nq}(M, S_{[\lambda]}\mathcal{V})$. We will say that elements of the above tensor product are sections of the holomorphic bundle $S_\lambda \mathbb{E}_n^* \otimes \mathbb{L}_{-\frac{m}{2}}$ *with coefficients in* $H^{nq}(M, S_{[\lambda]}\mathcal{V})$.

Note that the highest weight of the isotropy representation of the homogeneous vector bundle for the symplectic group coincides (up to a shift) after the addition or suppression of zeroes with the highest weight of the coefficient system for the orthogonal group. This correspondence of the highest weights between $O(p, q)$ and $Sp(n, \mathbb{R})$ agrees with the one obtained by Adams [Ad1].

On the other hand, we can construct cycles in M as follows. Recall that we can realize D as the set of negative q-planes in V:

$$D = \{z \subset V\colon\ \dim z = q, \quad (\ ,\ ) \mid z < 0\}.$$

Then for $\mathbf{x} = (x_1, \ldots, x_n) \in \underline{V}^n$ with positive definite inner product matrix $(\mathbf{x}, \mathbf{x}) = (x_i, x_j)_{i,j}$, we define a totally geodesic submanifold $D_\mathbf{x}$ by

$$D_\mathbf{x} = \{z \in D\colon\ z \perp span(\mathbf{x})\}.$$

This gives rise to a cycle $C_\mathbf{x}$ in M of dimension $(p-n)q$, and by summing over all $\mathbf{x}$ in a system of representatives of Γ-orbits in (a coset of) a lattice in $\underline{V}$ such that $\frac{1}{2}(\mathbf{x}, \mathbf{x}) = \beta > 0$, one obtains a composite cycle C_β. For β positive semidefinite of rank $t \leq n$, there is a similar construction to obtain cycles C_β of dimension $(p-t)q$. We can then assign coefficients to these cycles (see §4 for details) to obtain (relative) homology classes

$$C_{\beta,[\lambda]} \in S_\lambda(\mathbb{C}^n)^* \otimes H_{(p-t)q}(M, \partial M, S_{[\lambda]}\mathcal{V}),$$

i.e., for every vector $w \in S_\lambda(\mathbb{C}^n)$, we obtain a class

$$C_{\beta,[\lambda]}(w) \in H_{(p-t)q}(M, \partial M, S_{[\lambda]}\mathcal{V}).$$

Then for a cohomology class $\eta \in H_c^{(p-n)q}(M, S_{[\lambda]}\mathcal{V})$, the natural pairing gives a *vector*

$$\langle \eta \cup e_q^{n-t}, C_{\beta,[\lambda]} \rangle \in S_\lambda(\mathbb{C}^n)^*.$$

Here, for q even, e_q denotes a certain invariant q-form, the Euler form on D, and is zero if q is odd.

In the usual way, we can identify the space of holomorphic sections of the bundle $S_\lambda\mathbb{E}_n^* \otimes \mathbb{L}_{-\frac{m}{2}}$ over (the compactification of) M' with $Mod(\Gamma', S_\lambda(\mathbb{C}^n)^* \otimes \det^{-\frac{m}{2}})$, the space of holomorphic vector-valued Siegel modular forms for the representation $S_\lambda(\mathbb{C}^n)^* \otimes \det^{-\frac{m}{2}}$. Here $Mod(\Gamma', S_\lambda(\mathbb{C}^n)^* \otimes \det^{-\frac{m}{2}})$ is the space of holomorphic functions $f(\tau)$ on $\mathbb{H}_n$ with values in $S_\lambda(\mathbb{C}^n)^* \otimes \det^{-\frac{m}{2}}$, holomorphic at the cusps of M', such that

$$f(\gamma\tau) = (\rho_\lambda^* \otimes \det{}^{-m/2})({}^t j(\gamma,\tau)^{-1}) f(\tau).$$

Here ρ_λ is the action of $GL_n(\mathbb{C})$ on $S_\lambda(\mathbb{C}^n)$ and $j(\gamma,\tau) = c\tau + d$ is the usual automorphy factor for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$. Recall that for Siegel modular forms, the Fourier expansion is indexed by positive semidefinite $\beta \in Sym_n(\mathbb{Q})$, and note that the βth Fourier coefficient of such a form is now a *vector* in $S_\lambda(\mathbb{C}^n)^*$.

Our main result is

THEOREM 1.1. *The cohomology class $[\theta_{nq,[\lambda]}]$ is a holomorphic Siegel modular form for the representation $S_\lambda(\mathbb{C}^n)^* \otimes \det^{-\frac{m}{2}}$ with coefficients in $H^{nq}(M, S_{[\lambda]}\mathcal{V})$. Moreover, the Fourier expansion of $[\theta_{nq,[\lambda]}](\tau)$ is given by*

$$[\theta_{nq,[\lambda]}](\tau) = \sum_{t=0}^{n} \sum_{\substack{\beta \geq 0 \\ \text{rank}\,\beta = t}} \left(PD(C_{\beta,[\lambda]}) \cup e_q^{n-t} \right) e^{2\pi i tr(\beta\tau)},$$

where $PD(C_{\beta,[\lambda]})$ denotes the Poincaré dual class of $PD(C_{\beta,[\lambda]})$. Furthermore, if q is odd or if $i(\lambda) = n$, then $[\theta_{nq,[\lambda]}](\tau)$ is a cusp form.

This generalizes the main result of [KM4], where the generating series for the special cycles C_β with trivial coefficients was realized as a classical holomorphic Siegel modular form of weight $m/2$.

Pairing $[\theta_{nq,[\lambda]}]$ with cohomology and homology defines two maps, which we denote both by $\Lambda_{nq,[\lambda]}$, namely

$$\Lambda_{nq,[\lambda]}\colon\ H_c^{(p-n)q}(M, S_{[\lambda]}\mathcal{V}) \longrightarrow Mod(\Gamma', S_\lambda^* \otimes \det{}^{-\frac{m}{2}});$$
$$\Lambda_{nq,[\lambda]}\colon\ H_{nq}(M, S_{[\lambda]}\mathcal{V}) \longrightarrow Mod(\Gamma', S_\lambda^* \otimes \det{}^{-\frac{m}{2}}).$$

These pairings give rise to the following two reformulations of Theorem 1.1:

THEOREM 1.2. *For any cohomology class* $\eta \in H_c^{(p-n)q}(M, S_{[\lambda]}\mathcal{V})$ *and for any compact cycle* $C \in H_{nq}(M, S_{[\lambda]}\mathcal{V})$*, the generating series*

$$\sum_{t=0}^{n} \sum_{\beta \geq 0} \langle (\eta \cup e_q^{n-t}), C_{\beta,[\lambda]} \rangle e^{2\pi i \operatorname{tr}(\beta\tau)}$$

and

$$\sum_{t=0}^{n} \sum_{\beta \geq 0} \langle C, (C_{\beta,[\lambda]} \cap e_q^{n-t}) \rangle e^{2\pi i \operatorname{tr}(\beta\tau)}$$

define elements in $Mod(\Gamma', S_\lambda(\mathbb{C}^n)^* \otimes \det^{-\frac{m}{2}})$.

To illustrate our result, we consider the simplest example.

Example 1.3. Consider the weight $\lambda' = (\ell, \ell, \ldots, \ell)$ of $U(n)$ (so the number of ℓ's is n). Then $S_\lambda(\mathbb{C}^n) \simeq Sym^\ell(\bigwedge^n(\mathbb{C}^n))$ is one-dimensional, while $S_{[\lambda]}(V)$ can be realized as a summand in the harmonic tensors in $Sym^\ell(\bigwedge^n(V)) \subset V^{\otimes n\ell}$. For η a closed rapidly decreasing $S_{[\lambda]}\mathcal{V}$-valued smooth differential $(p-n)q$-form on M, the pairing $\langle [\eta], C_{\mathbf{x},[\lambda]} \rangle$ is given by the period

$$\langle [\eta], C_{\mathbf{x},[\lambda]} \rangle = \int_{C_{\mathbf{x}}} (\eta, (x_1 \wedge \cdots \wedge x_n)^\ell),$$

with the bilinear form $(\ ,\ )$ on V extended to $V^{\otimes n\ell}$. Then the generating series of these periods

$$\sum_{\substack{\mathbf{x} \in L^n \\ (\mathbf{x},\mathbf{x})>0 \\ \bmod \Gamma}} \left(\int_{C_{\mathbf{x}}} (\eta, (x_1 \wedge \cdots \wedge x_n)^\ell) \right) e^{\pi i tr((\mathbf{x},\mathbf{x})\tau)}$$

is a classical scalar-valued holomorphic Siegel cusp form of weight $\ell + m/2$. Here L is (a coset of) an integral lattice in $\underline{V}$.

For $n = 1$, several (sporadic) cases for generating series for periods over cycles with nontrivial coefficients as elliptic modular forms were already known: For signature $(2, 1)$ by Shintani [S], signature $(2, 2)$ by Tong [T] and Zagier [Z], and for signature $(2, q)$ by Oda [O] and Rallis and Schiffmann [RS]. For the unitary case of $U(p, q)$, see also [TW].

We have not tried to prove that the Siegel modular form associated to a cohomology class η or a cycle C is nonzero. However, for the case in which $G = SO_0(p, 1)$ the nonvanishing of the associated Siegel modular form (for a sufficiently deep congruence subgroup depending on β and λ) follows from [KM1] together with [M]. Indeed, first apply [KM1], Theorem 11.2, to reduce to the case where the cycle $C_{\beta,[\lambda]}$ consists of a single component $C_X \otimes \mathbf{x}_{[f(\lambda)]}$, by passing to a congruence subgroup, see §4.3. Then apply (the proof of) Theorem 6.4 of [M] where it is shown that for a sufficiently deep congruence subgroup the cycle $C_X \otimes \mathbf{x}_{[f(\lambda)]}$ is not a boundary.

For general orthogonal groups, the results of J. S. Li [L] suggest that again the Siegel modular form associated to a suitable η is nonzero. Indeed, Li [L] has used the theta correspondence (but not our special kernel $\theta_{nq,[\lambda]}$) to construct nonvanishing cohomology classes for $O(p,q)$ for the above coefficient systems (with some restrictions on λ). However it is possible that *all* the above cycles $C_{\beta,[\lambda]}$ are boundaries for some p, q and λ. This would be an unexpected development.

Finally, we would like to mention our motivation for the present work. For M not compact, we are interested in extending the lift $\Lambda_{nq,[\lambda]}$, say for $\lambda = 0$, the trivial coefficient case, to the *full* cohomology $H^{(n-p)q}(M,\mathbb{C})$. This would extend the results of Hirzebruch/Zagier [HZ], who, for Hilbert modular surfaces (essentially $\mathbb{Q}$-rank 1 for $O(2,2)$), lift the full cohomology $H^2(M,\mathbb{C})$ to obtain generating series for intersection numbers of cycles. In this process, cohomology classes and cycles with nontrivial coefficients naturally occur, as we now explain.

We let $\overline{M}$ denote the Borel-Serre compactification and let ∂M denote the Borel-Serre boundary of $\overline{M}$. We study the restriction of $\theta_{nq,0}$ to $\partial(\overline{M})$, which is glued together out of faces $e(P)$, one for each Γ-conjugacy class of proper parabolic $\mathbb{Q}$-subgroups P of G. In [FM1], [FM2] we show that the theta kernel $\theta_{nq,0}$ extends to $\overline{M}$. In fact, the restriction to $e(P)$ is given by a sum of theta kernels $\theta_{n(q-r),\lambda}$ for a nondegenerate subspace $W \subset V$ associated to an orthogonal factor of the Levi subgroup of P with values in $S_\lambda(W)$ for certain dominant weights λ.

The paper is organized as follows. In §2, we briefly review homology and cohomology with nontrivial coefficients needed for our purposes, while in §3, we review the construction of the finite dimensional representations of $GL_n(\mathbb{C})$ and $O(n)$ using the Schur functors S_λ and $S_{[\lambda]}$. We introduce the special cycles with coefficients in §4. In §5, we give the explicit construction of the Schwartz forms $\varphi_{nq,[\lambda]}$ underlying the theta series $\theta_{nq,[\lambda]}$. We give their fundamental properties and for the proofs, we reduce to the case of $n = 1$. §6 is the technical heart of the paper, in which we prove the fundamental properties of $\varphi_{nq,[\lambda]}$ for $n = 1$. Our main tool is the Fock model of the Weil representation, which we review in the appendix to this paper. Finally, in §7, we consider the global theta series $\theta_{nq,[\lambda]}$ and give the proof of the main result.

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order to produce generating functions for intersection numbers and periods that are vector-valued Siegel modular forms.

2. Homology and cohomology with local coefficients. In this section, we review the facts we need about homology and cohomology of manifolds (possibly with boundary) with coefficients in a flat bundle ("local coefficients") and "decomposable cycles." We refer the reader to [Ha], pages 330–336 for more details.

2.1. The definition of the groups. We now define the homology and cohomology groups of a manifold X with coefficients in E, a flat bundle over X. We will do this assuming that X is the underlying space of a connected simplicial complex K. We will define the *simplicial* homology and cohomology groups with values in E. By the usual subdivision argument one can prove that the resulting groups are independent of the triangulation K.

We define a p-chain with values in E to be a formal sum $\Sigma_{i-1}^{m} \sigma_i \otimes s_i$ where σ_i is an *oriented* p-simplex and s_i is a flat section over σ_i. We denote the group of such chains by $C_p(X, E)$. Before defining the boundary and coboundary operators we note that if t is a flat section of E over a face τ of a simplex σ then it extends to a unique flat section $e_{\sigma,\tau}(t)$ over σ. Similarly, if we have a flat section s over σ it restricts to a flat section $r_{\tau,\sigma}(s)$ over τ. Finally, if $\sigma = (v_0, \dots, v_p)$ we define the ith face σ_i by $\sigma_i = (v_0, \dots, \hat{v}_i, \dots, v_p)$. Here $\hat{v}_i$ means the ith vertex has been omitted.

We define the boundary operator $\partial_p\colon\ C_p(X, E) \longrightarrow C_{p-1}(X, E)$ for σ a p-simplex and s a flat section over X by

$$\partial_p(\sigma \otimes s) = \sum_{i=0}^{p} (-1)^i \sigma_i \otimes r_{\sigma_i,\sigma}(s).$$

Then $\partial_{p-1} \circ \partial_p = 0$ and we define the homology groups $H_\bullet(X, E)$ of X with coefficients in E in the usual way. These groups depend only on the topological space X and the flat bundle E.

In a similar way simplicial cohomology groups of X with coefficients in E are defined. A E-valued p-cochain on X with values in E is a function α which assigns to each p-simplex σ a flat section of E over σ. The coboundary $\delta_p\alpha$ of a p-cochain α is defined on a $(p+1)$-cochain σ by:

$$\delta_p\alpha(\sigma) = \sum_{i=0}^{p} (-1)^i e_{\sigma,\sigma_i}(\alpha(\sigma_i)).$$

Then $\delta_{p+1} \circ \delta_p = 0$, and we define the cohomology groups $H^\bullet(X, E)$ of X with coefficients in E in the usual way.

If A is a subspace of X, then the complex of simplicial chains with coefficients in $E \mid A$ is a subcomplex, and we define the relative homology groups

$H_\bullet(X, A, E)$ with coefficients in E to be the homology groups of the quotient complex. Similarly, we define the subcomplex of relative (to A) simplicial cochains with coefficients in E to be the complex of simplicial cochains that vanish on the simplices in A and define the relative cohomology groups $H^\bullet(X, A, E)$ to be the cohomology groups of the relative cochain complex.

2.2. Bilinear pairings. We first define the Kronecker pairing between homology and cohomology with local vector bundle coefficients. Let E, F and G be flat bundles over X. Assume that $\nu\colon\ E \otimes F \longrightarrow G$ is a parallel section of $Hom(E \otimes F, G)$. Let α be a p-cochain with coefficients in E and $\sigma \otimes s$ be a p-simplex with coefficients in F. Then the Kronecker index $\langle \alpha, \sigma \otimes s \rangle$ is the element of $H_0(X, G)$ defined by:

$$\langle \alpha, \sigma \otimes s \rangle = \nu(\alpha(\sigma) \otimes s).$$

The reader will verify that the Kronecker index descends to give a bilinear pairing

$$\langle\ ,\ \rangle\colon\ H^p(X, E) \otimes H_p(X, F) \longrightarrow H_0(X, G).$$

We note that if G is trivial then $H_0(X, G) \cong G_{x_0}$, here G_{x_0} denotes the fiber of G over x_0. In particular, we get a pairing

$$\langle\ ,\ \rangle\colon\ H^p(X, E^*) \otimes H_p(X, E) \longrightarrow \mathbb{R},$$

which is easily seen to be perfect. The coefficient pairing $E \otimes F \to G$ also induces cup products with local coefficients

$$\cup\colon\ H^p(X, E) \otimes H^q(X, F) \longrightarrow H^{p+q}(X, G)$$

and cap products with local coefficients (here we assume $m \geq p$)

$$\cap\colon\ H^p(X, E) \otimes H_m(X, F) \longrightarrow H_{m-p}(X, G).$$

These are defined in the usual way using the "front-face" and "back-face" of an ordered simplex and pairing the local coefficients using ν.

Remark 2.1. We define the cap product $\alpha \cap \sigma$ for α a p-cochain and σ a simplex by making α operate on the *back* p face of σ. This agrees with [Br], pp. 334–338 but does not agree with [Ha]. With this definition the adjoint formula

$$\langle \alpha \cup \beta, \sigma \rangle = \langle \alpha, \beta \cap \sigma \rangle \tag{2.1}$$

holds (rather than $\langle \alpha \cup \beta, \sigma \rangle = \langle \beta, \alpha \cap \sigma \rangle$), see [Br], Proposition 5.1 (iii).

The above pairings relativize in a fashion identical to the case of trivial coefficients.

Since the proof of Poincaré (Lefschetz) duality is a patching argument of local dualities (see [Ha], pp. 245–254), it goes through for local coefficients as well. Thus

THEOREM 2.2. *Let X be a compact oriented manifold with (possibly empty) boundary and (relative) fundamental class $[X, \partial X]$. Then we have an isomorphism*

$$\mathcal{D}\colon\ H^p(X,E) \longrightarrow H_{n-p}(X,\partial X,E)$$

given by

$$\mathcal{D}(\alpha) = \alpha \cap [X,\partial X].$$

Definition 2.3. Suppose $[a] \in H_p(X,\partial X,E)$. We will define the *Poincaré dual* of $[a]$ to be denoted $PD([a])$ by

$$PD([a]) = \mathcal{D}^{-1}([a]).$$

We can now define the intersection number of cycles with local coefficients, again following the conventions of [Br], see page 367.

Definition 2.4. Let E, F, G and ν be as above and $[a]$ and $[b]$ be homology classes with coefficients in E and F respectively. Then we define the intersection class $[a] \cdot [b] \in H_{\cdot}(X,G)$ by the formula

$$[a]\cdot[b] = \mathcal{D}(PD([b]) \cup PD([b])).$$

In order to help keep track of how the formula for intersection number depends on our convention in Remark 2.1 we note (proof left to the reader):

LEMMA 2.5.

$$[a]\cdot[b] = PD([b]) \cap [a].$$

So in the special case that $[a]$ *and* $[b]$ *have complementary dimensions we have*

$$[a]\cdot[b] = \langle PD([b]),[a]\rangle.$$

2.3. Decomposable cycles. There is a particularly simple construction of cycles with coefficients in E. Let Y be a compact oriented submanifold with (possibly empty) boundary $\partial Y \subset \partial X$ of X of codimension p and let s be a parallel section of the restriction of E to Y. Let $[Y,\partial Y]$ denote the relative fundamental cycle of Y so $[Y,\partial Y] = \Sigma_i \sigma_i$, a sum of oriented simplices.

Definition 2.6. $Y \otimes s$ denotes the $(n-p)$-chain with values in E given by

$$Y = \Sigma_i \sigma_i \otimes s_i$$

where s_i is the value of s on the first vertex of σ_i.

LEMMA 2.7. *$Y \otimes s$ is a relative $n-p$ cycle with coefficients in E, called a decomposable cycle.*

For motivation of the term *decomposable cycle* we refer the reader to [M], §3.2.1.

2.4. The de Rham theory of cohomology with local coefficients and the dual of a decomposable cycle. In this subsection we recall the de Rham representations of the cohomology groups $H^\bullet(X, E)$ and of the Poincaré dual class $PD(Y \otimes s)$.

From now on, X will always be smooth manifold.

A differential p-form ω with values in a vector bundle E is a section of the bundle $\bigwedge^p T^*(X) \otimes E$ over X. Thus ω assigns to a p-tuple of tangent vectors at $x \in X$ a point in the fiber of E over x. Suppose now that E admits a flat connection ∇. We can then make the graded vector space of smooth E-differential forms $A^*(X, E)$ into a complex by defining

$$\begin{aligned} d_\nabla(\omega)(X_1, X_2, \ldots, X_{p+1}) \;=\; & \sum_{i=1}^{p} (-1)^{i-1} \nabla_{X_i}(\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{p+1})) \\ & + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{p+1}). \end{aligned}$$

Here $X_i, 1 \le i \le p+1$, is a smooth vector field on X.

We now construct a map ι from $A^p(X, E)$ to the group of simplicial cochains $C^p(X, E)$ as follows. Let $\omega \in A^p(X, E)$ and σ be a p-simplex of K. Then in a neighborhood U of σ we may write $\omega = \sum_i \omega_i \otimes s_i$ where the s_i's are parallel sections of $E \mid U$ and the ω_i's are scalar forms. We then define

$$\langle \iota(\omega), \sigma \rangle = \sum_i \left(\int_\sigma \omega_i \right) s_i | \sigma.$$

The standard double-complex proof of de Rham's theorem due to Weil, see [BT], p. 138, yields:

THEOREM 2.8. *The integration map ι: $H^\bullet_{deRham}(X, E) \longrightarrow H^\bullet(X, E)$ is an isomorphism.*

Finally, we will need that the cohomology class $PD(Y \otimes s)$ has the following representation in de Rham cohomology with coefficients in E.

Let U be a tubular neighborhood of the oriented submanifold with boundary Y. We assume (by choosing a Riemannian metric) that we have a disk bundle $\pi\colon\ U \to Y$. Then a Thom form for Y is a closed form ω_Y where ω_Y is compactly supported along the fibers of π and has integral one along one and hence all fibers of π. It is standard that the extension of ω_Y to X by making it zero outside of U represents the Poincaré dual of the class of $[Y, \partial Y]$. The parallel section s of $E \mid Y$ extends to a parallel section of $E \mid U$ again denoted s. We extend $\omega_Y \otimes s$ to X by making it zero outside of U. We continue to use the notation $\omega_Y \otimes s$ for this extended form. We will see below that $\omega_Y \otimes s$ represents the Poincaré dual of $Y \otimes s$.

If $[a] \in H_p(X, \partial X, E)$, then the de Rham cohomology class $PD([a])$ is the class of $(n-p)$–forms characterized by the property that if a is a simplicial cycle representing $[a]$, then for any E^* valued p-form η vanishing on ∂X we have

$$\int_X \eta \wedge PD([a]) = \int_a \eta.$$

Remark 2.9. In abstract terms the above equation is

$$\langle [\eta] \cup PD([a]), [X, \partial X] \rangle = \langle [\eta], [a] \rangle.$$

Since the expression on the right-hand side of this formula is equal to $\langle [\eta], PD([a]) \cap [X, \partial X] \rangle$ our definition of Poincaré dual amounts to assuming the adjoint formula, (2.1), and hence amounts to assuming the "back p face" definition of the cap product.

LEMMA 2.10. *The de Rham cohomology class Poincaré dual to the cycle with coefficients $Y \otimes s$ is represented by the bundle-valued form $\omega_Y \otimes s$.*

Proof. We need to prove that for any E^*-valued closed $(n-p)$-form η vanishing on ∂X we have

$$\int_X \eta \wedge \omega_Y \otimes s = \int_{Y \otimes s} \eta = \int_Y \langle \eta, s \rangle.$$

But

$$\int_X \eta \wedge \omega_Y \otimes s = \int_X \langle \eta, s \rangle \wedge \omega.$$

But since s is parallel on U the scalar form $\langle \eta, s \rangle$ is closed, and the lemma follows because ω_Y is the Poincaré dual to $[Y, \partial Y]$. $\square$

3. Finite dimensional representations of $GL(n)$ and $O(n)$. In this section, we will review the construction of the irreducible finite dimensional (polynomial) representations of $GL(U)$ (resp. $O(U)$), where U is a complex vector space of dimension n (resp. a finite dimensional complex vector space of dimension n equipped with a nondegenerate symmetric bilinear form $(\, , \,)$).

3.1. Representations of the general linear group.

3.1.1. Schur functors. We recall that the symmetric group S_ℓ acts on the ℓ–fold tensor product $T^\ell(U)$ according to the rule that $s \in S_\ell$ acts on a decomposable element $v_1 \otimes \cdots \otimes v_\ell$ by moving v_i to the $s(i)$th position. Let $\lambda = (b_1, b_2, \ldots, b_n)$ be a partition of ℓ. We assume that the b_i's are arranged in decreasing order. We will use $D(\lambda)$ to denote the Young diagram associated to λ. We will identify the partition λ with the dominant weight λ for $GL(n)$ in the usual way.

For more details on what follows, see [FH], §4.2 and §6.1, [GW], §9.3.1–9.3.4 and [Boe], Ch. V, §5.

Standard fillings and the associated projections.

Definition 3.1. A standard filling $t(\lambda)$ of the Young diagram $D(\lambda)$ by the elements of the set $[\ell] = \{1, 2, \ldots, \ell\}$ is an assignment of each of the numbers in $[\ell]$ to a box of $D(\lambda)$ so that the entries in each row strictly increase when read from left to right and the entries in each column strictly increase when read from top to bottom. We will denote the set of standard fillings of $D(\lambda)$ by $S(\lambda)$. A Young diagram equipped with a standard filling will be called a standard tableau.

We let $t_0(\lambda)$ be the standard filling that assigns $1, 2, \ldots, \ell$ from left to right starting with the first row then moving to the second row etc.

We now recall the projection in $End(T^\ell(U))$ associated to a standard tableau T with ℓ boxes corresponding to a standard filling $t(\lambda)$ of a Young diagram $D(\lambda)$. Let P (resp. Q) be the group preserving the rows (resp. columns) of T. Define elements of the group ring of S_ℓ by $r_{t(\lambda)} = c_1 \sum_P p$ and $c_{t(\lambda)} = c_2 \sum_Q \epsilon(q) q$ where $c_1 = 1/|P|$ and $c_2 = 1/|Q|$, so $r_{t(\lambda)}$ and $c_{t(\lambda)}$ are idempotents. We let $\mathcal{P}$ (resp. $\mathcal{Q}$) be the projections operating on $T^\ell(U)$ obtained by acting by $r_{t(\lambda)}$ (resp. $c_{t(\lambda)}$). We put $s_{t(\lambda)} = c_3 r_{t(\lambda)} \cdot c_{t(\lambda)}$ (product in the group ring) where c_3 is chosen so that $s_{t(\lambda)}$ is an idempotent, see [FH], Lemma 4.26.

Remark 3.2. We have abused notation by not indicating the dependence of $\mathcal{Q}$ and $\mathcal{P}$ on the standard filling $t(\lambda)$. We will correct both these abuses by letting $\pi_{t(\lambda)}$ denote the projector obtained by correctly normalizing the previous product. We have defined $\pi_{t(\lambda)}$ to be $c_3 \mathcal{P}\mathcal{Q}$. However we could equally well use the projector $c_3 \mathcal{Q}\mathcal{P}$ as is done in [FH] (we have used this projector in Example 1.3 and Theorem 4.8). Indeed, the restriction of $\mathcal{Q}$ to the image of $c_3 \mathcal{P}\mathcal{Q}$ gives an isomorphism.

We now have:

THEOREM 3.3. *We have a direct sum decomposition*

$$T^{\ell}(U) = \bigoplus_{\lambda \in \mathcal{P}(\ell)} \bigoplus_{t(\lambda) \in S(\lambda)} \pi_{t(\lambda)} \left(T^{\ell}(U)\right),$$

where $\mathcal{P}(\ell)$ denotes the set of partitions of ℓ.

Furthermore we have, [GW], Theorem 9.3.9:

THEOREM 3.4. *For every standard filling $t(\lambda)$, the $GL(V)$–module $\pi_{t(\lambda)}(T^{\ell}(U))$ is irreducible with highest weight λ.*

Remark 3.5. If $t'(\lambda)$ is another standard filling then the permutation relating the two fillings induces an isomorphism of $\pi_{t(\lambda)}(T^{\ell}(U))$ and $\pi_{t'(\lambda)}(T^{\ell}(U))$.

For concreteness, we will define the Schur functor $S_{\lambda}(\cdot)$ by choosing $t(\lambda) = t_0(\lambda)$, whence $S_{\lambda}(U) = \pi_{t_0(\lambda)}(T^{\ell}(U))$. We obtain projections π_{λ}

$$\pi_{\lambda}\colon\ T^{\ell}(U) \longrightarrow S_{\lambda}(U)$$

and inclusions

$$\iota_{\lambda}\colon\ S_{\lambda}(U) \longrightarrow T^{\ell}(U).$$

Semistandard fillings and the associated basis of $S_{\lambda}(U)$.

Definition 3.6. A semistandard filling of $D(\lambda)$ by the set $[n] = \{1, 2, \ldots, n\}$ is an assignment of the numbers in $[n]$ to the boxes of $D(\lambda)$ such that the numbers in each row weakly increase and the numbers in each column strictly increase. We let $SS(\lambda, n)$ denote the set of semistandard fillings of $D(\lambda)$ by the elements of the set $[n]$.

Suppose $\mathbf{x} = (x_1, \ldots, x_n) \in U^n$ and $f(\lambda) \in SS(\lambda, n)$. Suppose a_{ij} is the jth entry in the ith column of the semistandard filling. Then $\mathbf{x}_{f(\lambda)}$, the word in $\mathbf{x}$ corresponding to $f(\lambda)$ (and the standard tableau $t_0(\lambda)$), is defined by

$$\mathbf{x}_{f(\lambda)} = x_{a_{11}} \otimes x_{a_{12}} \otimes \cdots \otimes x_{a_{kb_k}}.$$

We have:

THEOREM 3.7. *Let $u_1, \ldots, u_n$ be a basis for U and let $u = (u_1, \ldots, u_n)$. Then the set of vectors $\{\pi_{\lambda}(u_{\lambda})\colon f \in SS(\lambda, n)\}$ is a basis for $\pi_{(\lambda)}(T^{d}(U))$.*

For a simple proof of this theorem see [Boe], Theorem 5.3. Boerner proves the theorem using the idempotent $c_3\mathcal{P}\mathcal{Q}$ on $T^{\ell}(U)$ (actually, he considers $c_3\mathcal{Q}\mathcal{P}$ on $T^{\ell}(U^*)$, but his proof can be easily modified to give the theorem above).

3.2. Representations of the orthogonal group.

3.2.1. The harmonic Schur functors. We will follow [FH] in our description of the harmonic Schur functor $U \to S_{[\lambda]}(U)$ on an n-dimensional nondegenerate quadratic space $(U, (\ ,\ ))$ corresponding to a partition λ.

The harmonic projection. We extend the quadratic form $(\ ,\ )$ to $T^\ell(U)$ as the ℓ-fold tensor product and note that the action of S_ℓ on $T^\ell(U)$ is by isometries. For each pair $I = (i, j)$ of integers between 1 and ℓ we define the contraction operator C_I: $\otimes^\ell U \to \otimes^{\ell-2} U$ by

$$C_I(v_1 \otimes \cdots v_\ell) = \sum_k (v_i, e_k)(v_j, e_k) v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes \widehat{v_j} \otimes \cdots \otimes v_\ell$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis for $(\ ,\ )$. We also define the expansion operator A_I: $\otimes^{\ell-2} U \to \otimes^\ell U$ to be the adjoint of C_I, that is the operator that inserts the (dual of) the form $(\ ,\ )$ into the (i, j)th spots. We define the harmonic ℓ-tensors, to be denoted $U^{[\ell]}$, to be the kernel of all the contractions C_I. Following [FH], p. 263, we define the subspace $U^{[\ell]}_{\ell-2r}$ of $U^{\otimes \ell}$ by

$$U^{[\ell]}_{\ell-2r} = \sum A_{I_1} \circ \cdots A_{I_r} U^{[\ell-2r]}.$$

Carrying over the proof of [FH], Lemma 17.15 (and the exercise that follows it) from the symplectic case to the orthogonal case we have:

LEMMA 3.8. *We have a direct sum, orthogonal for* $(\ ,\ )$,

$$T^\ell(U) = U^{[\ell]} \oplus \oplus_{r=1}^{[\frac{\ell}{2}]} U^{[\ell]}_{\ell-2r}.$$

We define the harmonic projection $\mathcal{H}$: $T^\ell(U) \to U^{[\ell]}$ to be the orthogonal projection onto the harmonic ℓ-tensors $U^{[\ell]}$. The space of harmonic ℓ-tensors $U^{[\ell]}$ is invariant under the action of S_ℓ. Consequently we may apply the idempotents in the group algebra of S_ℓ corresponding to partitions to further decompose $U^{[\ell]}$ as an $O(U)$–module.

The harmonic Schur functors. We let $\lambda = (b_1, b_2, \dots, b_k)$ be a dominant weight of $SO(U)$ where $k = [\frac{n}{2}]$. Here our coordinates are relative to the standard basis $\{\epsilon_i\}$ of [Bou], Planche II and IV. We will also use λ to denote the corresponding partition of $\ell = \sum b_i$. Again following [FH], p. 296, we then define the harmonic Schur functor $S_{[\lambda]} U$ as follows.

Definition 3.9.

$$S_{[\lambda]}(U) = \mathcal{H}\mathcal{P}\mathcal{Q}T^\ell(U) = \mathcal{H}S_\lambda(U).$$

We then have see [FH], Theorem 19.22:

THEOREM 3.10. *The $O(U)$-module $S_{[\lambda]}(U)$ is irreducible with highest weight λ.*

Definition 3.11. We write $\pi_{[\lambda]} = \mathcal{H} \circ \pi_\lambda$ for the projection from $T^\ell(U)$ onto $S_{[\lambda]}(U)$. For a semistandard filling $f(\lambda)$, we also set for $\mathbf{x} = (x_1, \dots, x_n)$,

$$\mathbf{x}_{[f(\lambda)]} = \pi_{[\lambda]}\mathbf{x}_{f(\lambda)} \in S_{[\lambda]}(U).$$

In what follows, we will need the following:

LEMMA 3.12. (i) *$\mathcal{H}, \mathcal{P}$ and $\mathcal{Q}$ are self-adjoint relative to* (,).
(ii) *$\mathcal{H}$ commutes with $\mathcal{P}$ and $\mathcal{Q}$.*

Proof. It is clear that $\mathcal{H}$ is self-adjoint. The arguments for $\mathcal{P}$ and $\mathcal{Q}$ are the same. We give the one for $\mathcal{Q}$. We will use the symbol q to denote both the element $q \in Q$ and the corresponding operator on $T^\ell(V)$. Since q is an isometry we have $q^* = q^{-1}$. Hence we have $\mathcal{Q}^* = \sum \epsilon(q)q^* = \sum \epsilon(q^{-1})q^{-1} = \mathcal{Q}$. To prove that $\mathcal{H}$ commutes with $\mathcal{P}$ and $\mathcal{Q}$ it suffices to prove that $\mathcal{H}$ commutes with every element $g \in S_\ell$. But S_ℓ acts by isometries and preserves $U^{[\ell]}$. Consequently it commutes with orthogonal projection on $U^{[\ell]}$. □

The restriction of $S_{[\lambda]}(U)$ to $SO(U)$. The restriction of $S_{[\lambda]}(U)$ to $SO(U)$ remains irreducible unless the dimension of U is even, so $dim(U) = 2k$, and $i(\lambda) = k$, and in this case the restriction is the sum of two irreducible representations. For a precise statement the reader is referred to [FH], Theorem 19.22. Thus for these cases our cohomology classes will take values in the cohomology group of M with values in the coefficient systems associated to the above direct sum (considered as representations of $SO(p+q)$). However in case M is compact the classes obtained by our constructions for these exceptional cases will often be zero. We will deduce from [LS], Proposition 2.14, that this is indeed the case provided that λ is *regular* in the sense that the entries of λ are strictly decreasing. The argument divides into two cases: The case in which both p and q are even and the case in which both p and q are odd.

In case p and q are both even and λ regular it follows immediately from [LS], Proposition 2.14, that all cohomology groups for a coefficient system as above are zero except in the degree equal to the middle dimension $(pq)/2$ (note that $l_0(G) = 0$ in this case). But we claim that the codimensions of the cycles we construct with values in such coefficient systems are always larger than $(pq)/2$. Indeed, by Remark 4.7 in order to attach a nonzero local coefficient to the cycle $C_\mathbf{x}$ it is necessary that $dim(X) \geq i(\lambda)$. Hence

$$cod(C_\mathbf{x}) = dim(X)\, q \geq i(\lambda)\, q.$$

Since we are assuming that $i(\lambda)$ has its maximum value $(p+q)/2$ we obtain

$$cod(C_\mathbf{x}) \geq q(p+q)/2 > (pq)/2.$$

Thus the cycles we construct in this case are always boundaries.

Now suppose that p and q are both odd. It follows from [LS], Proposition 2.14, that all cohomology groups for a coefficient system as above are zero except in the degrees equal to the two middle dimensions $(pq-1)/2$ and $(pq+1)/2$ (note that $l_0(G) = 1$ in this case). Once again we have $i(\lambda) = (p+q)/2$ and $cod(C_{\mathbf{x}}) \geq q(p+q)/2$. Hence the cohomology classes we obtain as the Poincarè duals of our cycles with coefficients occur in degrees greater than those allowed by [LS] if

$$q(p+q)/2 > (pq+1)/2.$$

This inequality holds if and only if $q > 1$. But for the case $SO(p,1)$ with p odd *all* cohomology groups vanish provided only $i(\lambda) = (p+1)/2$, see [M], Theorem 1.2.

If λ is singular, it no longer follows from [VZ] that the cohomologies of the corresponding local coefficient systems are concentrated in the middle dimension(s). However it is still possible that all classes we construct in this case are boundaries.

4. Special cycles with local coefficients.

4.1. Arithmetic quotients for orthogonal groups. Let $\mathbb{K}$ be a totally real number field with Archimedean places $v_1, \ldots, v_r$ and associated embeddings $\lambda_1, \ldots, \lambda_r$ and let $\mathcal{O}$ be its ring of algebraic integers. Let $\underline{V}$ be an oriented vector space over $\mathbb{K}$ of dimension $m \geq 3$ with a nondegenerate bilinear form $(\ ,\ )$ and let V be the completion of $\underline{V}$ at v_1. We assume that the associated quadratic form has signature (p,q) at the completion v_1 and is positive definite at all other completions. Finally, we let L be an integral lattice in $\underline{V}$ and $L^\# \supseteq L$ its dual lattice.

Let $\underline{G}$ be the algebraic group whose $\mathbb{K}$-points is the group of orientation preserving isometries of determinant 1 of the form $(\ ,\ )$ and let $G := \underline{G}(\mathbb{R})$ its real points. We let $\Phi = \underline{G}(\mathcal{O})$ be the subgroup of $\underline{G}(\mathbb{K})$ consisting of those elements that take L into itself. We let $\mathfrak{b}$ be an ideal in $\mathcal{O}$ and let $\Gamma = \Gamma(\mathfrak{b})$ be the congruence subgroup of Φ of level $\mathfrak{b}$ (that is, the elements of Φ that are congruent to the identity modulo $\mathfrak{b}$). We fix a vector $h \in L^n$ once and for all and note that Γ operates on the coset $h + \mathfrak{b}L^n$.

We realize the symmetric space associated to V as the set of negative q-planes in V:

$$D \simeq \{z \subset V;\ \dim z = q \quad \text{and} \quad (\ ,\ )|_z < 0\}.$$

We denote the base point of D by z_0, and we have $D \simeq G/K$, where K is the maximal compact subgroup of G stabilizing z_0. Also note $\dim_{\mathbb{R}} D = pq$. For $z \in D$, we write $(\ ,\ )_z$ for the associated majorant. Finally, we write

$$M = \Gamma \backslash D$$

for the locally symmetric space.

4.2. Special cycles with trivial coefficients. Let $\mathbf{x} = \{x_1, x_2, \dots, x_n\} \in \underline{V}^n$ be an n-tuple of $\mathbb{K}$–rational vectors. We let $\underline{X}$ be the span of $\mathbf{x}$ and let X be the completion of $\underline{X}$ at λ_1. We write $(\mathbf{x}, \mathbf{x})$ for the n by n matrix with ijth entry equal to (x_i, x_j). We call $\mathbf{x}$ nondegenerate if $\operatorname{rank}(\lambda_i\mathbf{x}, \lambda_i\mathbf{x}) = \dim \underline{X}$ for all i and nonsingular if $\operatorname{rank}(\lambda_i\mathbf{x}, \lambda_i\mathbf{x}) = n$.

Assume $\mathbf{x}$ is nondegenerate with $\dim \underline{X} = t \leq n$ such that $(\ ,\ )|\underline{X}$ is positive definite. Let r_X be the isometric involution of V given by

$$r_X(v) = \begin{cases} -v & \text{if } v \in X \\ v & \text{if } v \in X^{\perp}. \end{cases}$$

We define the totally geodesic subsymmetric space D_X by

$$D_X = \{z \in D\colon\ (z, x_i) = 0, 1 \leq i \leq n\}.$$

Then D_X is the fixed-point set of r_X acting on D and has codimension $(n-t)q$ in D. We orient D_X as in [KM4], pp. 130–131. We also define subgroups G_X (resp. Γ_X) to be the stabilizer in G (resp. in Γ) of the subspace X. We define $G'_X \subset G_X$ to be the subgroup that acts trivially on X, and put $\Gamma'_X = \Gamma \cap G'_X$.

THEOREM 4.1. *There exists a congruence subgroup $\Gamma := \Gamma(\mathfrak{b})$ of Φ such that:*

(1) *$M = \Gamma\backslash D$ is an orientable manifold of dimension pq with finite volume, and*

(2) *for all X as above, the image C_X of D_X in M is the quotient $\Gamma_X\backslash D_X$ and defines a properly embedded orientable submanifold of codimension $(n-t)q$.*

The theorem will be a consequence of the existence of a "neat" congruence subgroup. We recall the definition of a neat subgroup of Γ.

Definition 4.2. ([B], p. 117) An element $g \in G$ is neat if the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of g is torsion free. In particular, if a root of unity z is an eigenvalue of a neat element then $z = 1$. A subgroup $\Gamma \subset G$ is neat if all the elements in Γ are neat.

PROPOSITION 4.3. (Proposition 17.4, [B]) *Let G be an algebraic group defined over $\mathbb{Q}$ and Γ an arithmetic subgroup. Then Γ admits a neat congruence subgroup.*

Theorem 4.1 is an immediate consequence of the following:

LEMMA 4.4. *If Γ is a neat subgroup, then Γ_X acts trivially on X, i.e., $\Gamma'_X = \Gamma_X$.*

Proof. We have a projection map $p_X\colon\ \Gamma_X \to O(X_1) \times O(X_2) \times \cdots \times O(X_r)$. Here by X_i we mean the ith completion of X. The ith completion of $(\ ,\ )$ restricted to X_i is positive definite for $1 \leq i \leq r$. Furthermore the splitting $V = X \oplus X^{\perp}$ is defined over $\mathbb{K}$. Thus the diagonal embedding of the intersection $L_X = L \cap X$ is a lattice in $\oplus_{i=1}^{r} X_i$ which is invariant under $p_X(\Gamma_X)$. Hence $p_X(\Gamma_X)$ is a discrete subgroup of a compact group hence a finite group. Hence if $\gamma \in p_X(\Gamma_X)$, then all eigenvalues of γ are roots of unity. Since Γ is neat all eigenvalues must be 1 and the lemma follows. □

We will later need:

Definition 4.5. The Riemannian exponential map from the total space of the normal bundle of D_X to D induces a fiber bundle $\pi_X\colon\ D \to D_X$ with totally geodesic fibers. The map π_X induces a quotient fibering $\pi_X\colon\ \Gamma_X\backslash D \to \Gamma_X\backslash D_X = C_X$, see [KM1]. A Thom form Φ_X for the cycle C_X is a closed integrable $(n-t)q$-form on $\Gamma_X\backslash D$ such that the integral of Φ_X over each fiber of π_X is 1. In particular, Φ_X is a Poincaré dual form for the cycle C_X in the noncompact submanifold $\Gamma_X\backslash D$.

Occasionally, we will also write $C_{\mathbf{x}}$ ($D_{\mathbf{x}}$) for C_X (D_X).

We introduce composite cycles as follows. For $\beta \in Sym_n(\mathbb{K})$, we set

$$\Omega_\beta = \left\{\mathbf{x} \in \underline{V}^n\colon\ \tfrac{1}{2}(\mathbf{x},\mathbf{x}) = \beta\right\}$$

and

$$\Omega_\beta^c = \{\mathbf{x} \in \Omega_\beta\colon\ \dim \lambda_i \underline{X} = \operatorname{rank}\beta \quad \text{for all } i\}.$$

We put

$$\mathcal{L}_\beta = \mathcal{L}_\beta(h,\mathfrak{b}) = (h + \mathfrak{b}L^n) \cap \Omega_\beta.$$

Then Γ acts on $\mathcal{L}_\beta^c = \mathcal{L}_\beta \cap \Omega_\beta^c$ with finitely many orbits and for β positive semidefinite (i.e., $\lambda_i(\beta) \geq 0$ for all i), we define

$$C_\beta = \sum_{\mathbf{x} \in \Gamma\backslash\mathcal{L}_\beta^c} C_X.$$

4.3. Special cycles with nontrivial coefficients. We now want to promote C_X to a (decomposable) cycle with coefficients for appropriate coefficient systems W by finding a nonzero parallel section of $\mathcal{W} \mid C_X$. Note that it is enough to find any Γ_X-fixed vector $w \in W$ since such a vector w gives rise to a parallel section s_w of $\mathcal{W} \mid C_X$ in the usual way. Namely, for $z \in C_X$, the section s_w for the bundle $C_X \times_{\Gamma_X} W \to C_X$ is given by $s_w(z) = (z,w)$. Thus s_w is constant, hence parallel. Furthermore, for such a vector w, we write $C_X \otimes w$ for $C_X \otimes s_w$.

The key point for us in constructing parallel sections is Lemma 4.4. Namely, the components $x_1,\ldots,x_n$ of $\mathbf{x}$ are all fixed by $\Gamma_X = \Gamma'_X$, hence any tensor word in these components will be fixed by Γ_X.

Definition 4.6. For $f(\lambda)$ a semistandard filling for $D(\lambda)$, we define special cycles with coefficients in $S_{[\lambda]}(V)$ by setting

$$C_{\mathbf{x},[f(\lambda)]} = C_X \otimes (\lambda_1\mathbf{x})_{[f(\lambda)]}.$$

We also define composite cycles $C_{\beta,[f(\lambda)]}$ analogously as before.

To lighten the notation, we will write $\mathbf{x}_{f(\lambda)}$ and $\mathbf{x}_{[f(\lambda)]}$ for $(\lambda_1\mathbf{x})_{f(\lambda)}$ and $(\lambda_1\mathbf{x})_{[f(\lambda)]}$.

However, there is an obstruction to the construction of nonzero sections.

Remark 4.7. ([M], Proposition 4.3) Let λ be the highest weight of W and let $i(\lambda)$ be the number of nonzero entries in λ (so $i(\lambda)$ is the number of rows in the associated partition). Then

$$dim(X) \geq i(\lambda)$$

is a necessary condition for the existence of a G'_X–invariant vector in W, i.e., to finding a nonzero parallel section of the restriction of the flat vector bundle $\mathcal{W}$ to the cycle C_X.

On the other hand, for $dim(X) \geq i(\lambda)$, we do have nonzero parallel sections along the submanifold C_X.

THEOREM 4.8. ([M], Theorem 4.13) *For a weight $\lambda = (b_1, \dots, b_{[\frac{m}{2}]})$, assume $i(\lambda) = k \leq n$. Let $f_0(\lambda)$ be the semistandard filling that puts 1's in the first row of $D(\mu)$, 2's in the second row etc.. Furthermore, assume that for $\mathbf{x} = (x_1, \dots, x_n)$, the first k vectors $x_1, \dots, x_k$ are linearly independent and satisfy $(x_i, x_j) = 0, i \neq j$. Then*

$$\mathbf{x}_{[f_0(\lambda)]} = \mathcal{H}\pi_\lambda(\mathbf{x}_{f_0(\lambda)}) = \mathcal{H}\mathcal{Q}x_1^{\otimes b_1} \otimes \cdots \otimes x_k^{\otimes b_k}$$

is a nonzero Γ'_X–invariant in $S_{[\lambda]}(V)$.

For later use, we record (by an analog of Lemma 2.10):

LEMMA 4.9. *Let η be a rapidly decreasing $\mathcal{S}_{[\lambda]}(\mathcal{V})$–valued closed $(p-n)q$ form on M. If Φ_X denotes a Thom form for the cycle C_X, then $\Phi_X \otimes \mathbf{x}_{[f(\lambda)]}$ satisfies*

$$\int_M \eta \wedge \left(\Phi_X \otimes \mathbf{x}_{[f(\lambda)]}\right) = \int_{C_{\mathbf{x},[f(\lambda)]}} \eta = \int_{C_X} (\eta, \mathbf{x}_{[f(\lambda)]}).$$

4.4. Cycle-valued homomorphisms on $T^\ell(\mathbb{Q}^n)$. We now construct composite cycles $C_{\beta,[\lambda]}$, which are homomorphisms from $S_\lambda(\mathbb{Q}^n)$ to $H_\bullet(M, S_{[\lambda]}(\mathcal{V}))$.

Definition 4.10. We define elements $C_{X,[\lambda]}(\cdot)$ of $Hom(S_\lambda(\mathbb{Q}^n), H_\bullet(M, S_{[\lambda]}(\mathcal{V})))$ by

$$C_{\mathbf{x},[\lambda]}(\epsilon_{f(\lambda)}) = C_X \otimes \mathbf{x}_{[f(\lambda)]}.$$

Here $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{Q}^n$. (Note that we could have defined the map on $T^\ell(\mathbb{Q}^n)$ instead and observed that it automatically factors through $S_\lambda(\mathbb{Q}^n)$). We then have composite cycles $C_{\beta,[\lambda]}(\cdot)$ as before by summing over all $\mathbf{x} \in \Gamma\backslash\mathcal{L}_\beta$.

Finally note that, if η is a rapidly decreasing $S_{[\lambda]}(\mathcal{V})$–valued closed $(p-n)q$ form on M, then the period $\int_{C_{\mathbf{x},[\lambda]}} \eta$ is the linear functional on $S_\lambda(\mathbb{C}^n)$ given by

$$\left(\int_{C_{\mathbf{x},[\lambda]}} \eta\right)(\pi_\lambda \epsilon_{f(\lambda)}) = \int_{C_X} (\eta, \mathbf{x}_{[f(\lambda)]}). \tag{4.1}$$

5. Special Schwartz forms. In this section, we will explicitly construct the Schwartz form $\varphi_{nq,[\lambda]}$ needed to construct the cohomology class $[\theta_{nq,[\lambda]}](\tau, z)$ alluded to in the introduction. As in the introduction we will choose a pair of highest weights λ for G and λ' for $U(n)$ which have the same nonzero entries. We let ℓ be the sum of the entries of λ (which equals the sum of the entries of λ').

5.1. A double complex for the Weil representation. In this section, V will denote a real quadratic space of dimension m and signature (p, q). We write $\mathcal{S}(V^n)$ for the space of (complex-valued) Schwartz functions on V^n. We denote by $G' = Mp(n, \mathbb{R})$ the metaplectic cover of the symplectic group $Sp(n, \mathbb{R})$ and let K' be the inverse image of the standard maximal compact $U(n) \subset Sp(n, \mathbb{R})$ under the covering map $Mp(n, \mathbb{R}) \to Sp(n, \mathbb{R})$. Note that K' admits a character $\det^{1/2}$, i.e., its square descends to the determinant character of $U(n)$. The embedding of $U(n)$ into $Sp(n, \mathbb{R})$ is given by $A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. We let $\omega = \omega_V$ be the Schrödinger model of the (restriction of the) Weil representation of $G' \times O(V)$ acting on $\mathcal{S}(V^n)$ associated to the additive character $t \mapsto e^{2\pi i t}$.

Let $\mathbb{H}_n = \{\tau = u + iv \in Sym_n(\mathbb{C})\colon\ v > 0\} \simeq Sp(n, \mathbb{R})/U(n)$ be the Siegel upper half space of genus n. We write $\mathfrak{g}'$ and $\mathfrak{k}'$ for the complexified Lie algebra of $Sp(n, \mathbb{R})$ and $U(n)$ respectively. We write the Cartan decomposition as $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$, and write $\mathfrak{p}' = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ for the decomposition of the tangent space of the base point $i1_n$ into the holomorphic and anti-holomorphic tangent spaces. We let $\bar{Z}_j$, $1 \le j \le n(n+1)/2$ be a basis of $\mathfrak{p}^-$ and let $\bar{\eta}_j$ be the dual basis. We let $\mathbb{C}(\chi_{m/2})$ be the 1-dimensional representation $\det^{m/2}$ of K'. We write $W_\ell = T^\ell(\mathbb{C}^n) \otimes \mathbb{C}(\chi_{m/2})$ considered as a representation of K' and let $\mathcal{W}_\ell$ be the G'-homogeneous vector bundle over $\mathbb{H}_n$ associated to W_ℓ. We also define $W_{\lambda'}$ and $\mathcal{W}_{\lambda'}$ in the same way using $S_{\lambda'}(\mathbb{C}^n)$ instead.

We pick an orthogonal basis $\{e_i\}$ of V such that $(e_\alpha, e_\alpha) = 1$ for $\alpha = 1, \dots, p$ and $(e_\mu, e_\mu) = -1$ for $\mu = p+1, \dots, p+q$. We will use "early" Greek letters (typically α and β) as subscripts to denote indices between 1 and p (for the "positive" variables) and "late" ones (typically μ and ν) to denote indices between $p+1$ and $p+q$ (for the "negative" ones).

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ its Cartan decomposition, where $Lie(K) = \mathfrak{k}$. Then $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{k}$ is isomorphic to the tangent space at the base point of $D \simeq G/K$. We denote by $X_{\alpha\mu}$ ($1 \le \alpha \le p$, $p+1 \le \mu \le p+q$) the elements of the standard basis of $\mathfrak{p}$ induced by the basis $\{e_i\}$ of V, i.e.,

$$X_{\alpha\mu}(e_i) = \begin{cases} e_\mu, & \text{if } i = \alpha \\ e_\alpha, & \text{if } i = \mu \\ 0, & \text{otherwise.} \end{cases}$$

We let $\omega_{\alpha\mu} \in \mathfrak{p}^*$ be the elements of the associated dual basis. Finally, we let $\mathcal{A}^k(D)$ be the space of (complex-valued) differential k forms on D.

We consider the graded associative algebra

$$C = \bigoplus_{i,j,\ell \geq 0} C_\ell^{i,j},$$

where

$$C_\ell^{i,j} = \left[W_\ell^* \otimes \bigwedge^i (\mathfrak{p}^-)^* \otimes \mathcal{S}(V^n) \otimes \bigwedge^j \mathfrak{p}^* \otimes T^\ell(V) \right]^{K' \times K},$$

where the multiplication $\cdot$ in C is given componentwise. For each ℓ, we have a double complex $(C_\ell^{\bullet,\bullet}, \bar\partial, d)$ with commuting differentials

$$\begin{aligned} \bar\partial &= \sum_{j=1}^{n(n+1)/2} 1 \otimes A(\bar\eta_j) \otimes \omega(\bar Z_j) \otimes 1 \otimes 1, \\ d &= d_{\mathcal{S}} + d_V, \end{aligned}$$

where

$$\begin{aligned} d_{\mathcal{S}} &= \sum_{\alpha,\mu} 1 \otimes 1 \otimes \omega(X_{\alpha\mu}) \otimes A(\omega_{\alpha\mu}) \otimes 1, \\ d_V &= \sum_{\alpha,\mu} 1 \otimes 1 \otimes 1 \otimes A(\omega_{\alpha\mu}) \otimes \rho(X_{\alpha\mu}). \end{aligned}$$

Here $A(\cdot)$ denotes the left multiplication, while ρ is the derivation action of $\mathfrak{g}$ on $T^\ell(V)$. Furthermore, K' acts on the first three tensor factors of $C_\ell^{i,j}$, while K acts on the last three. The actions on $S(V^n)$ are given by the Weil representation, while the actions on the other tensor factors are the natural ones.

We also have an analogous complex $C_{[\lambda]}^{\bullet,\bullet}$ by replacing W_ℓ^* and $T^\ell(V)$ with W_λ^* and $S_{[\lambda]}(V)$ respectively.

We call a d-closed element $\varphi \in C^{i,j}$ *holomorphic* if the cohomology class $[\varphi]$ is $\bar\partial$-closed, i.e., there exists an element $\psi \in C^{i+1,j-1}$ such that

$$\bar\partial\varphi = d\psi.$$

Note that the maps $\bar{\partial}$ and d correspond to the usual operators $\bar{\partial}$ and d under the isomorphism

$$[W_\ell^* \otimes \mathcal{A}^{0,i}(\mathbb{H}_n) \otimes \mathcal{S}(V^n) \otimes \mathcal{A}^j(D) \otimes T^\ell(V)]^{G'\times G} \to C_\ell^{i,j},$$

given by evaluation at the base points of $\mathbb{H}_n$ and D. We will frequently identify these two spaces, and we will use the same symbol for corresponding objects.

5.2. Special Schwartz forms. We construct for $n \le p$ a family of Schwartz functions $\varphi_{nq,\ell}$ on V^n taking values in $W_\ell^* \otimes \mathcal{A}^{nq}(D) \otimes T^\ell(V)$, which we interpret as the space of differential nq-forms on D which take values in $W_\ell^* \otimes T^\ell(V)$. That is, $\varphi_{nq,\ell} \in C_\ell^{0,j}$:

$$\begin{aligned}\varphi_{nq,\ell} &\in \left[W_\ell^* \otimes \mathcal{S}(V^n) \otimes \mathcal{A}^{nq}(D) \otimes T^\ell(V)\right]^{K'\times G} \\ &\simeq \left[W_\ell^* \otimes \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes T^\ell(V)\right]^{K'\times K}.\end{aligned}$$

These Schwartz forms are the generalization of the "scalar-valued" Schwartz forms considered by Kudla and Millson [KM2], [KM3], [KM4] to the coefficient case.

Starting with the standard Gaussian,

$$\varphi_0(\mathbf{x}) = e^{-\pi tr(x,x)_{z_0}} \in \mathcal{S}(V^n),$$

with $\mathbf{x} = (x_1, \dots, x_n) \in V^n$, the "scalar-valued" form $\varphi_{nq,0}$ is given by applying the operator

$$\begin{aligned}\mathcal{D}\colon\ \mathcal{S}(V^n) \otimes \bigwedge^{\bullet}(\mathfrak{p}^*) &\longrightarrow \mathcal{S}(V^n) \otimes \bigwedge^{\bullet+nq}(\mathfrak{p}^*), \\ \mathcal{D} = \frac{1}{2^{nq/2}} \prod_{i=1}^{n} \prod_{\mu=p+1}^{p+q} &\left[\sum_{\alpha=1}^{p} \left(x_{\alpha i} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha i}}\right) \otimes A(\omega_{\alpha\mu})\right],\end{aligned}$$

to $\varphi_0 \otimes 1 \in [S(V^n) \otimes \bigwedge^0(\mathfrak{p}^*)]^K$:

$$\varphi_{nq,0} = \mathcal{D}(\varphi_0 \otimes 1).$$

Note that this is $2^{nq/2}$ times the corresponding quantity in [KM4]. We have

$$\varphi_{nq,0} \in C_0^{0,nq} = \left[\mathbb{C}(\chi_{-m/2}) \otimes \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*)\right]^{K'\times K}.$$

Here the K-invariance is immediate, while the K'-invariance is Theorem 3.1 in [KM2].

We let

$$\mathcal{A} = End_{\mathbb{C}}\left(\mathcal{S}(V^n) \otimes \bigwedge^{\bullet}(\mathfrak{p}^*) \otimes T(V)\right),$$

where $T(V) = \bigoplus_{\ell=0}^{\infty} T^{\ell}(V)$ denotes the (complexified) tensor algebra of V. Note that $\mathcal{A}$ is an associative $\mathbb{C}$-algebra by composition. We now define for $1 \leq i \leq n$ another differential operator $\mathcal{D}_i \in \mathcal{A}$ by

$$\mathcal{D}_i = \frac{1}{2}\sum_{\alpha=1}^{p}\left(x_{\alpha i} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha i}}\right) \otimes 1 \otimes A(e_\alpha).$$

Here $A(e_\alpha)$ denotes the left multiplication by e_α in $T(V)$. Note that the operator $\mathcal{D}_i$ is clearly K-invariant. We introduce a homomorphism $T\colon\ \mathbb{C}^n \to \mathcal{A}$ by

$$T(\epsilon_i) = \mathcal{D}_i,$$

where $\epsilon_1, \ldots, \epsilon_n$ denotes the standard basis of $\mathbb{C}^n$. Let $m_\ell\colon\ T^{\ell}\mathcal{A} \to \mathcal{A}$ be the ℓ-fold multiplication. We now define

$$\mathcal{T}_\ell\colon\ T^{\ell}(\mathbb{C}^n) \longrightarrow \mathcal{A}$$

by

$$\mathcal{T}_\ell = m_\ell \circ \left(\bigotimes{}^{\ell} T\right).$$

We identify

$$\mathrm{Hom}_{\mathbb{C}}\left(W_\ell, \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes T^{\ell}(V)\right) \simeq W_\ell^* \otimes \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes T^{\ell}(V),$$

and use the same symbols for corresponding objects.

Definition 5.1. We define

$$\varphi_{qn,\ell} \in \mathrm{Hom}_{\mathbb{C}}\left(W_\ell, \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes T^{\ell}(V)\right)^K$$

by

$$\varphi_{nq,\ell}(w) = \mathcal{T}_\ell(w)\varphi_{nq,0}$$

for $w \in T^{\ell}(\mathbb{C}^n)$. We put $\varphi_{nq,\ell} = 0$ for $\ell < 0$.

Note that the symmetric group S_ℓ on ℓ letters is acting on $T^{\ell}(\mathbb{C}^n)$ and $T^{\ell}(V)$ in the natural fashion. We will now show that $\varphi_{nq,\ell}$ is an equivariant map with respect to S_ℓ. More precisely, we have

PROPOSITION 5.2.

$$\varphi_{nq,\ell} \in \mathrm{Hom}_{\mathbb{C}}\left(W_\ell, \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes T^\ell(V)\right)^{S_\ell \times K},$$

that is, for any $s \in S_\ell$, we have

$$\varphi_{nq,\ell} \circ s = (1 \otimes 1 \otimes s)\varphi_{nq,\ell}.$$

Proof. We first need:

LEMMA 5.3. *Let $s \in S_\ell$. Then*

$$\mathcal{T}_\ell \circ s = (1 \otimes 1 \otimes s) \circ \mathcal{T}_\ell.$$

Proof. Let $i_1, i_2, \ldots, i_\ell \in \{1, \ldots, n\}$. Then

$$\left(\bigotimes{}^\ell T\right)\left(s(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})\right) = s\left(T(\epsilon_{i_1}) \otimes \cdots \otimes T(\epsilon_{i_\ell})\right),$$

where s on the right-hand side permutes the factors of $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$. Hence

$$\mathcal{T}_\ell\left(s(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})\right) = m_\ell\left(s\left(T(\epsilon_{i_1}) \otimes \cdots \otimes T(\epsilon_{i_\ell})\right)\right),$$

thus $\mathcal{T}_\ell\left(s(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})\right)$ takes the factors of $\left(\bigotimes{}^\ell T\right)\left(s(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})\right)$, permutes them according to s and then multiplies them in $\mathcal{A}$. The product $\mathcal{T}_\ell(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})$ is a sum of tensor products of products of certain differential operators in $\mathcal{S}(V^n)$ with products of the e_α's in $T(V)$. But the differential operators in $\mathcal{S}(V^n)$ commute with each other so that the rearrangement of the D_i's has no effect on this factor in the tensor product. Hence s only acts on the third factor of $\mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes T^\ell(V)$. □

The proposition now follows easily. For $w \in T^\ell(\mathbb{C}^n)$, we have

$$\varphi_{nq,\ell}(sw) = \mathcal{T}_\ell(sw)\varphi_{nq,0} = \left((1\otimes 1\otimes s)\mathcal{T}_\ell(w)\right)\varphi_{nq,0} = (1\otimes 1\otimes s)\left(\mathcal{T}_\ell(w)\varphi_{nq,0}\right). \quad \square$$

This will now enable us to define the Schwartz forms $\varphi_{nq,[\lambda]}$. We first note:

LEMMA 5.4. *For any standard filling $t(\lambda)$ of $D(\lambda)$, the composition*

$$(1 \otimes 1 \otimes \pi_{t(\lambda)}) \circ \varphi_{nq,\ell}\colon\ T^\ell\mathbb{C}^n \longrightarrow \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes S_{t(\lambda)}(V)$$

descends to a map

$$\pi_{t(\lambda)}T^{\ell}(\mathbb{C}^n) \longrightarrow \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes \pi_{t(\lambda)}T^{\ell}(V).$$

Proof. We have $(1 \otimes 1 \otimes s_{f(\lambda)})^2 = 1 \otimes 1 \otimes s_{f(\lambda)}$. Since $\varphi_{nq,\ell}$ is equivariant with respect to S_ℓ, we have

$$(1 \otimes 1 \otimes \pi_{t(\lambda)}) \circ \varphi_{nq,\ell}(w_1 \otimes \cdots \otimes w_\ell) = (1 \otimes 1 \otimes \pi_{t(\lambda)}) \circ \varphi_{nq,\ell}(\pi_{t(\lambda)}(w_1 \otimes \cdots \otimes w_\ell))$$

for all $w_i \in \mathbb{C}^n, 1 \leq i \leq \ell$. □

We use the lemma for the standard filling $t_0(\lambda)$ to introduce $\varphi_{nq,[\lambda]}$.

Definition 5.5. We define

$$\varphi_{nq,[\lambda]} \in \mathrm{Hom}_{\mathbb{C}}\left(S_\lambda(\mathbb{C}^n), \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes S_{[\lambda]}(V)\right)^K$$

by

$$\varphi_{nq,[\lambda]}(w) = (1 \otimes 1 \otimes \pi_{[\lambda]})(\varphi_{nq,\ell}(\iota_\lambda)(w)),$$

the projection onto $S_{[\lambda]}(V)$, the harmonic tensors in $S_\lambda(V)$.

5.3. Fundamental Properties of the Schwartz forms. We will now state the four basic properties of our Schwartz forms. These are:

- K'-invariance; thus $\varphi_{nq,\ell} \in C^{0,nq}_{\ell}$.
- d-closedness; thus $\varphi_{nq,\ell}$ defines a cohomology class $[\varphi_{nq,\ell}]$.
- The holomorphicity of $[\varphi_{nq,[\lambda]}]$.
- A recursion formula relating $[\varphi_{nq,\ell}]$ to $[\varphi_{nq,\ell-1}]$.

The first three properties are the generalizations of the properties of $\varphi_{nq,0}$ in [KM2], [KM3], [KM4], the trivial coefficient case. Except for the K'-invariance, we will reduce the statements to the case of $n = 1$. Our main tool in proving these properties will be then the Fock model of the Weil representation. We will carry out the proofs for the K'-invariance and for the other statements in the case of $n = 1$ in the next section.

THEOREM 5.6. *The forms $\varphi_{nq,\ell}$ and $\varphi_{nq,[\lambda]}$ are K'-invariant, i.e.,*

$$\varphi_{nq,\ell} \in C^{0,nq}_{\ell} = [W_\ell^* \otimes \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes T^{\ell}(V)]^{K' \times K}$$

and

$$\varphi_{nq,[\lambda]} \in C^{0,nq}_{[\lambda]} = [W_\lambda^* \otimes \mathcal{S}(V^n) \otimes \bigwedge^{nq}(\mathfrak{p}^*) \otimes S_{[\lambda]}(V)]^{K' \times K}.$$

In particular, for $n = 1$, we have

$$\varphi_{q,\ell} \in \left[\mathbb{C}(\chi_{-\ell-m/2}) \otimes \mathcal{S}(V) \otimes \bigwedge^{q}(\mathfrak{p}^*) \otimes T^{\ell}(V)\right]^{K' \times K}.$$

Proof. We consider the first statement using the Fock model in the next section. The second statement follows from the first by projecting onto $S_{[\lambda]}(V)$. □

The K'-invariance of the Schwartz forms will enable us in Section 7 to construct theta series using the forms $\varphi_{nq,[\lambda]}$.

THEOREM 5.7. *The forms $\varphi_{nq,\ell}$ and $\varphi_{nq,[\lambda]}$ define closed differential forms on D, i.e.,*

$$d\varphi_{nq,\ell}(\mathbf{x}) = 0$$

for all $\mathbf{x} \in V^n$. In particular, $\varphi_{nq,[\lambda]}(\mathbf{x})$ defines a (de Rham) cohomology class

$$[\varphi_{nq,[\lambda]}(\mathbf{x})] \in H^{nq}\left(D, \mathrm{Hom}_{\mathbb{C}}(S_{\lambda}(\mathbb{C}^n), S_{[\lambda]}(V))\right).$$

Proof. We will prove the case $n = 1$ in the next section using the Fock model of the Weil representation. For general n, it is enough to show that $\varphi_{nq,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})$ is closed for any n-tuple $(\epsilon_{i_1}, \dots, \epsilon_{i_\ell})$. By the S_ℓ-equivariance of $\varphi_{nq,\ell}$ we can assume that $i_1 \leq \cdots \leq i_\ell$, so that $\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell} = \epsilon_1^{\otimes \ell_1} \otimes \cdots \otimes \epsilon_n^{\otimes \ell_n}$ for some nonnegative integers $\ell_1, \dots, \ell_n$. But this implies that

$$\varphi_{nq,\ell}(\epsilon_1^{\otimes \ell_1} \otimes \cdots \otimes \epsilon_n^{\otimes \ell_n})(\mathbf{x}) = \varphi_{q,\ell_1}(x_1) \wedge \cdots \wedge \varphi_{q,\ell_n}(x_n).$$

Here the wedge $\wedge$ means the usual wedge for $\mathcal{A}(D)$ and the tensor product in the other slots. This reduces the closedness of $\varphi_{nq,\ell}$ to the case $n = 1$. □

To state the last two properties of the forms $\varphi_{nq,[\lambda]}$, we first need to introduce some more notation. We define a map

$$\sigma\colon\ \mathbb{C}^n \longrightarrow \left(V^n\right)^* \otimes \bigwedge^{\bullet} \mathfrak{p}^* \otimes V$$

by

$$\sigma(\epsilon_i) = \sum_{j=1}^{m} x_{ij} \otimes 1 \otimes e_j.$$

Here the $x_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$ are the standard coordinate functions on V^n. Thus under the identification of $(V^n)^* \otimes \bigwedge^{\bullet}\mathfrak{p}^* \otimes V$ with $Hom(V^n, \bigwedge^{\bullet}\mathfrak{p}^* \otimes V)$ we have

$$\sigma(\epsilon_i)(\mathbf{x}) = \sum_{j=1}^{m} x_{ij}(1 \otimes e_j) \ = \ 1 \otimes \sum_{j=1}^{m} x_{ij}e_j = 1 \otimes x_i.$$

Now the x_{ij} are numbers, the coordinates of the n-tuple of vectors $\mathbf{x}$.

We let σ_ℓ be the ℓth outer tensor power of σ, and for λ a partition of $[\ell]$, we put

$$\sigma_\lambda = \sigma_\ell \circ \iota_\lambda \colon \ S_\lambda\mathbb{C}^n \longrightarrow (V^n)^* \otimes {\bigwedge}^{\bullet}\mathfrak{p}^* \otimes T(V).$$

Note that we do not need to distinguish between λ' and λ because only the nonzero parts of the partition matter here.

LEMMA 5.8. (i) *Let f be a semistandard filling of $D(\lambda)$. Then*

$$\sigma_\lambda(\epsilon_{f(\lambda)})(\mathbf{x}) = 1 \otimes \mathbf{x}_{f(\lambda)}$$

for any $\epsilon = \epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell} \in T^\ell(\mathbb{C}^n)$. In particular,

$$\sigma_\lambda \colon \ S_\lambda\mathbb{C}^n \longrightarrow \left(V^n\right)^* \otimes {\bigwedge}^{\bullet}\mathfrak{p}^* \otimes S_\lambda(V).$$

(ii) *The map σ_λ is $GL_n(\mathbb{C})$-invariant, i.e., for $a \in GL_n(\mathbb{C})$, we have*

$$\sigma((a^{-1}\epsilon)_{f(\lambda)})(\mathbf{x}a) = \sigma(\epsilon_{f(\lambda)})(\mathbf{x}).$$

Proof. For (i), first note

$$\sigma_\ell(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})(\mathbf{x}) = 1 \otimes (x_{i_1} \otimes \cdots \otimes x_{i_\ell}).$$

Indeed,

$$\begin{aligned}\sigma_\ell(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})(\mathbf{x}) = \sigma_\ell(\epsilon_{i_1}) \circ \cdots \circ \sigma_\ell(\epsilon_{i_\ell})(\mathbf{x}) \ &= \ (1 \otimes x_{i_1}) \circ \cdots \circ (1 \otimes x_{i_\ell}) \\ &= \ 1 \otimes (x_{i_1} \otimes \cdots \otimes x_{i_\ell}).\end{aligned}$$

But now for $s \in S_\ell$ and $w_i \in \mathbb{C}^n$, $1 \leq i \leq \ell$, we have

$$\sigma_\ell\left(s(w_1 \otimes \cdots \otimes w_\ell)\right) = (1 \otimes s)\sigma(w_1 \otimes \cdots \otimes w_\ell),$$

which gives immediately

$$\sigma_\ell(\epsilon_{f(\lambda)})(\mathbf{x}) = 1 \otimes \mathbf{x}_{f(\lambda)},$$

as claimed. (ii) follows easily from $\sigma(a^{-1}\epsilon)(\mathbf{x}a) = \sigma(\epsilon)(\mathbf{x})$. □

We can therefore define $\sigma_{[\lambda]}$ by post-composing with the harmonic projection $\pi_{[\lambda]}$ onto $S_{[\lambda]}(V)$, and we have

$$\sigma_{[\lambda]}(\pi_\lambda \epsilon_{f(\lambda)})(\mathbf{x}) = 1 \otimes \mathbf{x}_{[f(\lambda)]}.$$

Via left multiplication we can interpret σ (and similarly $\sigma_\ell, \sigma_\lambda, \sigma_{[\lambda]}$) as a map from $\mathbb{C}^n$ to $\mathcal{A}$ (and we do not distinguish between these two interpretations). With this identification and considering $\mathcal{S}(V^n)\otimes\bigwedge^\bullet\mathfrak{p}^*\otimes T(V)$ as the $\bigwedge^\bullet\mathfrak{p}^*\otimes T(V)$-valued Schwartz functions on V^n, we have

$$(\sigma(\epsilon_i)\varphi)(\mathbf{x}) = (1 \otimes A(x_i))\varphi(\mathbf{x}).$$

For $v \in V$, we let $A_j(v)$: $T^{\ell-1}(V) \to T^\ell(V)$ be the insertion of v into the jth spot. We let A_{jk}: $T^{\ell-2}(V) \to T^\ell$

$$A_{jk}(f) = \sum_{\alpha=1}^{p} A_j(e_\alpha)A_k(e_\alpha) - \sum_{\mu=p+1}^{p+q} A_j(e_\mu)A_k(e_\mu)$$

be the insertion of the dual of the form $(\ ,\ )$ into the (j,k)th spot, and we put

$$A(f) = \frac{1}{2}\sum_{j=1}^{\ell}\sum_{k=1}^{\ell-1} A_{jk}(f).$$

One of the fundamental properties of the scalar-valued Schwartz form $\varphi_{nq,0}$ is that for $(\mathbf{x},\mathbf{x})$ positive semidefinite, $\varphi_{nq,0}(\mathbf{x})$ gives rise to a Thom form for the special cycle C_X. In view of Lemma 4.9, we now relate $\varphi_{nq,[\lambda]}(\mathbf{x})$ to $(\sigma_{[\lambda]}\varphi_{nq,0})(\mathbf{x})$.

THEOREM 5.9. (i) *Let $n = 1$ and let σ_j be the operator on $\mathcal{S}(V) \otimes \bigwedge^\bullet\mathfrak{p}^* \otimes T(V)$ defined by $\sigma_j(x) = 1 \otimes 1 \otimes A_j(x)$. Then for each $j = 1, \dots, \ell$, we have in cohomology*

$$[\varphi_{q,\ell}] = [\sigma_j\varphi_{q,\ell-1}] + \frac{1}{4\pi}\sum_{k=1}^{\ell-1}[A_{jk}(f)\varphi_{q,\ell-2}]$$

for all $x \in V$. In particular,

$$[\varphi_{q,[\ell]}] = [\sigma_{[\ell]}\varphi_{q,0}].$$

(ii) *For general n, we have in cohomology*

$$[\varphi_{nq,[\lambda]}] = [\sigma_{[\lambda]}\varphi_{nq,0}],$$

i.e., for all semistandard fillings f,

$$[\varphi_{nq,[\lambda]}(\epsilon_{f(\lambda)})(\mathbf{x})] = \left[(1 \otimes \mathbf{x}_{[f(\lambda)]})\varphi_{nq,0}(\mathbf{x})\right],$$

where $\epsilon = \epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell} \in T^\ell(\mathbb{C}^n)$.

Proof. We prove (i) in the next section via the Fock model. For (ii), we first see by (i) that up to exact forms we have

$$\begin{aligned}\varphi_{nq,\ell}(\epsilon_1^{\otimes \ell_1} \otimes \cdots \otimes \epsilon_n^{\otimes \ell_n})(\mathbf{x}) = &\left(\sigma(x_1)\varphi_{q,\ell_1-1}(x_1) + \frac{1}{4\pi}\sum_{k=1}^{\ell_1-1} A_{jk}(f)\varphi_{q,\ell_1-1}(x_1)\right)\\ &\wedge \cdots\\ &\wedge\left(\sigma(x_n)\varphi_{q,\ell_n-1}(x_n) + \frac{1}{4\pi}\sum_{k=1}^{\ell_n-1} A_{jk}(f)\varphi_{q,\ell_n-1}(x_n)\right),\end{aligned}$$

and we get a similar statement for $\varphi_{nq,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})$ by the S_ℓ-equivariance of $\varphi_{nq,\ell}$. Iterating and using Lemma 5.8 then gives a statement for $\varphi_{nq,\ell}(\mathbf{x})$ with coefficients in $S_\lambda(V)$ analogous to (ii)—up to terms coming from the metric. Projecting to $S_{[\lambda]}(V)$ now gives the claim. □

One of the main results of [KM4] is that the scalar-valued cohomology class $[\varphi_{nq,0}]$ is holomorphic, i.e, $[\overline{\partial}\varphi_{nq,0}] = 0$. We will now show that the more general cohomology classes $[\varphi_{nq,[\lambda]}]$ are holomorphic as well.

For $n = 1$, we have $\mathfrak{g}' = \mathfrak{sl}_2(\mathbb{C})$, and the anti-holomorphic tangent space $\mathfrak{p}^-$ is spanned by the element $L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1\end{pmatrix}$.

THEOREM 5.10. (i) *Let* $n = 1$. *Then in cohomology, we have*

$$[\omega(L)\varphi_{q,\ell}] = \frac{-1}{4\pi}[A(f)\varphi_{q,\ell-2}].$$

In particular,

$$[\overline{\partial}\varphi_{q,[\ell]}] = 0.$$

(ii) *For general n, we have*

$$[\overline{\partial}\varphi_{nq,[\lambda]}] = 0.$$

Proof. We will prove (i) in the next section. (ii) follows from (i) by generalizing the argument given for the scalar valued case in [KM4], Theorem 5.2. First note that we have to show $[\overline{\partial}_{ij}\varphi_{nq,[\lambda]}] = 0$ for all $n(n+1)/2$ partial derivatives $\overline{\partial}_{ij}$ ($i \leq j$) in $\mathfrak{p}^-$. By (i), we see

$$(1 \otimes 1 \otimes \pi_{[\lambda]})\overline{\partial_{ii}}\varphi_{nq,\ell}(\epsilon_1^{\otimes \ell_1} \otimes \cdots \otimes \epsilon_n^{\otimes \ell_n}) = 0$$

up to an exact form. By the S_ℓ-equivariance, we then see

$$(1 \otimes 1 \otimes \pi_{[\lambda]})\overline{\partial_{ii}}\varphi_{nq,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell}) = 0,$$

again, up to an exact form. This gives the desired vanishing for the anti-holomorphic tangent space $\mathfrak{p}_0^-$ of $\mathbb{H} \times \cdots \times \mathbb{H}$, naturally embedded into $\mathbb{H}_n$. By the K'-invariance of $\varphi_{nq,\ell}$ we now see that $(1 \otimes 1 \otimes c_{[\lambda]})\varphi_{nq,\ell}$ is annihilated by the $Ad\, K'$ orbit of $\mathfrak{p}_0^-$ inside $\mathfrak{p}^-$, which is all of $\mathfrak{p}^-$. □

6. Proof of the fundamental properties of the Schwartz forms. The purpose of this section is to prove the K'-invariance of $\varphi_{nq,\ell}$ and for $n = 1$ the other fundamental properties of $\varphi_{q,\ell}$ given in the previous section. Our main tool will be the Fock model of the Weil representation, which we review in the appendix.

By abuse of notation we will frequently use in the following the same symbols for corresponding objects and operators in the two models.

6.1. The Schwartz forms in the Fock model and the K'-invariance. For multi-indices $\underline{\alpha} = (\alpha_1, \dots, \alpha_q)$ and $\underline{\beta} = (\beta_1, \dots, \beta_\ell)$, (usually suppressing their length), we will write

$$\begin{aligned} \omega_{\underline{\alpha}} &= \omega_{\alpha_1 p+1} \wedge \cdots \wedge \omega_{\alpha_q p+q}, \\ z_{\underline{\alpha} j} &= z_{\alpha_1 j} \cdots z_{\alpha_q j}, \\ e_{\underline{\beta}} &= e_{\beta_1} \otimes \cdots \otimes e_{\beta_\ell}. \end{aligned}$$

Here we have returned to our original notation, denoting the standard basis elements of V by e_α and e_μ. In the Fock model $\mathcal{F}$, the “scalar-valued” Schwartz form $\varphi_{nq,0}$ becomes with this notation

$$\begin{aligned} \varphi_{nq,0} &= \frac{1}{2^{nq/2}} \left(\frac{-i}{2\pi}\right)^{nq} \sum_{\underline{\alpha_1}, \dots, \underline{\alpha_n}} z_{\underline{\alpha_1} 1} \cdots z_{\underline{\alpha_n} n} \otimes \omega_{\underline{\alpha_1} 1} \wedge \cdots \wedge \omega_{\underline{\alpha_n} n} \otimes 1 \\ &\in \mathcal{F} \otimes {\textstyle\bigwedge}^{nq}(\mathfrak{p}^*) \otimes T^0(V) \end{aligned}$$

We define

$$\varphi_{0,\ell} \in \mathrm{Hom}_{\mathbb{C}}\,(T^\ell(\mathbb{C}^n), \mathcal{F} \otimes {\textstyle\bigwedge}^{0}(\mathfrak{p}^*) \otimes T^\ell(V))$$

by

$$\varphi_{0,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell}) = \left(\frac{-i}{4\pi}\right)^{n\ell} \sum_{\underline{\beta}} z_{\beta_1 i_1} \cdots z_{\beta_\ell i_\ell} \otimes 1 \otimes e_{\underline{\beta}}.$$

We then easily see:

LEMMA 6.1.

$$\varphi_{nq,\ell} = \varphi_{nq,0} \cdot \varphi_{0,\ell},$$

where the multiplication is the natural one in $\mathrm{Hom}_{\mathbb{C}}\left(T(\mathbb{C}^n), \mathcal{F} \otimes \bigwedge^{\bullet}(\mathfrak{p}^*) \otimes T(V)\right)$.

We should note that only in the Fock model we have such a "splitting" of $\varphi_{nq,\ell}$ into the product of two elements. We do not have an analogous statement in the Schrödinger model (only in terms of operators acting on the Gaussian φ_0).

THEOREM 6.2. (Theorem 5.6) *The form* $\varphi_{nq,\ell}$ *is* K'*-invariant.*

Proof. We show this on the Lie algebra level. The element $k' = \frac{1}{2i} w'_j \circ w''_k \in \mathfrak{k} \simeq \mathfrak{gl}_n(\mathbb{C})$ is the endomorphism of $\mathbb{C}^n$ mapping ϵ_j to ϵ_k and annihilating the other basis elements. To show $\omega(k')\varphi_{nq,\ell} = 0$, we need to show

$$\omega(k')\left(\varphi_{nq,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})\right) = \varphi_{nq,\ell}(k'(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})).$$

From Lemma A.1 we see

$$\begin{aligned} \omega(k')\left(\varphi_{nq,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})\right) \;=\; & \omega(k')\varphi_{nq,0} \cdot \varphi_{0,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell}) \\ & + \varphi_{nq,0} \cdot \sum_{\alpha=1}^{p} z_{\alpha k} \frac{\partial}{\partial z_{\alpha j}} \left(\varphi_{0,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})\right). \end{aligned}$$

We have $\omega(k')\varphi_{nq,0} = 0$, since $\varphi_{nq,0} \in \left[\mathbb{C}(\chi_{-m/2}) \otimes \mathcal{F} \otimes \bigwedge^{nq}(\mathfrak{p}^*)\right]^{K'}$ by [KM2], Theorem 5.1. On the other hand, one easily sees

$$\sum_{\alpha=1}^{p} z_{\alpha k} \frac{\partial}{\partial z_{\alpha j}} \left(\varphi_{0,\ell}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})\right) = \varphi_{0,\ell}(k'(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_\ell})).$$

The assertion follows. $\square$

6.2. The Schwartz forms for $n = 1$**.** For $n = 1$, we consider the forms $\varphi_{q,\ell}$, $\varphi_{q,0}$, and $\varphi_{0,\ell}$ to be in $\mathcal{F} \otimes \bigwedge^{\bullet}(\mathfrak{p}^*) \otimes T(V)$, and we have

$$\varphi_{q,\ell} = c_{q,\ell} \sum_{\underline{\alpha},\underline{\beta}} z_{\underline{\alpha}} z_{\underline{\beta}} \otimes \omega_{\underline{\alpha}} \otimes e_{\underline{\beta}}.$$

Here $c_{q,\ell} = 2^{q/2}(-i/4\pi)^{q+\ell}$. Also

$$\varphi_{q,0} = 2^{q/2}(-i/4\pi)^q \sum_{\underline{\alpha}} z_{\underline{\alpha}} \otimes \omega_{\underline{\alpha}} \otimes 1$$

and

$$\varphi_{0,\ell} = (-i/4\pi)^\ell \sum_{\underline{\beta}} z_{\underline{\beta}} \otimes 1 \otimes e_{\underline{\beta}}.$$

For later use, we note that Theorem 6.2 for $n = 1$ boils down to

$$\sum_{\alpha=1}^{p} \left(z_\alpha \frac{\partial}{\partial z_\alpha} \otimes 1 \otimes 1 \right) \varphi_{q,\ell} = (q+\ell)\,\varphi_{q,\ell}, \tag{6.1}$$

which follows directly from

$$\sum_{\alpha=1}^{p} z_\alpha \frac{\partial}{\partial z_\alpha} \varphi_{q,0} = q\,\varphi_{q,0} \qquad \text{and} \qquad \sum_{\alpha=1}^{p} z_\alpha \frac{\partial}{\partial z_\alpha} \varphi_{0,\ell} = \ell\,\varphi_{0,\ell}. \tag{6.2}$$

6.3. Closedness. Similarly to the Schrödinger model, the differentiation d in the Lie algebra complex $\mathcal{F} \otimes \bigwedge^*(\mathfrak{p}^*) \otimes S^\ell(V)$ is given by $d = d_{\mathcal{F}} + d_V$ with

$$\begin{aligned} d_{\mathcal{F}} &= \sum_{\alpha,\mu} \omega(X_{\alpha\mu}) \otimes A(\omega_{\alpha\mu}) \otimes 1 \qquad \text{and} \\ d_V &= \sum_{\alpha,\mu} 1 \otimes A(\omega_{\alpha\mu}) \otimes \rho(X_{\alpha\mu}). \end{aligned} \tag{6.3}$$

Furthermore, we write $d_{\mathcal{F}} = d'_{\mathcal{F}} + d''_{\mathcal{F}}$ with

$$\begin{aligned} d'_{\mathcal{F}} &= -4\pi \sum_{\alpha,\mu} \frac{\partial^2}{\partial z_\alpha \partial z_\mu} \otimes A(\omega_{\alpha\mu}) \otimes 1, \\ d''_{\mathcal{F}} &= \frac{1}{4\pi} \sum_{\alpha,\mu} z_\alpha z_\mu \otimes A(\omega_{\alpha\mu}) \otimes 1. \end{aligned} \tag{6.4}$$

THEOREM 6.3. (Theorem 5.7) *The form $\varphi_{q,\ell}$ is closed. More precisely,*

$$d'_{\mathcal{F}}\varphi_{q,\ell} = d''_{\mathcal{F}}\varphi_{q,\ell} = 0$$

and

$$d_V\varphi_{q,\ell} = 0.$$

Proof. First note that $d'_{\mathcal{F}}\varphi_{q,\ell} = 0$ is obvious from (6.4). From the "scalar-valued" case, see [KM2], we have $d_{\mathcal{F}}\varphi_{q,0} = d''_{\mathcal{F}}\varphi_{q,0} = 0$. In fact, this can be seen directly by (6.4), since one easily checks

$$\sum_{\alpha} (z_\alpha \otimes A(\omega_{\alpha\mu}) \otimes 1)\, \varphi_{q,0} = 0 \tag{6.5}$$

for any μ. We then easily see

$$d'_{\mathcal{F}}\varphi_{q,\ell} = \left(d'_{\mathcal{F}}\varphi_{q,0}\right) \cdot \varphi_{0,\ell} = 0.$$

For the action of d_V, we first note

$$(1 \otimes 1 \otimes \rho(X_{\alpha\mu}))\, \varphi_{0,\ell} = \frac{-i}{4\pi} \sum_{k=1}^{\ell} (z_\alpha \otimes 1 \otimes A_k(e_\mu))\, \varphi_{0,\ell-1}. \tag{6.6}$$

(6.3) then implies

$$d_V\varphi_{q,\ell} = \frac{-i}{4\pi} \sum_{\alpha,\mu} \sum_{k=1}^{\ell} (z_\alpha \otimes A(\omega_{\alpha\mu}) \otimes 1)\, \varphi_{q,0} \cdot (1 \otimes 1 \otimes A_k(e_\mu))\, \varphi_{0,\ell-1} = 0$$

by (6.5). □

6.4. Recursion. We will now show Theorem 5.9 (i), the recursive formula for the cohomology class $[\varphi_{q,\ell}]$.

For $j \geq 1$, we define operators $A_j(\sigma)$ by

$$A_j(\sigma) = i \sum_{\alpha} \left(\frac{\partial}{\partial z_\alpha} - \frac{1}{4\pi} z_\alpha\right) \otimes 1 \otimes A_j(e_\alpha) \;-\; i \sum_{\mu} \left(\frac{\partial}{\partial z_\mu} - \frac{1}{4\pi} z_\mu\right) \otimes 1 \otimes A_j(e_\mu).$$

We write $A(\sigma) = A_1(\sigma)$, and by Lemma A.3 we note that this is the image in the Fock model of the operator $A(\sigma)$ in the Schrödinger model.

For $j \geq 1$, we introduce operators h'_j by

$$h'_j = \sum_{\alpha,\mu} \frac{\partial}{\partial z_\alpha} \otimes A^*(\omega_{\alpha\mu}) \otimes A_j(e_\mu).$$

We write $h' = h'_1$. Here $A^*(\omega_{\alpha\mu})$ denotes the (interior) multiplication with $X_{\alpha\mu}$, i.e., $A^*(\omega_{\alpha\mu})(\omega_{\alpha'\mu'}) = \delta_{\alpha\alpha'}\delta_{\mu\mu'}$. We define a $(q-1)$-form $\Lambda^{(j)}_{q,\ell}$ by

$$\Lambda^{(j)}_{q,\ell} = \frac{-i}{p+q+\ell-1} h'_j \varphi_{q,\ell}.$$

We write $\Lambda_{q,\ell} = \Lambda^{(1)}_{q,\ell}$.

THEOREM 6.4. (Theorem 5.9) *For any $1 \le j \le \ell$, we have*

$$\varphi_{q,\ell} = A_j(\sigma)\varphi_{q,\ell-1} + d\Lambda^{(j)}_{q,\ell-1} + \frac{1}{4\pi}\sum_{k=1}^{\ell-1} A_{jk}(f)\varphi_{q,\ell-2}.$$

For the proof of Theorem 6.4, we first compute $A_j(\sigma)\varphi_{q,\ell-1}$:

LEMMA 6.5. *For any $1 \le j \le \ell$, we have*

$$A_j(\sigma)\varphi_{q,\ell-1} = \varphi_{q,\ell} + A_j + B_j + C_j^+.$$

Here

$$A_j = \frac{i}{4\pi}\left(\sum_{\mu=p+1}^{p+q} z_\mu \otimes 1 \otimes A_j(e_\mu)\right)\varphi_{q,\ell-1},$$

$$B_j = i\sum_{\alpha=1}^{p}\left(\frac{\partial}{\partial z_\alpha}\otimes 1 \otimes 1\right)\varphi_{q,0}\cdot(1\otimes 1\otimes A_j(e_\alpha))\,\varphi_{0,\ell-1},$$

$$C_j^+ = \frac{1}{4\pi}\sum_{k=1}^{\ell-1} A_{jk}(f_+)\varphi_{q,\ell-2},$$

where $A_{jk}(f_+)$ is the insertion $\sum_{\alpha=1}^{p} A_j(e_\alpha)A_k(e_\alpha)$ in the jth and kth position in $T(V)$.

Proof. We write $A_j(\sigma) = A'_j(\sigma) + A''_j(\sigma)$ with $A'_j(\sigma) = \frac{-i}{4\pi}\sum_\alpha z_\alpha \otimes 1 \otimes A_j(e_\alpha) + \frac{i}{4\pi}\sum_\mu z_\mu \otimes 1 \otimes A_j(e_\mu)$. We immediately see

$$A'_j(\sigma)\varphi_{q,\ell-1} = \varphi_{q,\ell} + A_j.$$

On the other hand, we have

$$\begin{aligned} A''_j(\sigma)\varphi_{q,\ell-1} &= i\sum_{\alpha=1}^{p}\left(\frac{\partial}{\partial z_\alpha}\otimes 1 \otimes A_j(e_\alpha)\right)(\varphi_{q,0}\cdot\varphi_{0,\ell-1}) \\ &= B_j + i\varphi_{q,0}\cdot\left(\sum_{\alpha=1}^{p}\left(\frac{\partial}{\partial z_\alpha}\otimes 1\otimes A_j(e_\alpha)\right)\right)\varphi_{0,\ell-1}. \end{aligned}$$

But the last term is equal to C_j^+. Indeed, a little calculation gives

$$\left(\frac{\partial}{\partial z_\alpha}\otimes 1 \otimes 1\right)\varphi_{0,\ell-1} = \frac{-i}{4\pi}\sum_{k=1}^{\ell-1}(1\otimes 1\otimes A_k(e_\alpha))\,\varphi_{0,\ell-2}, \tag{6.7}$$

from which the claim follows. □

Therefore Theorem 6.4 will follow from:

PROPOSITION 6.6. *For any $1 \leq j \leq \ell$, we have*

$$d\Lambda^{(j)}_{q,\ell-1} = -(A_j + B_j + C_j^-),$$

where

$$C_- = \frac{1}{4\pi} \sum_{k=1}^{\ell-1} A_{jk}(f_-)\varphi_{q,\ell-2}$$

with $A_{jk}(f_-) = \sum_{\mu=1}^{p} A_j(e_\mu)A_k(e_\mu)$.

Proof. Since $\varphi_{q,\ell-1}$ is closed, we have

$$(p+q+\ell-2)d\Lambda_{q,\ell-1} = dh'_j\varphi_{q,\ell-1} = \{d, h'_j\}\varphi_{q,\ell-1},$$

where $\{A, B\}$ denotes the anticommutator $AB + BA$. It is easy to see that

$$\{d'_{\mathcal{F}}, h'_j\}\varphi_{q,\ell-1} = 0,$$

so that we only need to compute $\{d''_{\mathcal{F}}, h'_j\}$ and $\{d_V, h'_j\}$.

LEMMA 6.7. *As operators on* $\mathcal{F} \otimes \bigwedge^\bullet(\mathfrak{p}^*) \otimes T(V)$,

$$\text{(i)} \quad 4\pi i\{d''_{\mathcal{F}}, h'_j\} = \sum_{\alpha,\mu} z_\mu \frac{\partial}{\partial z_\alpha} z_\alpha \otimes 1 \otimes A_j(e_\mu) - \sum_{\mu,\nu} z_\mu \otimes D_{\mu\nu} \otimes A_j(e_\nu).$$

$$\text{(ii)} \quad i\{d_V, h'_j\} = \sum_{\alpha,\mu} \frac{\partial}{\partial z_\alpha} \otimes 1 \otimes A_j(e_\mu)\rho(X_{\alpha\mu}) + \sum_{\alpha,\beta} \frac{\partial}{\partial z_\beta} \otimes D_{\alpha\beta} \otimes A_j(e_\alpha).$$

Here the operators $D_{\alpha\beta}$ and $D_{\mu\nu}$ are the derivations of $\bigwedge^\bullet(\mathfrak{p}^*)$ determined by

$$D_{\alpha\beta}\omega_{\gamma\mu} = \delta_{\beta\gamma}\omega_{\alpha\mu} \qquad \textit{and} \qquad D_{\mu\nu}\omega_{\alpha\lambda} = \delta_{\lambda\nu}\omega_{\alpha\mu}.$$

Proof. For (i), using the definitions of the operators and $\frac{\partial}{\partial z_\beta} z_\alpha = z_\alpha \frac{\partial}{\partial z_\beta} + \delta_{\alpha\beta}$, we easily see

$$4\pi i\{d''_{\mathcal{F}}, h'_j\} = \sum_{\substack{\alpha,\mu\\ \beta,\nu}} z_\mu z_\alpha \frac{\partial}{\partial z_\beta} \otimes \{A_{\alpha\mu}, A^*_{\beta\nu}\} \otimes A_j(e_\nu) + \delta_{\alpha\beta} z_\mu \otimes A^*_{\beta\nu}A_{\alpha\mu} \otimes A_j(e_v).$$

Here and in the following we write $A_{\alpha,\mu}$ for $A(\omega_{\alpha,\mu})$. The Clifford identities imply

$$\{A_{\alpha\mu}, A^*_{\beta\nu}\} = \delta_{\alpha\beta}\delta_{\mu\nu}$$

and

$$A^*_{\alpha\nu}A_{\alpha\mu} = \begin{cases} I - A_{\alpha\mu}A^*_{\alpha\nu}, & \text{if } \mu = \nu \\ -A_{\alpha\mu}A^*_{\alpha\nu}, & \text{if } \mu \neq \nu. \end{cases}$$

Note

$$\sum_{\alpha} A_{\alpha\mu}A^*_{\alpha\nu} = D_{\mu\nu} \qquad \text{and} \qquad \sum_{\mu} A_{\alpha\mu}A^*_{\beta\mu} = D_{\alpha\beta}. \tag{6.8}$$

We therefore obtain

$$\sum_{\alpha,\mu} z_\mu \left(z_\alpha \frac{\partial}{\partial z_\alpha} + 1 \right) \otimes 1 \otimes A_j(e_\mu) \;-\; \sum_{\alpha,\mu,\nu} z_\mu \otimes A_{\alpha\mu}A^*_{\alpha\nu} \otimes A_j(e_\nu),$$

and the assertion follows from (6.8).

For (ii), first note $\rho(X_{\alpha\mu})A_j(e_\nu) = \delta_{\mu\nu}A_j(e_\alpha) + A_j(e_v)\rho(X_{\alpha\mu})$. One then obtains

$$\begin{aligned} \{d_V, h'_j\} &= \sum_{\substack{\alpha,\mu \\ \beta,\nu}} \frac{\partial}{\partial z_\beta} \otimes \{A_{\alpha\mu}, A^*_{\beta\nu}\} \otimes A_j(e_v)\rho(X_{\alpha\mu}) \;+\; \frac{\partial}{\partial z_\beta} \otimes A_{\alpha\mu}A^*_{\beta\nu} \otimes \delta_{\mu\nu}A_j(e_\alpha) \\ &= \sum_{\alpha,\mu} \frac{\partial}{\partial z_\alpha} \otimes 1 \otimes A_j(e_\mu)\rho(X_{\alpha\mu}) \;+\; \sum_{\alpha,\beta} \frac{\partial}{\partial z_\beta} \otimes D_{\alpha\beta} \otimes A_j(e_\alpha), \end{aligned}$$

by (6.8). □

Proposition 6.6 now follows from:

LEMMA 6.8.

(i) $\{d''_{\mathcal{F}}, h'_j\}\varphi_{q,\ell-1} \;=\; -(p+q+\ell-2)A_j.$

(ii) $\{d_V, h'_j\}\varphi_{q,\ell-1} \;=\; -(p+q+\ell-2)(B_j + C^-_j).$

Proof. We use Lemma 6.7. For (i), we first have

$$\begin{aligned} \sum_{\alpha,\mu} \left(z_\mu \frac{\partial}{\partial z_\alpha} z_\alpha \otimes 1 \otimes A_j(e_\mu) \right) \varphi_{q,\ell-1} &= (p+q+\ell-1) \left(\sum_{\mu} z_\mu \otimes 1 \otimes A_j(e_\mu) \right) \varphi_{q,\ell-1} \\ &= -4\pi i(p+q+\ell-1)A_j, \end{aligned}$$

which follows immediately from

$$\begin{aligned} \sum_{\alpha} \left(\frac{\partial}{\partial z_\alpha} z_\alpha \otimes 1 \otimes 1 \right) \varphi_{q,\ell-1} \;&=\; \sum_{\alpha} \left((z_\alpha \frac{\partial}{\partial z_\alpha} + 1) \otimes 1 \otimes 1 \right) \varphi_{q,\ell-1} \\ &=\; (p+q+\ell-1)\varphi_{q,\ell-1} \end{aligned} \tag{6.9}$$

by (6.1). Furthermore, by [KM4], Lemma 8.2 we have

$$(1 \otimes D_{\mu\nu} \otimes 1)\varphi_{q,0} = \delta_{\mu\nu}\varphi_{q,0},$$

and therefore

$$\begin{aligned}\sum_{\mu,\nu} \left(z_\mu \otimes D_{\mu\nu} \otimes A_j(e_\nu)\right) \varphi_{q,\ell-1} &= \sum_{\mu,\nu} (1 \otimes D_{\mu\nu} \otimes 1)\varphi_{q,0} \cdot \left(z_\mu \otimes 1 \otimes A_j(e_\nu)\right) \varphi_{0,\ell} \\ &= \left(\sum_{\mu} z_\mu \otimes 1 \otimes A_j(e_\mu)\right) \varphi_{q,\ell-1} = -4\pi i A_j.\end{aligned}$$

Lemma 6.7 now gives (i). For (ii), by (6.6) we first note

$$(1 \otimes 1 \otimes \rho(X_{\alpha\mu})\, \varphi_{q,\ell-1} = \frac{-i}{4\pi} \sum_{k=1}^{\ell-1} (z_\alpha \otimes 1 \otimes A_k(e_\mu))\varphi_{q,\ell-2}. \tag{6.10}$$

Thus, using (6.9), we see

$$\begin{aligned}\sum_{\alpha,\mu} \left(\frac{\partial}{\partial z_\alpha} \otimes 1 \otimes A_j(e_\mu)\rho(X_{\alpha\mu})\right) \varphi_{q,\ell-1} &= \frac{-i}{4\pi} \left(\sum_{k=1}^{\ell-1} A_{jk}(f_-)\right) (p+q+\ell-2)\varphi_{q,\ell-2} \\ &= -i(p+q+\ell-2)C_j^-.\end{aligned}$$

Finally, by [KM4], Lemma 8.2, we have

$$(1 \otimes D_{\alpha\beta} \otimes 1)\varphi_{q,0} = (z_\beta \frac{\partial}{\partial z_\alpha} \otimes 1 \otimes 1)\varphi_{q,0}.$$

Hence

$$\begin{aligned}&\sum_{\alpha,\beta} \left(\frac{\partial}{\partial z_\beta} \otimes D_{\alpha\beta} \otimes A_j(e_\alpha)\right) \varphi_{q,\ell-1} \\ &= \sum_{\alpha,\beta} \left(\frac{\partial}{\partial z_\beta} z_\beta \otimes 1 \otimes A_j(e_\alpha)\right) \left(\left(\frac{\partial}{\partial z_\alpha} \otimes 1 \otimes 1\right) \varphi_{q,0}\right) \cdot \varphi_{0,\ell-1} \\ &= \sum_{\alpha,\beta} \left(\left(\frac{\partial}{\partial z_\beta} z_\beta \frac{\partial}{\partial z_\alpha} \otimes 1 \otimes 1\right) \varphi_{q,0}\right) \cdot (1 \otimes 1 \otimes A_j(e_\alpha))\, \varphi_{0,\ell-1} \\ &\quad + \sum_{\alpha,\beta} ((\frac{\partial}{\partial z_\alpha} \otimes 1 \otimes 1)\varphi_{q,0}) \cdot \left(\frac{\partial}{\partial z_\beta} z_\beta \otimes 1 \otimes A_j(e_\alpha)\right) \varphi_{0,\ell-1}.\end{aligned} \tag{6.11}$$

The second term, using (6.9), is equal to $-i(\ell-1)B$, while for the first we have

$$\sum_{\alpha,\beta}\left(\left(\frac{\partial}{\partial z_\beta}(\frac{\partial}{\partial z_\alpha}z_\beta-\delta_{\alpha\beta})\otimes 1\otimes 1\right)\varphi_{q,0}\right)\cdot(1\otimes 1\otimes A_j(e_\alpha))\,\varphi_{0,\ell-1},$$

and this, using (6.9) again, equals to $-i(p+q-1)B$. This finishes the proof of (ii). ◻

This concludes the proof of Proposition 6.6 and hence the proof of Theorem 6.4! ◻

6.5. Holomorphicity. We will now show Theorem 5.10(i), i.e., that the cohomology class $[\varphi_{q,[\ell]}]$ is holomorphic.

Following [KM4], we define another operator h on $\mathcal{F}\otimes\bigwedge^\bullet(\mathfrak{p}^*)\otimes S^\bullet(V)$ by

$$h = \sum_{\alpha,\mu} z_\mu\frac{\partial}{\partial z_\alpha}\otimes A^*(\omega_{\alpha\mu})\otimes 1.$$

Definition 6.9. We introduce a $(q-1)$-form $\psi_{q,\ell}$ by

$$\psi_{q,\ell} := \frac{-1}{2(p+q-1)}\left(h\varphi_{q,0}\right)\cdot\varphi_{0,\ell}.$$

It will be convenient to note that we could have also defined $\psi_{q,\ell}$ by letting h act on $\varphi_{q,\ell}$:

LEMMA 6.10.

$$\psi_{q,\ell} = \frac{-1}{2(p+q+\ell-1)}h\varphi_{q,\ell}.$$

Proof. We have

$$\begin{aligned}
h\varphi_{q,\ell} &= \left(h\varphi_{q,0}\right)\cdot\varphi_{0,\ell}+\sum_{\alpha,\mu}(z_\mu\otimes A^*(\omega_{\alpha\mu})\otimes 1)\varphi_{q,0} \\
&\quad\cdot\left(\frac{\partial}{\partial z_\alpha}\otimes 1\otimes 1\right)\varphi_{0,\ell} && (6.12)\\
&= \left(h\varphi_{q,0}\right)\cdot\varphi_{0,\ell}-\frac{i}{4\pi}\sum_{j=1}^{\ell}h_j''\varphi_{q,\ell-1}, && (6.13)
\end{aligned}$$

with $h_j'' = \sum_{\alpha,\mu} z_\mu \otimes A^*(\omega_{\alpha\mu}) \otimes A_j(e_\alpha)$. But now one easily checks that

$$\frac{-1}{4\pi} h_j'' \varphi_{q,\ell-1} = \varphi'_{q,0} \cdot \varphi_{0,\ell}$$

with

$$\varphi'_{q,0} = \sum_{\mu,\alpha_1,\dots,\alpha_{q-1}} (-1)^{\mu-p-1} z_\mu z_{\underline{\alpha}} \otimes \omega_{\alpha_1 p+1} \wedge \cdots \wedge \widehat{\omega_{\alpha_{\mu-p}\mu}} \wedge \cdots \omega_{\alpha_{q-1} p+q} \otimes 1.$$

On the other hand, we also compute

$$h\varphi_{q,0} = (p+q-1)\varphi'_{q,0}.$$

From this the lemma follows. □

THEOREM 6.11. (Theorem 5.10) *The action of the lowering operator L on $\varphi_{q,\ell}$ is given by*

$$\omega(L)\varphi_{q,\ell} = d\left(\psi_{q,\ell} + \frac{1}{2}\sum_{j=1}^{\ell} \Lambda^{(j)}_{q,\ell-1}\right) - \frac{1}{4\pi} A(f)\varphi_{q,\ell-2}.$$

Remark 6.12. This theorem is the generalization of one of the main points in [KM4] for $\ell = 0$, the trivial coefficient case. Namely, [KM4], Lemma 8.3 states

$$\omega(L)\varphi_{q,0} = d\psi_{q,0}. \tag{6.14}$$

Proof of Theorem 6.11. We first compute the left-hand side:

LEMMA 6.13.

$$\omega(L)\varphi_{q,\ell} = \big(\omega(L)\varphi_{q,0}\big) \cdot \varphi_{0,\ell} \; - \; \sum_{j=1}^{\ell} B_j \; - \; \frac{1}{4\pi} A(f_+)\varphi_{q,\ell-2},$$

with B_j as in Lemma 6.5.

Proof. By Lemma A.1 we have

$$\omega(L)\varphi_{q,\ell} \;=\; \frac{-1}{8\pi} \sum_{\mu} \left(z_\mu^2 \otimes 1 \otimes 1\right) \varphi_{q,\ell} \tag{6.15}$$

$$+2\pi c_{q,\ell} \sum_{\substack{\gamma \\ \underline{\alpha},\underline{\beta}}} \left(\frac{\partial^2}{\partial z_\gamma^2} z_{\underline{\alpha}}\right) z_{\underline{\beta}} \otimes \omega_{\underline{\alpha}} \otimes e_{\underline{\beta}} \tag{6.16}$$

$$+4\pi c_{q,\ell} \sum_{\substack{\gamma \\ \underline{\alpha},\underline{\beta}}} \left(\frac{\partial}{\partial z_\gamma} z_{\underline{\alpha}}\right)\left(\frac{\partial}{\partial z_\gamma} z_{\underline{\beta}}\right) \otimes \omega_{\underline{\alpha}} \otimes e_{\underline{\beta}} \tag{6.17}$$

$$+2\pi c_{q,\ell} \sum_{\substack{\gamma \\ \underline{\alpha},\underline{\beta}}} z_{\underline{\alpha}} \left(\frac{\partial^2}{\partial z_\gamma^2} z_{\underline{\beta}}\right) \otimes \omega_{\underline{\alpha}} \otimes e_{\underline{\beta}}. \tag{6.18}$$

The first two terms ((6.15) and (6.16) give $\left(\omega(L)\varphi_{q,0}\right) \cdot \varphi_{0,\ell}$. By (6.7), the third term (6.17) is equal to

$$-i \sum_\gamma \left(\frac{\partial}{\partial z_\gamma} \otimes 1 \otimes 1\right) \varphi_{q,0} \cdot \sum_{j=1}^{\ell} (1 \otimes 1 \otimes A_j(e_\gamma))\varphi_{0,\ell-1} = -\sum_{j=1}^{\ell} B_j.$$

For the fourth term (6.18) in the sum above, we apply (6.7) twice and obtain

$$\begin{aligned} &\frac{-i}{2}\varphi_{q,0} \cdot \sum_{j=1}^{\ell}\sum_\gamma \left(\frac{\partial}{\partial z_\gamma} \otimes 1 \otimes A_j(e_\gamma)\right)\varphi_{0,\ell-1} \\ &= \frac{-1}{8\pi}\varphi_{q,0} \cdot \sum_{j=1}^{\ell}\sum_{k=1}^{\ell-1}\sum_\gamma (1 \otimes 1 \otimes A_j(e_\gamma)A_k(e_\gamma))\varphi_{0,\ell-2} = \frac{-1}{4\pi}A(f_+)\varphi_{0,\ell-2}. \end{aligned}$$

□

We now compute $d\psi_{q,\ell}$:

LEMMA 6.14.

$$\text{(i)} \qquad d_{\mathcal{F}}\psi_{q,\ell} = \left(d_{\mathcal{F}}\psi_{q,0}\right) \cdot \varphi_{0,\ell} - \frac{1}{2}\sum_{j=1}^{\ell} B_j.$$

$$\text{(ii)} \qquad d_V\psi_{q,\ell} = \frac{1}{2}\sum_{j=1}^{\ell} A_j.$$

with A_j and B_j as in Lemma 6.5.

Proof. For (i), we first observe

$$d''_{\mathcal{F}}\left(\left(h\varphi_{q,0}\right) \cdot \varphi_{0,\ell}\right) = \left(d''_{\mathcal{F}} h\varphi_{q,0}\right) \cdot \varphi_{0,\ell}. \tag{6.19}$$

For $d'_{\mathcal{F}}$ we get

$$d'_{\mathcal{F}}\left((h\varphi_{q,0})\cdot\varphi_{0,\ell}\right) = -4\pi\sum_{\alpha,\mu}\left(\frac{\partial}{\partial z_\alpha}\otimes 1\otimes 1\right) \times\left[\left(\left(\frac{\partial}{\partial z_\mu}\otimes A_{\alpha\mu}\otimes 1\right)h\varphi_{q,0}\right)\cdot\varphi_{0,\ell}\right] \tag{6.20}$$

$$= (d'_{\mathcal{F}}h\varphi_{q,0})\cdot\varphi_{0,\ell} -4\pi\sum_{\alpha,\mu}\left(\left(\frac{\partial}{\partial z_\mu}\otimes A_{\alpha\mu}\otimes 1\right)h\varphi_{q,0}\right) \cdot\left(\left(\frac{\partial}{\partial z_\alpha}\otimes 1\otimes 1\right)\varphi_{0,\ell}\right). \tag{6.21}$$

For the first term in (6.21) we see

$$\begin{aligned}\sum_{\mu}\left(\frac{\partial}{\partial z_\mu}\otimes A_{\alpha\mu}\otimes 1\right)(h\varphi_{q,0}) &= \sum_{\substack{\mu,\nu\\ \beta}}\left(\frac{\partial}{\partial z_\beta}\frac{\partial}{\partial z_\mu}z_\nu\otimes A_{\alpha\mu}A^*_{\beta\mu}\otimes 1\right)\varphi_{q,0}\\ &= \sum_{\beta,\mu}\left(\frac{\partial}{\partial z_\beta}\otimes A_{\alpha\mu}A^*_{\beta\nu}\otimes 1\right)\varphi_{q,0}\\ &= \sum_{\beta}\left(\frac{\partial}{\partial z_\beta}\otimes D_{\alpha\beta}\otimes 1\right)\varphi_{q,0}.\end{aligned}$$

Combining this with (6.7) we obtain for (6.21)

$$i\sum_{j=1}^{\ell}\sum_{\alpha,\beta}\left(\left(\frac{\partial}{\partial z_\beta}\otimes D_{\alpha\beta}\otimes A_j(e_\alpha)\right)\varphi_{q,0}\right)\cdot\varphi_{0,\ell-1}.$$

But this is exactly (up to a constant) the term (6.11), i.e., (6.21) is equal to $(p+q-1)\sum_{j=1}^{\ell}B_j$. This together with (6.19), (6.20) and collecting the constants implies (i).

For (ii), we will use Lemma 6.10. We easily see

$$\{d_V,h\} = \sum_{\alpha,\mu}z_\mu\frac{\partial}{\partial z_\alpha}\otimes 1\otimes\rho(X_{\alpha\mu}).$$

By (6.6) and (6.1), we get

$$\begin{aligned}d_V h\varphi_{q,\ell} &= \{d_V,h\}\varphi_{q,\ell}\\ &= \frac{-i}{4\pi}\sum_{j=1}^{\ell}\sum_{\alpha,\mu}\left(z_\mu\frac{\partial}{\partial z_\alpha}z_\alpha\otimes 1\otimes A_j(e_\mu)\right)\varphi_{q,\ell-1}\\ &= -\sum_{j=1}^{\ell}(p+q+\ell-1)A_j.\end{aligned}$$

This implies (ii). □

We are now in the position to finish the proof of Theorem 6.11: Combining Lemma 6.13, Lemma 6.14 and Proposition 6.6, we get

$$\begin{aligned}\omega(L)\varphi_{q,\ell}-d\left(\psi_{q,\ell}+\frac{1}{2}\sum_{j=1}^{q}\Lambda_{q,\ell-1}^{(j)}\right) &= (\omega(L)\varphi_{q,0})\cdot\varphi_{0,\ell}-\sum_{j=1}^{\ell}B_j-\frac{1}{4\pi}A(f_+)\varphi_{q,\ell-2}\\ &\quad -(d_{\mathcal{F}}\psi_{q,0})\cdot\varphi_{0,\ell}+\frac{1}{2}\sum_{j=1}^{\ell}B_j-\frac{1}{2}\sum_{j=1}^{\ell}A_j\\ &\quad +\frac{1}{2}\sum_{j=1}^{\ell}(A_j+B_j+C_j^-)\\ &= -\frac{1}{4\pi}A(f)\varphi_{q,\ell-2},\end{aligned}$$

via (6.14) and $A(f_-)\varphi_{q,\ell-1}=\frac{1}{2}\sum_{j=1}^{\ell}C_j^-$, since $A(f)=A(f_+)-A(f_-)$. □

7. Main result. In this section, we first construct the cohomology class $[\theta_{nq,[\lambda]}]$ and use the fundamental properties of $\varphi_{nq,[\lambda]}$ to derive our main result.

First note that over $\mathbb{R}$, the Weil representation action of the standard Siegel parabolic in $Sp(n,\mathbb{R})$ on $\varphi\in\mathcal{S}(V^n)$ is given by

$$\omega\left(\begin{pmatrix} a & 0\\ 0 & {}^ta^{-1}\end{pmatrix}\right)\varphi(\mathbf{x})=(\det a)^{m/2}\varphi(\mathbf{x}a)$$

for $a\in GL_n^+(\mathbb{R})$, and

$$\omega\left(\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix}\right)\varphi(\mathbf{x})=e^{\pi i tr(b(\mathbf{x},\mathbf{x}))}\varphi(\mathbf{x})$$

for $b\in Sym_n(\mathbb{R})$. Here $\mathbf{x}=(x_1,\cdots,x_n)\in V^n$, as before.

Globally, we let $\mathbb{A}=\mathbb{A}_{\mathbb{K}}$ be the ring of adeles of $\mathbb{K}$. Let $G'(\mathbb{A})$ be the two-fold cover $Sp(n,\mathbb{A})$, which acts on $S(\underline{V}^n(\mathbb{A}))$ via the (global) Weil representation $\omega=\omega_{\underline{V}}$.

For $g'\in G'(\mathbb{A})$, we define the standard theta kernel associated to a Schwartz function $\varphi=\varphi_f\otimes\varphi_\infty\in S(V_{\mathbb{A}}^n)$ by

$$\theta(g',\varphi):=\sum_{\mathbf{x}\in\underline{V}^n}\omega(g')\varphi(\mathbf{x}).$$

By abuse of notation, we let $\varphi_{nq,[\lambda]}=\otimes_{i=1}^{r}\varphi_{v_i}=\varphi_\infty$, where φ_{v_i} is the Schwartz form $\varphi_{nq,[\lambda]}$ at the first infinite place and the standard Gaussian φ_0 at the other infinite places, and we define $\varphi_{nq,\ell}$ in the same way. For the finite places, we let φ_f correspond to the characteristic function of $h+\mathfrak{b}L^n$.

Given $\tau = (\tau_1, \ldots, \tau_r) \in \mathbb{H}_n^r$ we let $g'_\tau \in Sp(n, \mathbb{K}_\infty)$ be a standard element which moves the base point $(i, \ldots, i) \in \mathbb{H}_n^r$ to τ, i.e.,

$$g'_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix},$$

with $v = a({}^t a^{-1})$. We consider $g'_\tau \in G'(\mathbb{A})$ in the natural way.

We write ρ_λ for the representation action of $GL_n(\mathbb{C})$ on $S_\lambda(\mathbb{C}^n)$.

For $\tau \in \mathbb{H}_n^r$, we then define

$$\begin{aligned} \theta_{nq,[\lambda]}(\tau, z) &= N_{\mathbb{Q}}^{\mathbb{K}}(\det(a)^{-m})\rho_\lambda^*(a)\theta(g'_\tau, \varphi_f \otimes \varphi_{nq,[\lambda]}) \\ &= \sum_{\mathbf{x} \in h + \mathfrak{b}L^n} \rho_\lambda^*(a)\varphi_{nq,[\lambda]}(\mathbf{x}a)e_*((\mathbf{x}, \mathbf{x})u/2), \end{aligned}$$

and similarly, $\theta_{nq,\ell}(\tau, z)$. Here

$$e_*(A) = \exp\left(2\pi i \sum_{j=1}^{r} tr(\lambda_i(A))\right),$$

for $A \in M_{n,n}(\mathbb{K})$.

We denote by $\mathcal{A}(\Gamma', S_\lambda(\mathbb{C}^n)^* \otimes \det^{-m/2})$ the space of vector-valued (not necessarily holomorphic) Hilbert-Siegel modular forms of degree n for a congruence subgroup Γ' and for the representation $(\rho_\lambda^* \otimes \det^{-m/2}, \det^{-m/2}, \cdots, \det^{-m/2})$.

Then Theorem 5.6 and the standard theta machinery give us:

PROPOSITION 7.1.

$$\theta_{nq,[\lambda]}(\tau, z) \in \mathcal{A}(\Gamma', S_\lambda(\mathbb{C}^n)^* \otimes \det{}^{-m/2}) \otimes A^{nq}(M, S_{[\lambda]}\mathcal{V}),$$

i.e, $\theta_{nq,[\lambda]}(\tau, z)$ is a vector-valued non-holomorphic Hilbert-Siegel modular form for the representation $(\rho_\lambda^* \otimes \det^{-m/2}, \det^{-m/2}, \ldots, \det^{-m/2})$ with values in the $S_{[\lambda]}(\mathcal{V})$-valued closed differential nq-forms of the manifold M.

The Fourier expansion of $\theta_{nq,[\lambda]}(\tau)$ is given by

$$\theta_{nq,[\lambda]}(\tau) = \sum_{\beta \in Sym_n(\mathbb{K})} \theta_{\beta,nq,[\lambda]}(v)e_*(\beta\tau),$$

with

$$\theta_{\beta,nq,[\lambda]}(v) = \sum_{\mathbf{x} \in \mathcal{L}_\beta} \rho_\lambda^*(a)\varphi_{nq,[\lambda]}(\mathbf{x}a)e_*(-i(\mathbf{x}, \mathbf{x})v/2).$$

Definition 7.2. If η is a rapidly decreasing $S_{[\lambda]}(\mathcal{V})$-valued closed $(p-n)q$ form on M representing a class $[\eta] \in H_c^{(p-n)q}(M, S_{[\lambda]}\mathcal{V})$, we define

$$\Lambda_{nq,[\lambda]}(\tau, \eta) = \int_M \eta \wedge \theta_{nq,[\lambda]}(\tau) \in \mathcal{A}(\Gamma', S_\lambda(\mathbb{C}^n)^* \otimes \det{}^{-m/2}).$$

We define $\Lambda_{nq,\ell}(\tau, \eta)$ in the same fashion for η taking values in $T^\ell(\mathcal{V})$. If η is $S_{[\lambda]}(\mathcal{V})$-valued, then we have

$$\Lambda_{nq,\ell}(\tau, \eta) = \Lambda_{nq,[\lambda]}(\tau, \eta).$$

Before we can state our main result, we need a bit more notation. For $q = 2k$ even, we let e_q be the Euler form of the symmetric space D (which is the Euler class of the tautological vector bundle over D, i.e., the fiber over a point $z \in D$ is given by the negative q-plane z) and zero for q odd. Here e_q is normalized such that it is given in $\bigwedge^q(\mathfrak{p}^*)$ by

$$e_q = \left(-\frac{1}{4\pi}\right)^k \frac{1}{k!} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) \Omega_{p+\sigma(1),p+\sigma(2)} \cdots \Omega_{p+\sigma(2k-1),p+\sigma(2k)},$$

with

$$\Omega_{\mu\nu} = \sum_{\alpha=1}^{p} \omega_{\alpha\mu} \wedge \omega_{\alpha\nu}.$$

Remark 7.3. The main result of [KM4] is that in the scalar valued case, the generating series

$$\sum_{t=0}^{n} \sum_{\substack{\beta \geq 0 \\ \text{rank}\, \beta = t}} \left(PD(C_\beta) \cup e_q^{n-t}\right) e_*(\beta\tau),$$

is a classical Hilbert-Siegel modular form of weight $m/2$. The key point is that for nondegenerate $\mathbf{x}$ such that $(\mathbf{x}, \mathbf{x})$ positive semidefinite of rank t, $\varphi_{nq,0}(\mathbf{x})$ is (essentially) a Thom form for the cycle $C_X \cap e_q^{n-t}$. One has

$$\int_{\Gamma_X \backslash D} \eta \wedge \varphi_{nq,0}(\mathbf{x}a) = \left(\int_{C_X} \eta \wedge e_q^{n-t}\right) e_*(i(\mathbf{x}, \mathbf{x})v/2).$$

Remark 7.4. Actually, in [KM4] the noncompact hyperbolic case of signature $(p, 1)$ with $n = p - 1$ is excluded (when the cycles are infinite geodesics). In the following, we will also exclude this case. Note however that for signature $(2, 1)$ and $n = 1$ this restriction was removed in [FM1], and our result will also hold in that particular case.

The following two theorems are the generalization of the main result of [KM4].

THEOREM 7.5. *The cohomology class $[\theta_{nq,[\lambda]}]$ is holomorphic; i.e., it defines a holomorphic Siegel modular form of genus n with values in $S_\lambda(\mathbb{C}^n)^* \otimes \chi(-m/2)$ and with coefficients in $H^{nq}(M, S_{[\lambda]}\mathcal{V})$.*

Proof. This follows immediately from Theorem 5.10: We have $[\overline{\partial}\varphi_{nq,[\lambda]}] = 0$, thus $[\overline{\partial}\theta_{nq,[\lambda]}](\tau) = 0$. □

THEOREM 7.6. *The Fourier expansion of $[\theta_{nq,[\lambda]}](\tau)$ is given by*

$$[\theta_{nq,[\lambda]}](\tau) = \sum_{t=0}^{n} \sum_{\substack{\beta \geq 0 \\ \text{rank } \beta = t}} \left(PD(C_{\beta,[\lambda]}) \wedge e_q^{n-t} \right) e_*(\beta\tau),$$

where $PD(C_{\beta,[\lambda]})$ denotes the Poincaré dual class of $PD(C_{\beta,[\lambda]})$. Furthermore, if q is odd or $i(\lambda) = n$, then $[\theta_{nq,[\lambda]}](\tau)$ is a cusp form.

This is equivalent to:

THEOREM 7.7. *For η a rapidly decreasing closed $q(n-p)$ form on M with values in $S_{[\lambda]}(\mathcal{V})$, the generating series*

$$\Lambda_{nq,[\lambda]}(\tau,\eta) = \sum_{t=0}^{n} \sum_{\substack{\beta \geq 0 \\ \text{rank } \beta = t}} \int_{C_{\beta,[\lambda]}} \left(\eta \wedge e_q^t \right) e_*(\beta\tau)$$

is a holomorphic Siegel modular form of type $(\det^{-m/2} \otimes \rho_\lambda^*, \det^{-m/2}, \cdots, \det^{-m/2})$.

Note that by Theorem 3.7 a basis of $S_\lambda(\mathbb{C}^n)$ is given by $\pi_\lambda \epsilon_{f(\lambda)}$, where $f(\lambda)$ runs through the semistandard fillings $SS(\lambda, n)$. With respect to this basis, we define the $\pi_\lambda f(\lambda)$ component $(\Lambda_{[\lambda]}(\tau,\eta))_{\pi_\lambda f(\lambda)}$ by

$$(\Lambda_{nq,[\lambda]}(\tau,\eta))_{\pi_\lambda f(\lambda)} = \left(\int_M \eta \wedge \theta_{nq,[\lambda]}(\tau) \right) (\epsilon_{f(\lambda)}) .$$

Note here that the value of Λ at $\epsilon_{f(\lambda)}$ and $\pi_\lambda f(\lambda)$ is the same. We then have

$$\begin{aligned} (\Lambda_{nq,[\lambda]}(\tau,\eta))_{\pi_\lambda f(\lambda)} &= \sum_{t=0}^{n} \sum_{\substack{\beta \geq 0 \\ \text{rank } \beta = t}} \int_{C_{\beta,[f(\lambda)]}} \left(\eta \wedge e_q^t \right) e_*(\beta\tau) \\ &= \sum_{t=0}^{n} \sum_{\substack{\mathbf{x} \in \mathcal{L}_t^c \\ (\mathbf{x},\mathbf{x}) \geq 0 \\ \text{mod } \Gamma}} \int_{C_\mathbf{x}} \left((\eta, \mathbf{x}_{f(\lambda)}) \wedge e_q^t \right) e_*((\mathbf{x},\mathbf{x})\tau/2), \end{aligned}$$

where

$$\mathcal{L}_t^c = \{\mathbf{x} \in h + \mathfrak{b}L^n\colon \ \mathrm{rank}\,(\mathbf{x},\mathbf{x}) = t; \mathbf{x} \textit{ nondegenerate}\}.$$

Proof. We denote the β Fourier coefficient of $(\Lambda_{nq,[\lambda]}(\tau,\eta))_{f(\lambda)}$ by

$$a_\beta = \left(\int_M \eta \wedge \sum_{\mathbf{x}\in\mathcal{L}_\beta} \rho_\lambda^*(a)\varphi_{nq,[\lambda]}(\mathbf{x}a)\right)((a^{-1}\epsilon)_{f(\lambda)})e_*(-i\beta v).$$

We first note:

LEMMA 7.8. *Assume that β* not *positive semidefinite. Then*

$$a_\beta = 0.$$

Proof. For $n > 1$, this follows from the Koecher principle, since $\Lambda_{[\lambda]}(\tau,\eta)$ is holomorphic. For $n = 1$, so that $\beta < 0$, the vanishing follows from the vanishing in the trivial coefficient case by an argument similar to the positive definite coefficient, see Lemma 7.9 below. □

For β positive semidefinite, we write

$$a_\beta^c = \left(\int_M \eta \wedge \sum_{\mathbf{x}\in\mathcal{L}_\beta^c} \rho_\lambda^*(a)\varphi_{nq,[\lambda]}(\mathbf{x}a)\right)((a^{-1}\epsilon)_{f(\lambda)})e_*(-i\beta v)$$

for the contribution of the closed orbits and $a_\beta^d = a_\beta - a_\beta^c$ for the degenerate part.

LEMMA 7.9. *Assume β is positive semidefinite of rank t. Then*

$$a_\beta^c = \int_{C_{\beta,[f(\lambda)]}} \left(\eta \wedge e_q^{n-t}\right).$$

Proof. By the usual unfolding argument, we obtain

$$\begin{aligned} a_\beta^c e_*(i\beta v) &= \left(\int_{\Gamma\backslash D} \eta \wedge \sum_{x\in\mathcal{L}_\beta^c} \varphi_{nq,[\lambda]}(\mathbf{x}a)\right)((a^{-1}\epsilon)_{f(\lambda)}) \\ &= \left(\int_{\Gamma\backslash D} \eta \wedge \sum_{x\in\Gamma\backslash\mathcal{L}_\beta^c}\ \sum_{\gamma\in\Gamma_X\backslash\Gamma} \gamma^*\varphi_{nq,[\lambda]}(\mathbf{x}a)\right)((a^{-1}\epsilon)_{f(\lambda)}) \end{aligned}$$

$$= \sum_{x \in \Gamma \backslash \mathcal{L}_\beta^c} \left(\int_{\Gamma_x \backslash D} \eta \wedge \varphi_{nq,[\lambda]}(\mathbf{x}a) \right) ((a^{-1}\epsilon)_{f(\lambda)}).$$

But now by Theorem 5.9 and Lemma 5.8, we have

$$[\varphi_{nq,[\lambda]}(\mathbf{x}a)((a^{-1}\epsilon)_{f(\lambda)})] = \left[(1 \otimes 1 \otimes \mathbf{x}_{[f(\lambda)]})\varphi_{nq,0}(\mathbf{x}a)\right].$$

Thus

$$\begin{aligned} \left(\int_{\Gamma_x \backslash D} \eta \wedge \rho_\lambda^*(a)\varphi_{nq,[\lambda]}(\mathbf{x}a) \right)(\epsilon_{f(\lambda)}) &= \int_{\Gamma_x \backslash D} \eta \wedge (1 \otimes 1 \otimes \mathbf{x}_{[f(\lambda)]})\varphi_{nq,0}(\mathbf{x}a) \\ &= \int_{\Gamma_x \backslash D} (\eta, \mathbf{x}_{[f(\lambda)]}) \wedge \varphi_{nq,0}(\mathbf{x}a) \\ &= \left(\int_{C_X} (\eta, \mathbf{x}_{[f(\lambda)]}) \wedge e_q^t \right) e_*(i(\mathbf{x},\mathbf{x})v/2), \end{aligned}$$

by Remark 7.3. This implies $a_\beta^c = \int_{C_{\beta,[f(\lambda)]}} \eta \wedge e_q^t$, as claimed. □

It remains to show:

LEMMA 7.10. *Assume β is positive semidefinite. Then*

$$a_\beta^d = 0.$$

Proof. The recursion formula reduces this to the analogous statement for the singular coefficients in the scalar-valued case in the same way as in Lemma 7.9. The lemma then follows from the vanishing of those coefficients in the scalar-valued case, see [KM4], §4. We leave the details to the reader. □

This concludes the proof of the theorem. □

Appendix A. The Fock model. We briefly review the construction of the Fock model of the (infinitesimal) Weil representation of the symplectic Lie algebra $\mathfrak{sp}(W \otimes \mathbb{C})$, where $(W, \langle\, ,\, \rangle)$ denotes a nondegenerate real symplectic space of dimension $2N$. We follow [Ad2], [KM4]. We let J_0 be a positive definite complex structure on W, i.e., the bilinear form given by $\langle w_1, J_0 w_2 \rangle$ is positive definite. Let $e_1, \dots, e_N; f_1, \dots, f_N$ be a standard symplectic basis of W so that $J_0 e_j = f_j$ and $J_0 f_j = -e_j$. We decompose

$$W \otimes \mathbb{C} = W' \oplus W''$$

into the $+i$ and $-i$ eigenspaces under J_0. Then $w'_j = e_j - if_j$ and $w''_j = e_j + if_j$, $j = 1, \dots, N$ form a basis for W' and W'' respectively. We identify $Sym^2(W)$ with $\mathfrak{sp}(W)$ via

$$(x \circ y)(z) = \langle x, z \rangle y + \langle y, z \rangle x.$$

Given $\lambda \in \mathbb{C}$, we define the quantum algebra (see [Ho]) $\mathcal{W}_\lambda$ to be the tensor algebra $T(W \otimes \mathbb{C})$ modulo the two sided ideal generated by the elements of the form $x \otimes y - y \otimes x - \lambda\langle x, y \rangle 1$. We let p: $T(W \otimes \mathbb{C}) \to \mathcal{W}_\lambda$ be the quotient map. Since $T(W \otimes \mathbb{C})$ is graded, we have a filtration $F^\bullet$ on $\mathcal{W}_\lambda$, and we easily that $[F^k\mathcal{W}_\lambda, F^{k'}\mathcal{W}_\lambda] \subset F^{k+k'-2}\mathcal{W}_\lambda$. Hence $F^2\mathcal{W}_\lambda$ is a Lie algebra. Furthermore, we have a split extension of Lie algebras

$$0 \longrightarrow F^1\mathcal{W}_\lambda \longrightarrow F^2\mathcal{W}_\lambda \longrightarrow \mathfrak{sp}(W \otimes \mathbb{C}) \longrightarrow 0.$$

Here the second map is $p(x \otimes y) \mapsto \lambda(x \circ y) \in Sym^2(W \otimes \mathbb{C}) \simeq \mathfrak{sp}(W \otimes \mathbb{C})$, while the splitting map j: $Sym^2(W \otimes \mathbb{C}) \to F^2\mathcal{W}_\lambda$ is given by

$$j(x \circ y) = \frac{1}{2\lambda}\left(p(x)p(y) + p(y)p(x)\right).$$

(Note the sign error in [KM4], p. 151.)

We let $\mathcal{J}$ be the left ideal in $\mathcal{W}_\lambda$ generated by W'. The projection p induces an isomorphism of the symmetric algebra $Sym^\bullet(W'')$ with $\mathcal{W}_\lambda/\mathcal{J}$. We denote by ρ_λ the action of $\mathcal{W}_\lambda$ on $\mathcal{W}_\lambda/\mathcal{J} \simeq Sym^\bullet(W'')$ given by left multiplication. We now identify $Sym^\bullet(W'')$ with the polynomial functions $\mathcal{P}(\mathbb{C}^N) = \mathbb{C}[z_1, \dots, z_N]$ on W' via $z_j(w''_k) = \langle w'_j, w''_k \rangle = 2i\delta_{jk}$ and observe that then the action of $W \subset \mathcal{W}_\lambda$ on $\mathcal{P}(\mathbb{C}^N)$ is given by

$$\rho_\lambda(w''_j) = z_j \qquad \text{and} \qquad \rho_\lambda(w'_j) = 2i\lambda\frac{\partial}{\partial z_j}.$$

This gives the action of $\mathcal{W}_\lambda$, and we obtain an action $\omega_\lambda = \rho_\lambda \circ j$ of $\mathfrak{sp}(W \otimes \mathbb{C})$ on $\mathcal{P}(\mathbb{C}^N)$. This is the Fock model of the Weil representation with central character λ.

We now let V be a real quadratic space of signature (p, q) (for the moment, we change notation and denote the standard basis elements by v_α and v_μ), and let W be a real symplectic space over $\mathbb{R}$ of dimension $2n$ (with standard symplectic basis e_j and f_j, $j = 1, \dots, n$). We consider the symplectic space $\mathbb{W} = V \otimes W$ of dimension $2n(p + q)$, and note that $\mathbb{J} = \theta \otimes J$ defines a positive definite complex structure on $\mathbb{W}$. Here θ is the Cartan involution with respect to the above basis of V, while J is the positive define complex structure with respect to the above symplectic basis of W. Then the $+i$-eigenspace $\mathbb{W}'$ of $\mathbb{J}$ is spanned by the $v_\alpha \otimes w'_j$ and $v_\mu \otimes w''_j$, while the $-i$ eigenspace $\mathbb{W}''$ is spanned by the $v_\alpha \otimes w''_j$ and $v_\mu \otimes w'_j$.

We naturally have $\mathfrak{o}(V)\times\mathfrak{sp}(W)\subset\mathfrak{sp}(V\otimes W)$, and one easily checks that the inclusions $j_1\colon\ \mathfrak{o}(V)\simeq\bigwedge^2(V)\to\mathfrak{sp}(V\otimes W)\simeq Sym^2(V\otimes W)$ and $j_2\colon\ \mathfrak{sp}(W)\to\mathfrak{sp}(V\otimes W)\simeq Sym^2(V\otimes W)$ are given by

$$j_1(v_1\wedge v_2) = \frac{1}{2i}\left[\sum_{j=1}^{n}(v_1\otimes w_j')\circ(v_2\otimes w_j'') - \sum_{j=1}^{n}(v_1\otimes w_j'')\circ(v_2\otimes w_j')\right]$$

$$j_2(w_1\circ w_2) = \sum_{\alpha=1}^{p}(v_\alpha\otimes w_1)\circ(v_\alpha\otimes w_2) - \sum_{\mu=p+1}^{p+q}(v_\mu\otimes w_1)\circ(v_\mu\otimes w_2),$$

with $v_1,v_2\in V$ and $w_1,w_2\in W$, see [KM4], Lemma 7.3.

We write $\mathcal{F}=\mathcal{P}(\mathbb{C}^{n(p+q)})$ for the Fock model $\mathcal{F}$ of the (infinitesimal) Weil representation of $\mathfrak{sp}(V\otimes W)$. We denote the variables in $\mathcal{P}(\mathbb{C}^{n(p+q)})$ by $z_{\alpha j}$ corresponding to $v_\alpha\otimes w_j''$ and $z_{\mu j}$ corresponding to $v_\mu\otimes w_j'$. For $n=1$ we drop the subscript $j(=1)$. We have

$$\rho_\lambda(v_\alpha\otimes w_j') = 2i\lambda\frac{\partial}{\partial z_{\alpha j}}, \qquad \rho_\lambda(v_\alpha\otimes w_j'') = z_{\alpha j},$$
$$\rho_\lambda(v_\mu\otimes w_j'') = 2i\lambda\frac{\partial}{\partial z_{\mu j}}, \qquad \rho_\lambda(v_\mu\otimes w_j') = z_{\mu j}.$$

We easily obtain the following formulas for the action of $\mathfrak{o}(V)\times\mathfrak{sp}(W)$ in $\mathcal{F}$: (the formulas differ from the ones given in [KM4] by a sign due to the sign error mentioned above).

LEMMA A.1. *For the symplectic group, we note that in the decomposition $\mathfrak{sp}(W\otimes\mathbb{C})=\mathfrak{k}'\oplus\mathfrak{p}^+\oplus\mathfrak{p}^-$, $\mathfrak{k}'\simeq\mathfrak{gl}_n\mathbb{C}$ is spanned by the elements of the form $w_j'\circ w_k''$, $\mathfrak{p}^+$ is spanned by $w_j''\circ w_k''$ and $\mathfrak{p}^-$ is spanned by $w_j'\circ w_k'$ $(1\le j,k\le n)$. Then*

$$\omega(w_j'\circ w_k'') = 2i\left[\sum_{\alpha=1}^{p} z_{\alpha k}\frac{\partial}{\partial z_{\alpha j}} - \sum_{\mu=p+1}^{p+q} z_{\mu j}\frac{\partial}{\partial z_{\mu k}}\right] + i(p-q)\delta_{jk},$$

$$\omega(w_j''\circ w_k'') = \frac{1}{\lambda}\sum_{\alpha=1}^{p} z_{\alpha j}z_{\alpha k} + 4\lambda\sum_{\mu=p+1}^{p+q}\frac{\partial^2}{\partial z_{\mu j}\partial z_{\mu k}},$$

$$\omega(w_j'\circ w_k') = -4\lambda\sum_{\alpha=1}^{p}\frac{\partial^2}{\partial z_{\alpha j}\partial z_{\alpha k}} - \frac{1}{\lambda}\sum_{\mu=p+1}^{p+q} z_{\mu j}z_{\mu k}.$$

Note that for $n=1$, we have $\mathfrak{sp}(W\otimes\mathbb{C})\simeq\mathfrak{sl}_2(\mathbb{C})$, and (for $\lambda=2\pi i$) the action of $L:=\frac{1}{2}\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}=\frac{-i}{4}w_1'\circ w_1'$ and $R:=\frac{1}{2}\begin{pmatrix}1 & i\\ i & -1\end{pmatrix}=\frac{i}{4}w_1''\circ w_1''$ correspond to the classical Maass lowering and raising operators on the upper half plane.

LEMMA A.2. *For the orthogonal group $\mathfrak{o}(V) = \mathfrak{k} \oplus \mathfrak{p}$, we write $X_{rs} = v_r \wedge v_s \in \bigwedge^2(V) \simeq \mathfrak{o}(V)$. So $\mathfrak{k}$ is spanned by $X_{\alpha\beta}$ and $X_{\mu\nu}$, while $\mathfrak{p}$ is spanned by the $X_{\alpha\mu}$. Then*

$$\begin{aligned}
\omega(X_{\alpha\beta}) &= -\sum_{j=1}^{n} z_{\alpha j}\frac{\partial}{\partial z_{\beta j}} - z_{\beta j}\frac{\partial}{\partial z_{\alpha j}},\\
\omega(X_{\mu\nu}) &= \sum_{j=1}^{n} z_{\mu j}\frac{\partial}{\partial z_{\nu j}} - z_{\nu j}\frac{\partial}{\partial z_{\mu j}},\\
\omega(X_{\alpha\mu}) &= 2i\lambda\sum_{j=1}^{n}\frac{\partial^2}{\partial z_{\alpha j}\partial z_{\mu j}} - \frac{1}{2i\lambda}\sum_{j=1}^{n} z_{\alpha j}z_{\mu j}.
\end{aligned}$$

We now give the intertwiner of the Schrödinger model with the Fock model for $\lambda = 2\pi i$. The K'-finite vectors of the Schrödinger model form the polynomial Fock space $S(V^n) \subset \mathcal{S}(V^n)$ which consists of those Schwartz functions on V^n of the form $p(\mathbf{x})\varphi_0(\mathbf{x})$, where $p(\mathbf{x})$ is a polynomial function on V^n and $\varphi_0(\mathbf{x})$ is the standard Gaussian on V^n. On the other hand, we define an action of the quantum algebra $\mathbb{W}_\lambda$ on $S(V^n)$ by

$$\begin{aligned}
\omega(v_\alpha \otimes e_j) &= 2\pi i x_{\alpha j}, &\qquad \omega(v_\alpha \otimes f_j) &= -\tfrac{\partial}{\partial x_{\alpha j}},\\
\omega(v_\mu \otimes e_j) &= -2\pi i x_{\mu j}, &\qquad \omega(v_\mu \otimes f_j) &= -\tfrac{\partial}{\partial x_{\mu j}},
\end{aligned}$$

which has central character $\lambda = 2\pi i$. As before, we obtain an action of $\mathfrak{s}p(V \otimes W)$, and this is the infinitesimal action of the Schrödinger model of the Weil representation introduced in the previous section. For $\lambda = 2\pi i$, we then have a unique $\mathbb{W}_\lambda$-intertwining operator $\iota\colon\ S(V^n) \to \mathcal{P}(\mathbb{C}^{n(p+q)})$ satisfying $\iota(\varphi_0) = 1$ ($\mathbb{W}'$ annihilates $1 \in \mathcal{P}(\mathbb{C}^{n(p+q)})$ and $\varphi_0 \in S(V^n)$).

LEMMA A.3. *The intertwining operator between the Schrödinger and the Fock model satisfies*

$$\begin{aligned}
\iota\left(x_{\alpha j} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha j}}\right)\iota^{-1} &= -i\frac{1}{2\pi}z_{\alpha j}, &\qquad \iota\left(x_{\alpha j} + \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha j}}\right)\iota^{-1} &= 2i\frac{\partial}{\partial z_{\alpha j}},\\
\iota\left(x_{\mu j} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\mu j}}\right)\iota^{-1} &= i\frac{1}{2\pi}z_{\mu j}, &\qquad \iota\left(x_{\mu j} + \frac{1}{2\pi}\frac{\partial}{\partial x_{\mu j}}\right)\iota^{-1} &= -2i\frac{\partial}{\partial z_{\mu j}}.
\end{aligned}$$

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