# Dynamics on Lorentz manifolds
# ENSET Oran, November 4-9, 2006

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**Abstract**

This course was given at ENSET Oran, November 4-9, 2006. It is an introduction to isometric actions of Lie groups on Lorentz manifolds. Several examples are presented, followed by general definitions and basic facts about Lorentz manifolds, simple Lie groups, and proper versus non-proper actions. The main goal is a dynamical argument for nonproper isometric actions of simple Lie groups, orignally due to Kowalsky [Kow], which has been used in many proofs in the subject. The second topic is totally geodesic lightlike hypersurfaces associated to actions with non-compact stabilizers. We prove some results towards the theorem of [Kow], along the lines of [ADZ] and [DMZ], that any simple Lie group with finite center, acting nonproperly by isometries of a Lorentz manifold, is locally isomorphic to $O(1, n)$ for some $n \geq 2$ or $O(2, n)$, for $n \geq 3$.

We will assume the reader has familiarity with multivariable calculus and linear algebra. Manifolds, Lorentz metrics, Lie groups, and connections will all be introduced, although in some cases only briefly, and further references will be given.

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## 1 First examples of Lorentz manifolds and isometric actions

Consider the symmetric bilinear form on $\mathbf{R}^n$

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

given by the matrix

$$B_{1,n-1} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

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This bilinear form will be referred to as the *standard inner product of type* $(1, n-1)$. Any inner product given by a matrix

$$B = g^t B_{1,n-1} g \qquad g \in GL(n, \mathbf{R})$$

is of *type* $(1, n-1)$.

The linear automorphisms of $\mathbf{R}^n$ preserving $B_{1,n-1}$ form the group

$$O(1, n-1) = \{g \in GL(n, \mathbf{R}) \ : \ g^t B_{1,n-1} g = B_{1,n-1}\}$$

**Definition 1.1** *A subspace $P \subset \mathbf{R}^n$ of dimension $k$, where $\mathbf{R}^n$ is equipped with an inner product $\langle \ , \ \rangle$ of type $(1, n-1)$, is called* spacelike, Lorentzian, *or* degenerate, *according as the restriction of $\langle \ , \ \rangle$ to $P$ is positive-definite, type $(1, k-1)$, or positive-semidefinite, respectively. Degenerate subspaces are also known as* lightlike *subspaces.*

Note that, in case $P$ is a degenerate subspace, the dimension of the *kernel*—the subspace of all $v \in P$ such that $\langle v, u\rangle = 0$ for all $u \in P$—is 1.

**Exercise 1.2**

- *Prove that, for any $c \in \mathbf{R}$, the group $O(1, n-1)$ acts transitively on the set of vectors $v \in \mathbf{R}^n$ with $\langle v, v\rangle = c$.*
- *A plane $P \subset \mathbf{R}^n$ is* degenerate, spacelike, *or* Lorentzian *if $P$ has a basis $u, v$ with $|u \wedge v|$ zero, positive, or negative, respectively, where*

  $$|u \wedge v| = \langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2$$

  *Show that $O(1, n-1)$ acts transitively on degenerate planes, spacelike planes, and Lorentzian planes.*

## 1.1 Constant-curvature models

The Lorentz models of constant zero, positive, and negative sectional curvature are *Minkowski space*, *de Sitter space*, and *anti de-Sitter space*, respectively. These are analogous to the sphere, flat Euclidean space, and hyperbolic space in Riemannian geometry. The standard basis of $\mathbf{R}^n$ will be denoted $e_1, \ldots, e_n$ below.

*Minkowski space* $\mathrm{Min}^n$ of dimension $n$ is $\mathbf{R}^n$ with the translation-invariant Lorentz metric given by the standard inner product of type $(1, n-1)$.

**Definition 1.3** *A* diffeomorphism *of* $\mathbf{R}^n$ *is a smooth map* $f : \mathbf{R}^n \to \mathbf{R}^n$ *with a smooth inverse* $f^{-1}$*. An* isometry *of* $Min^n$ *is a diffeomorphism* $f$ *of* $\mathbf{R}^n$ *satisfying*

$$\langle f_{*x}u, f_{*x}v\rangle = \langle u, v\rangle$$

*for all* $x, u, v \in \mathbf{R}^n$*.*

**Proposition 1.4** *The isometries of* $Min^n$ *fixing the origin form the group* $O(1, n-1)$*. The full isometry group is the semi-direct product*

$$Isom(Min^n) = \mathbf{R}^n \rtimes O(1, n-1)$$

*and* $Min^n$ *is a homogeneous space* $\mathbf{R}^n \rtimes O(1, n-1)/O(1, n-1)$*.*

**Proof:** It is clear that $\mathbf{R}^n \rtimes O(1, n-1) \subseteq \mathrm{Isom}(\mathrm{Min}^n)$. Now suppose that $f \in \mathrm{Isom}(\mathrm{Min}^n)$. By post-composing with a translation, we may assume that $f$ fixes the origin. Then $f_{*0}$ is a linear isometry of $T_0(\mathrm{Min}^n)$, which is isometric to $\mathbf{R}^n$ with the inner product $B_{1,n-1}$. Therefore, $f_{*0}$ coincides with an element of $O(1, n-1)$.

The key fact is that $f$ must carry straight lines to straight lines, because these are the *geodesics* in $\mathrm{Min}^n$. (Geodesics will be defined for general Lorentz manifolds in section 9 below.) Then $f$ must be linear, so $f = f_{*0} \in O(1, n-1)$. ♢

*De Sitter space* $\mathrm{dS}^n$ of dimension $n$ is the subset of $\mathbf{R}^{n+1}$

$$\{x \in \mathbf{R}^{n+1} \ : \ x^t B_{1,n} x = 1\}$$

with the induced Lorentz metric: the tangent space $T_x\mathrm{dS}^n = x^\perp$, and the restriction of $B_{1,n}$ to $x^\perp$ is of type $(1, n-1)$. Indeed, let $\gamma(t)$ be a curve with

$$\gamma(t)^t B_{1,n}\gamma(t) = \langle\gamma(t), \gamma(t)\rangle = 1$$

for all $t$ and $\gamma(0) = x$. Then

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \langle\gamma(t), \gamma(t)\rangle &= 0 \\ &= \langle\gamma'(0), \gamma(0)\rangle + \langle\gamma(0), \gamma'(0)\rangle \\ &= 2\langle\gamma'(0), x\rangle\end{aligned}$$

Since any $v \in T_x\mathrm{dS}^n$ equals $\gamma'(0)$ for such a curve $\gamma$, we conclude that $T_x\mathrm{dS}^n = x^\perp$.

**Definition 1.5** *A map* $f : dS^n \to dS^n$ *is an* isometry *if* $f$ *extends to a smooth map* $\tilde{f}$ *with a smooth inverse in a neighborhood of* $dS^n$ *in* $\mathbf{R}^{n+1}$*, and*

$$\langle f_{*x} u f_{*x} v \rangle = \langle u, v \rangle$$

*for all* $x \in dS^n$ *and* $u, v \in T_x dS^n$.

The group $O(1, n)$ acts isometrically and transitively on $\mathrm{dS}^n$.

**Proposition 1.6** *The group of isometries of* $dS^n$ *fixing* $e_{n+1}$ *is isomorphic to* $O(1, n-1)$*. The full isometry group is*

$$Isom(dS^n) \cong O(1, n)$$

*and* $dS^n$ *is a homogeneous space* $O(1, n)/O(1, n-1)$.

**Proof:** Let $f$ be an isometry of $\mathrm{dS}^n$. By post-composing with an element of $O(1, n)$, we may assume that $f$ belongs to the stabilizer of $p = e_{n+1}$. The derivative $f_{*p}$ is a linear isometry of $T_p\mathrm{dS}^n = p^\perp$ and coincides with an element of the stabilizer $O(1, n)(p) \cong O(1, n-1)$ of $p$ in $O(1, n)$. Then post-composing with another element of $O(1, n)$, we may assume $f_{*p}$ is trivial.

Now the key fact is that $f$ must carry each intersection $P \cap \mathrm{dS}^n$, where $P$ is a linear plane in $\mathbf{R}^{n+1}$, to $P' \cap \mathrm{dS}^n$, for some linear plane $P'$ in $\mathbf{R}^{n+1}$, because these intersections are the geodesics of $\mathrm{dS}^n$. Suppose that $\gamma(t)$ is such a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v \in T_p\mathrm{dS}^n$. Then the plane $P$ containing $\gamma$ is spanned by $e_{n+1}$ and $v$. Because $f_{*p}$ is trivial, $f$ preserves the intersection $P \cap \mathrm{dS}^n$ for every plane $P$ containing $e_{n+1}$, and it actually preserves each geodesic $\gamma(t) \subset P \cap \mathrm{dS}^n$ with its parametrization, provided $\langle \gamma'(0), \gamma'(0) \rangle \neq 0$. Then $f$ must be trivial in a neighborhood of $e_{n+1}$.

The argument above shows in fact that the set of $x \in \mathrm{dS}^n$ with $f(x) = x$ and $f_{*x} = Id$ is open. But this set is also closed, and $\mathrm{dS}^n$ is connected, so $f$ is trivial. $\diamondsuit$

*Anti de Sitter space* $\mathrm{AdS}^n$ of dimension $n$ is the subset of $\mathbf{R}^{n+1}$

$$\{x \in \mathbf{R}^{n+1} \ : \ x^t B_{2,n-1} x = -1\}$$

where $B_{2,n-1}$ is a bilinear form of type $(2, n-1)$:

$$\langle u, v \rangle = -u_1 v_1 - u_2 v_2 + u_3 v_3 + \cdots + u_{n+1} v_{n+1}$$

The tangent space $T_x\mathrm{AdS}^n = x^\perp$ as above, and the restriction of $B_{2,n-1}$ to it is of type $(1, n-1)$.

Isometries of $\mathrm{AdS}^n$ are defined analogously to those of $\mathrm{dS}^n$. The group $O(2, n-1)$ consists of all linear automorphisms of $\mathbf{R}^{n+1}$ preserving $B_{2,n-1}$. It acts isometrically and transitively on $\mathrm{AdS}^n$.

**Proposition 1.7** *The group of isometries of $AdS^n$ fixing $e_1$ is isomorphic to $O(1, n-1)$. The full isometry group is*

$$Isom(AdS^n) \cong O(2, n)$$

*and $AdS^n$ is a homogeneous space $O(2, n-1)/O(1, n-1)$.*

The proof of this proposition is similar to the proof for $\mathrm{dS}^n$, because again the geodesics of $\mathrm{AdS}^n \subset \mathbf{R}^{n+1}$ are intersections with linear planes in $\mathbf{R}^{n+1}$.

Each of the above examples has constant *sectional curvature*. The sectional curvature at each point is a rational function on the nondegenerate tangent planes at that point, and it is invariant by isometries. The fact that the spaces above are all homogeneous with isotropy $O(1, n-1)$ implies that all Lorentzian tangent planes have the same sectional curvature. Now because the sectional curvature is rational and constant on a nonempty open set, it is constant everywhere. The definition of sectional curvature will be given in section 9 below.

# 2 Algebraic Lie groups, Lie algebras, and the adjoint representation

## 2.1 Definition of linear Lie group and Lie algebra

**Definition 2.1** *An* algebraic Lie group *$G$ is a subgroup of $GL(n, \mathbf{R}) \subset \mathbf{R}^{n^2}$ comprising the common zeroes of a finite set of polynomials on $\mathbf{R}^{n^2}$.*

**Example.** The group $SL(n, \mathbf{R})$ is defined by the polynomial

$$\det g - 1 = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} g_{1,\sigma(1)} \cdots g_{n,\sigma(n)} - 1 = 0$$

where $S_n$ is the group of all permutations of $\{1, \ldots, n\}$, and $\mathrm{sgn}(\sigma) \in \{1, -1\}$ is the sign of the permutation $\sigma$.

**Example.** The group $O(1,n)$ is defined by

$$g^t B_{1,n} g = B_{1,n}$$

which is equivalent to the vanishing of the following polynomials

$$\begin{array}{rcll} -g_{11}^2 + g_{21}^2 + \cdots + g_{n+1,1}^2 + 1 & = & 0 & \\ -g_{1i}^2 + g_{2i}^2 + \cdots + g_{n+1,i}^2 - 1 & = & 0 & i = 2, \ldots, n+1 \\ -g_{1i}g_{1j} + g_{2i}g_{2j} + \cdots + g_{n+1,i}g_{n+1,j} & = & 0 & i \neq j,\ i,j = 1, \ldots, n \end{array}$$

**Exercise 2.2** *Find the polynomial equations defining the group $O(p,q)$, where*

$$O(p,q) = \{g \in GL(p+q, \mathbf{R}) \ : \ g^t B_{p,q} g = B_{p,q}\}$$

Note that matrix multiplication $\mu : (g,h) \mapsto gh$ is given by polynomials in the entries of $g$ and $h$:

$$(gh)_{ij} = \sum_k g_{ik} h_{kj}$$

When $G$ is a subgroup of $SL(n, \mathbf{R})$, then inversion $\iota : g \mapsto g^{-1}$ is also given by polynomials in the entries of $g$, by Cramer's formula:

$$(g^{-1})_{ij} = (-1)^{i+j} \det\, g(\hat{j}, \hat{i})$$

where $g(\hat{j}, \hat{i})$ is the $(n-1) \times (n-1)$ matrix obtained by removing the $j^{th}$ row and $i^{th}$ column from $g$.

Now let $G$ be a linear Lie group. The tangent space to the identity $T_1 G$ is a subspace of the endomorphisms $\mathrm{End}(\mathbf{R}^n)$. There is a bracket

$$[X, Y] = XY - YX$$

making $\mathrm{End}(\mathbf{R}^n)$ into what is called a *Lie algebra*.

**Proposition 2.3** *The derivative of inversion $\iota$ is $-Id$ on $T_1 G$.*

**Proof:** Let $\alpha(t)$ be a curve in $G$ with $\alpha(0) = 1$ and $\alpha'(0) = X$. For all $t$,

$$\alpha(t)\iota(\alpha(t)) = \alpha(t)\alpha(t)^{-1} = 1$$

Differentiation yields

$$\alpha'(0) + \iota_{*1}(\alpha'(0)) = 0$$

which implies $\iota_{*1}(X) = -X$. $\diamondsuit$

**Proposition 2.4** *If $G < GL(n, \mathbf{R})$ is a linear Lie group, then $T_1G$ is a sub-Lie algebra of End$(\mathbf{R}^n)$—in particular, it is closed under bracket.*

**Proof:** Let $\beta(t)$ in $G$ be such that $\beta(0) = 1$, and $\beta'(0) = Y$. The conjugate $\beta(t)^{-1}X\beta(t) \in T_1G$ for all $t$, so

$$\beta(t)^{-1}X\beta(t)$$

is a curve in $T_1G$. The derivative

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \beta(t)^{-1}X\beta(t) = -YX + XY$$

and belongs to $T_1G$. $\diamondsuit$

Now we can define an algebraic Lie algebra.

**Definition 2.5** *An algebraic Lie algebra $\mathfrak{g}$ is a subalgebra of End$(\mathbf{R}^n)$ with underlying vector space equal to $T_1G$, for $G$ an algebraic Lie group.*

The bracket in an algebraic Lie algebra satisfies the *Jacobi identity*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

**Example.** For $G = SL(2, \mathbf{R})$, the tangent space $T_1G$ can be computed by differentiating an arbitrary curve

$$\gamma(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \in SL(2, \mathbf{R}) \qquad \gamma(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Differentiating

$$\det(\gamma(t)) = a(t)d(t) - b(t)c(t) = 1$$

gives, when $t = 0$,

$$a'(0) + d'(0) = 0$$

or $\mathrm{tr}(\gamma'(0)) = 0$. Thus $T_1SL(2, \mathbf{R})$ consists of all traceless $2 \times 2$ real matrices.

**Exercise 2.6** *Show that the Lie algebra $\mathfrak{sl}(n, \mathbf{R})$ of $SL(n, \mathbf{R})$ is the algebra of all $n \times n$ traceless matrices.*

**Example.** A curve in the algebraic Lie group $O(p,q)$ satisfies for all $t$

$$\gamma(t)^t B_{p,q} \gamma(t) = B_{p,q}$$

Assuming that $\gamma(0) = 1$, the derivative is

$$\gamma'(0)^t B_{p,q} + B_{p,q} \gamma'(0)$$

So $T_1 O(p,q)$ consists of all $(p+q) \times (p+q)$ matrices $X$ with

$$X B_{p,q} + B_{p,q} X = 0$$

If $p = 0$, then $B_{p,q} = I_q$, and the equation

$$X^t + X = 0$$

defines the algebra $\mathfrak{o}(q)$, consisting of all skew-symmetric matrices.

## 2.2 Left-invariant vector fields and one-parameter subgroups

**Definition 2.7** *A* one-parameter subgroup *in an algebraic Lie group $G$ is a homomorphism $\gamma : \mathbf{R} \to G$—that is*

$$\gamma(t+s) = \gamma(t)\gamma(s)$$

*for all $s, t \in \mathbf{R}$.*

We will show that one-parameter subgroups are solutions of ODEs, and so always exist and are unique with given initial values. Then we will give an explicit solution in an algebraic Lie group.

For any $X \in T_1 G$ and $g \in G$, the translate $gX \in T_g G$. Then associated to any $X \in T_1 G$ is a *left-invariant vector field* satisfying

$$g(X(h)) = X(gh) \qquad \forall\ g, h \in G$$

Denote by $\gamma_X(t)$ the integral curve for $X$ passing through 1: there is some $\epsilon > 0$, such that, for each $t \in (-\epsilon, \epsilon)$,

$$\gamma'_X(t) = X(\gamma_X(t)) = \gamma_X(t) X$$

Now fix $t_0 \in (-\epsilon, \epsilon)$. For any $t$ such that $t, t_0+t \in (-\epsilon, \epsilon)$, the curve $\gamma_X(t_0)\gamma_X(t)$ satisfies

$$\begin{aligned}(\gamma_X(t_0)\gamma_X(t))'(t) &= \gamma_X(t_0)\gamma_X'(t) \\ &= \gamma_X(t_0)\gamma_X(t)X\end{aligned}$$

Thus both $\gamma_X(t_0+t)$ and $\gamma_X(t_0)\gamma_X(t)$ satisfy the ODE

$$\gamma'(t) = \gamma(t)X$$

Both curves have initial value 1, so, by uniqueness of solutions of ODEs, they must be equal:

$$\gamma_X(t_0+t) = \gamma_X(t_0)\gamma_X(t)$$

whenever all three are defined. But now $\gamma_X(t)$ can be defined for all $t \in \mathbf{R}$, yielding a one-parameter subgroup in $G$.

Let $X \in \mathrm{End}(\mathbf{R}^n)$, and consider the power series

$$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots$$

If each entry of $X$ is bounded in absolute value by $c > 0$, then the entries of $e^X$ are bounded by

$$\frac{1}{n}(e^{nc} - 1) + 1$$

For $s, t \in \mathbf{R}$,

$$e^{(t+s)X} = e^{tX}e^{sX}$$

so $e^{tX}$ is a one-parameter subgroup of $GL(n, \mathbf{R})$. Note that $e^0 = I$, so $e^X$ is invertible with inverse $e^{-X}$. If $X \in T_1G$, then

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}e^{tX}\right|_s &= X + sX^2 + \frac{s^2}{2}X^3 + \cdots \\ &= e^{sX}X\end{aligned}$$

so $e^{tX}$ is a one-parameter subgroup in $G$. The map $X \mapsto e^{tX}$ is called the *exponential map*. The derivative of the exponential map $T_1G \to T_1G$ is the identity. By the inverse function theorem, the exponential map is a diffeomorphism from a neighborhood of 0 in $T_1G$ to a neighborhood of 1 in $G$. Such a neighborhood of 1 in $G$ will be called a *normal neighborhood* of the identity.

## 2.3 Adjoint representation

**Definition 2.8** *The* adjoint representation *of $G$ on $\mathfrak{g}$ is*

$$Ad(g)(X) = gXg^{-1}$$

If $G$ is an algebraic Lie group, then the kernel of Ad is exactly the center $Z(G)$ of $G$. For each $g \in G$, the adjoint $\mathrm{Ad}(g)$ is a Lie algebra automorphism:

$$[\mathrm{Ad}(g)X, \mathrm{Ad}(g)(Y)] = \mathrm{Ad}(g)([X, Y])$$

for all $X, Y \in \mathfrak{g}$.

Let $e^{tX}$ be a one-parameter subgroup of $G$ and $g \in G$. Then

$$ge^{tX}g^{-1} = e^{\mathrm{Ad}(g)(tX)}$$

because both are one-parameter subgroups with initial tangent vector $\mathrm{Ad}(g)(X)$.

**Definition 2.9** *A connected algebraic Lie group $G$ is* simple *if $dim(G) > 1$ and the adjoint representation is irreducible—that is, the only invariant subspaces are $0$ and $\mathfrak{g}$. An algebraic Lie group is simple if the connected component of the identity is simple. An algebraic Lie algebra $\mathfrak{g}$ is simple if $G$ is simple.*

**Proposition 2.10** *If $G$ is simple, then $Z(G)$ is discrete—that is, every $z \in Z(G)$ has a neighborhood $U$ in $G$ with $U \cap Z(G) = \{z\}$.*

**Proof:** Let $U$ be a normal neighborhood of $e$ in $G$ and suppose that $z = e^X \in Z(G) \cap U$. For any $g \in G$ sufficiently close to 1,

$$gzg^{-1} = e^{\mathrm{Ad}(g)X} = e^X$$

which implies that $\mathrm{Ad}(g)(X) = X$ for all $g \in G^0$, the identity component of $G$. But then $\mathbf{R}X$ would be a nontrivial invariant subspace of $\mathfrak{g}$, contradicting simplicity. Therefore, $U \cap Z(G) = \{1\}$. Now for any $z \in Z(G)$, the translate $zU$ is a neighborhood of $z$ with $zU \cap Z(G) = \{z\}$. ♢

# 3 Lorentz manifolds and isometries

There are many books providing a good introduction to differentiable manifolds and pseudo-Riemannian metrics. Among them are [Sp], [dC], and [ON].

A smooth $n$-dimensional manifold is a topological space that is locally diffeomorphic to $\mathbf{R}^n$.

**Definition 3.1** *An $n$-*dimensional manifold *$M$ is a paracompact Hausdorff topological space with a collection of charts $(\varphi, U)$, where $U$ is an open subset of $M$, the union of all $U$ covers $M$, and $\varphi$ is a homeomorphism from $U$ to an open subset of $\mathbf{R}^n$. The charts must satisfy a smooth compatibility condition: if $(\varphi, U)$ and $(\psi, V)$ are two charts, then the* transition map

$$\psi \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

*is a smooth map on its domain of definition in $\mathbf{R}^n$.*

**Definition 3.2** *Let $M$ be a smooth $n$-dimensional manifold, and $x \in M$. The* tangent space at *$x$, denoted $T_x M$, is the collection of all $(\varphi(x), v)$ where $(\varphi, U)$ is a chart with $x \in U$, and $v \in \mathbf{R}^n$, subject to the equivalence relation*

$$(\varphi(x), v) \cong (\psi(x), w) \;\Leftrightarrow\; (\psi \circ \varphi^{-1})_{*\varphi(x)}(v) = w$$

The tangent space $T_x M$ is a vector space. The sum

$$(\varphi(x), v) + (\varphi(x), v') = (\varphi(x), v + v')$$

The sum is well-defined, because if $(\psi(x), w) \cong (\varphi(x), v)$ and $(\psi(x), w') \cong (\varphi(x), v')$, then

$$\begin{aligned} (\psi \circ \varphi^{-1})_{*\varphi(x)}(v + v') &= (\psi \circ \varphi^{-1})_{*\varphi(x)}(v) + (\psi \circ \varphi^{-1})_{*\varphi(x)}(v') \\ &= w + w' \end{aligned}$$

Similarly, scalar multiplication

$$a(\varphi(x), v) = (\varphi(x), av)$$

is well-defined.

**Example.** The subset of $\mathbf{R}^{n+1}$ underlying $\mathrm{dS}^n$, the set of $x \in \mathbf{R}^{n+1}$ with

$$x^t B_{1,n} x = 1$$

is a manifold. A global chart is given by

$$\varphi(x_1, \ldots, x_{n+1}) = \frac{1}{e^{x_1}\sqrt{1 + x_1^2}}(x_2, \ldots, x_{n+1})$$

The image of this chart is $\mathbf{R}^n \backslash \{0\}$.

**Example.** An algebraic Lie group $G \subset GL(n, \mathbf{R})$ is a manifold. Let $U$ be a normal neighborhood of the identity, and denote by $\exp^{-1}$ the inverse of the exponential map. Then for any $g \in G$, the pair $(\exp^{-1} \circ g^{-1}, gU)$ is a chart with image an open neighborhood of 0 in $T_1G$, which can be identified with $\mathbf{R}^{\dim G}$. If $gU \cap hU \neq \emptyset$, the transition map is

$$\exp^{-1} \circ h^{-1} \circ g \circ \exp$$

which is a composition of smooth maps, so is smooth.

**Exercise 3.3** *Show that the subset of* $\mathbf{R}^{n+1}$ *underlying* $AdS^n$ *is a smooth manifold.*

If $M$ and $N$ are two smooth manifolds, then a map $f : M \to N$ is smooth, if, for every $x \in M$, and every pair of charts $(\varphi, U)$ on $M$ and $(\psi, V)$ on $N$ with $x \in U$ and $f(x) \in V$, the composition

$$\psi \circ f \circ \varphi^{-1}$$

is smooth.

Suppose that $(\varphi(x), v)$ and $(\psi(x), w)$ define the same tangent vector in $T_xM$. Suppose that $f$ is a smooth map from $M$ to $N$, and $(\theta, V)$ is a chart in $N$ at $f(x)$. Then

$$\begin{aligned}
(\theta \circ f \circ \varphi^{-1})_{*\varphi(x)}(v) &= (\theta \circ f \circ \varphi^{-1})_{*\varphi(x)} \circ (\varphi \circ \psi^{-1})_{*\psi(x)}(w) \\
&= (\theta \circ f \circ \psi^{-1})_{*\psi(x)}(w)
\end{aligned}$$

so the tangent vector $(\theta(f(x)), (\theta \circ f \circ \varphi^{-1})_{*\varphi(x)}(v))$ is independent of the choice of $(\varphi(x), v)$. Similarly, it is independent of the choice of $\theta$.

**Definition 3.4** *For* $f : M \to N$ *a smooth map of manifolds, the* derivative *of* $f$ *at* $x \in M$ *is*

$$\begin{aligned}
f_{*x} &: T_xM \to T_{f(x)}N \\
f_{*x} &: (\varphi(x), v) \mapsto (\theta(f(x)), (\theta \circ f \circ \varphi^{-1})_{*\varphi(x)}(v))
\end{aligned}$$

*where* $(\theta, V)$ *is any chart of* $N$ *with* $f(x) \in V$.

**Definition 3.5** *A* Lorentz metric *on a smooth manifold* $M$ *is a smoothly varying inner product* $\langle\ ,\ \rangle_x$ *of type* $(1, n-1)$ *on each tangent space* $T_xM$.

**Definition 3.6** *A smooth map $f : M \to N$ is an* isometry *if $f$ has a smooth inverse, and, for every $x \in M$, and $u, v \in T_x M$,*

$$\langle u, v \rangle_x = \langle f_{*x} u, f_{*x} v \rangle_{f(x)}$$

**Example.** Suppose that $\rho : \widetilde{M} \to M$ is a covering map. Suppose further that $\widetilde{M}$ is a Lorentz manifold, and that all deck transformations are isometries. Then $M$ admits the structure of a Lorentz manifold such that $\rho$ is a local isometry—every $x \in M$ has a neighborhood $U$ such that $\rho$ maps $\rho^{-1}(U)$ isometrically to $U$. Given $x \in M$ and $u, v \in T_x M$, define

$$\langle u, v \rangle_x = \langle \rho_{*x}^{-1} u, \rho_{*x}^{-1} v \rangle_{\rho^{-1}(x)}$$

Every Lorentz manifold has special curves called *geodesics*, that are the analogues of straight lines in $\mathrm{Min}^n$ and planar sections in $\mathrm{dS}^n$ and $\mathrm{AdS}^n$. Any isometry of a Lorentz manifold must carry geodesics to geodesics.

# 4 Lie groups and Lie algebras

**Definition 4.1** *A* Lie group *is a group $G$ that is also a manifold such that multiplication and inversion are smooth.*

Denote by $L_g$ left multiplication by $g$:

$$\begin{aligned} L_g &: G \to G \\ L_g &: h \mapsto gh \end{aligned}$$

and by $R_g$ right multiplication by $g$.

Given $X \in T_1 G$, there is a *left-invariant vector field* $X$ with $X(g) = (L_g)_{*1}(X)$. Then $X$ satisfies

$$X(gh) = (L_g)_{*h}(X(h))$$

For a left-invariant vector field $X$, integral curves $\gamma_X$ through 1 satisfy the ODE

$$\gamma_X'(t) = X(\gamma_X(t)) = \gamma_X(t)_{*1}(X)$$

and they are one-parameter subgroups, as in the case of algebraic Lie groups. The map $X \mapsto \gamma_X(1)$ is defined on a neighborhood of 0 in $T_1 G$ and is called the *exponential map*; the element $\gamma_X(1)$ is often denoted $e^X$.

The *adjoint representation* of $G$ on $T_1 G$ is

$$\mathrm{Ad}(g)(X) = (R_{g^{-1}})_{*g}(L_g)_{*1}(X)$$

**Exercise 4.2** *Verify that Ad is a homomorphism $G \to GL(\mathfrak{g})$:*

$$Ad(gh) = Ad(g) \circ Ad(h)$$

*(hint: use that right and left multiplication commute:*

$$R_{h^{-1}} \circ L_g = L_g \circ R_{h^{-1}}$$

*for all $g, h \in G$.)*

The adjoint corresponds under the exponential map to conjugation

$$ge^{tX}g^{-1} = e^{\mathrm{Ad}(g)(tX)}$$

because both curves are one-parameter subgroups with the same initial tangent vector.

Given $X, Y \in T_1G$, the *bracket* can be defined as the bracket $[X, Y]$ of the corresponding left-invariant vector fields on $G$. Recall that the bracket

$$[X, Y] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 Y(\gamma_X(t)) - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 X(\gamma_Y(t))$$

where $\gamma_X(t)$ and $\gamma_Y(t)$ are integral curves for $X, Y$, respectively. The bracket satisfies, for any diffeomorphism $f$,

$$f_*([X, Y]) = [f_*X, f_*Y]$$

where $(f_*X)(x) = f_{*f^{-1}(x)}(X(f^{-1}(x)))$. If $X, Y$ are left-invariant vector fields and $g \in G$, then

$$(L_g)_*([X, Y]) = [(L_g)_*X, (L_g)_*Y] = [X, Y]$$

so the bracket is again left-invariant. The bracket is bilinear and skew-symmetric. We leave it to the interested reader to verify that, for $G$ a Lie group, the bracket on $T_1G$ satisfies the Jacobi identity.

**Definition 4.3** *A* Lie algebra $\mathfrak{g}$ *is a (finite-dimensional) vector space with a skew-symmetric bracket satisfying the Jacobi identity:*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

The bracket in a Lie algebra could also be defined as follows, for $X, Y \in T_1G$:

$$[X, Y] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Ad}(e^{tX})(Y)$$

Note that when $G$ is an algebraic Lie group, then $[X, Y] = XY - YX$. To prove that the definition above in terms of the adjoint is the same as the bracket of the corresponding left-invariant vector fields, let $X, Y \in T_1G$ and $e^{tX}, e^{tY}$ be the corresponding one-parameter subgroups. Then

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Ad}(e^{tX})(Y) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (R_{e^{-tX}})_*(L_{e^{tX}})_*Y \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (R_{e^{-tX}})_*(Y(e^{tX})) \end{aligned}$$

where $Y$ is the left-invariant vector field with $Y(1) = Y$. Now we have

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (R_{e^{-tX}})_*(Y(e^{tX})) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 Y(e^{tX}) + \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (R_{e^{-tX}})_*\right)(Y) \\ &= XY + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 e^{sY}e^{-tX} \\ &= XY + \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 (L_{e^{sY}})_*(-X) \\ &= XY - \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 X(e^{sY}) \\ &= XY - YX \end{aligned}$$

Finally, the adjoint $\mathrm{Ad}(g)$ respects brackets because

$$[(R_{g^{-1}})_*(L_g)_*X, (R_{g^{-1}})_*(L_g)_*Y] = (R_{g^{-1}})_*(L_g)_*[X, Y]$$

so the adjoint representation is by Lie algebra automorphisms.

# 5 Compact homogeneous examples

Next we will construct two examples of compact homogeneous Lorentz manifolds. Both will be quotients of an algebraic Lie group with a bi-invariant Lorentz metric. In this case, the geodesics through the identity are the one-parameter subgroups.

## 5.1 Isometry group $SL(2, \mathbf{R})$

Consider the bilinear form $\langle X, Y\rangle_1 = \mathrm{tr} XY$ on $\mathfrak{sl}(2, \mathbf{R})$. Because the trace is symmetric, this form is symmetric and $\mathrm{Ad}(SL(2, \mathbf{R}))$-invariant. It is of type $(1, 2)$. Identifying $\mathfrak{sl}(2, \mathbf{R})$ with $T_1 SL(2, \mathbf{R})$, we obtain a Lorentz metric $\mu$ on $SL(2, \mathbf{R})$ defined by

$$\langle u, v\rangle_g = \langle g^{-1}u, g^{-1}v\rangle_1$$

This metric is obviously left-$SL(2\mathbf{R})$-invariant. The adjoint-invariance of $\langle\ ,\ \rangle_1$ implies that it is also right-$SL(2, \mathbf{R})$-invariant. The identity component

$$\mathrm{Isom}^0(SL(2, \mathbf{R}), \mu) \cong SL(2, \mathbf{R}) \times_Z SL(2, \mathbf{R})$$

the quotient of $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ by the group generated by $(-I_2, -I_2)$. To prove this claim, let $f \in \mathrm{Isom}^0(SL(2, \mathbf{R}), \mu)$. By post-composing with an element of $SL(2, \mathbf{R}) \times \{1\}$, we may assume that $f(1) = 1$. Now

$$\mathrm{Ad}(SL(2, \mathbf{R})) \cong SL(2, \mathbf{R})/Z(SL(2, \mathbf{R})) \cong PSL(2, \mathbf{R})$$

But $PSL(2, \mathbf{R}) \cong O^0(1, 2)$—both are the identity component of the isometry group of the hyperbolic plane, in the upper half-plane and hyperboloid models, respectively. So by post-composing with an element of $\mathrm{Ad}(SL(2, \mathbf{R}))$, the stabilizer in $SL(2, \mathbf{R}) \times_Z SL(2, \mathbf{R})$ of 1, we may assume that $f_{*1}$ is trivial. Then $f$ fixes all one-parameter subgroups through 1, which implies that $f$ is trivial on a neighborhood of 1. This argument shows in fact that the set on which $f(x) = x$ and $f_{*x} = Id$ is open in $SL(2, \mathbf{R})$; since it is also closed and $SL(2, \mathbf{R})$ is connected, we conclude that $f$ is trivial.

**Exercise 5.1** *Show that $SL(2, \mathbf{R})$ with this Lorentz metric is isometric to $AdS^3$. Conclude that $O^0(2, 2) \cong SL(2, \mathbf{R}) \times_Z SL(2, \mathbf{R})$.*

Now let $\Gamma$ be a *cocompact lattice* in $SL(2, \mathbf{R})$—a discrete subgroup such that the quotient

$$M = SL(2, \mathbf{R})/\Gamma$$

is compact. The quotient $M$ is a compact homogeneous Lorentz manifold.

Assuming $I_2 \in \Gamma$, the connected isometry group $\mathrm{Isom}^0(M) \cong PSL(2, \mathbf{R})$. From covering space theory, any $f \in \mathrm{Isom}^0(M)$ can be lifted to $\tilde{f}$ on $SL(2, \mathbf{R})$ such that $\tilde{f}\gamma = \gamma\tilde{f}$ for all $\gamma \in \Gamma$. But the centralizer of $\Gamma \subset \{1\} \times SL(2, \mathbf{R})$ is just $SL(2, \mathbf{R}) \times \{\pm I_2\}$, by the Borel density theorem (see [Zi1] 3.2.5).

## 5.2 Isometry group a warped Heisenberg group

The second example of a homogeneous compact Lorentz manifold will come from a bi-invariant metric on a solvable Lie group. Consider the *Heisenberg group* $H$ of $3 \times 3$ upper-triangular matrices. The tangent space $T_1 H$ is the vector space of $3 \times 3$ nilpotent upper-triangular matrices. A basis for this vector space is

$$X = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{bmatrix}$$

The bracket in the Lie algebra $\mathfrak{h}$ spanned by $X, Y, Z$ is

$$\begin{aligned} [X,Y] &= XY - YX = Z \\ [X,Z] &= [Y,Z] = 0 \end{aligned}$$

Thus the center $\mathfrak{z}(\mathfrak{h})$ is spanned by $Z$.

Now let $S$ be the semi-direct product

$$\begin{aligned} S &= S^1 \ltimes H \\ e^{i\theta} \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} e^{-i\theta} &= \begin{bmatrix} 1 & x\cos\theta - y\sin\theta & z - \frac{1}{2}xy\cos 2\theta + \frac{1}{4}(x^2 - y^2)\sin 2\theta \\ & 1 & y\cos\theta + x\sin\theta \\ & & 1 \end{bmatrix} \end{aligned}$$

The Lie algebra $\mathfrak{s}$ is generated by $\mathfrak{h}$ and another element $W$ with

$$[W,X] = Y \qquad [W,Y] = -X \qquad [W,Z] = 0$$

The adjoint representation of $S$ on $\mathfrak{s}$ is given by

$$\begin{aligned} \mathrm{Ad}(e^{i\theta})(W + aX + bY + Z) &= W + (a\cos\theta - b\sin\theta)X + (b\cos\theta + a\sin\theta)Y + Z \\ \mathrm{Ad}\left(\begin{bmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{bmatrix}\right)(W) &= W - xY - \frac{1}{2}x^2 Z \\ \mathrm{Ad}\left(\begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix}\right)(W) &= W + yX - \frac{1}{2}y^2 Z \\ \mathrm{Ad}\left(\begin{bmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{bmatrix}\right)(W) &= W \end{aligned}$$

Now the following inner product on $\mathfrak{s}$ is $\mathrm{Ad}(S)$-invariant

$$\begin{aligned}\langle X, X\rangle &= \langle Y, Y\rangle = \langle W, Z\rangle = 1 \\ \langle W, W\rangle &= \langle Z, Z\rangle = \langle X, Y\rangle = 0 \\ span\{Z, W\} &\perp span\{X, Y\}\end{aligned}$$

This inner product is of type $(1,3)$, and determines a bi-invariant Lorentz metric on $S$.

Now let $\Gamma$ be a cocompact lattice in $H$—for example, the group of all matrices in $H$ with integral entries. The quotient $M = S/\Gamma$ is a compact Lorentz manifold on which $S$ acts isometrically. The subgroup $Z(H) \cap \Gamma$ acts trivially.

Next we will show that $\mathrm{Isom}^0(M) \cong S/(Z(H) \cap \Gamma)$. Let $f \in \mathrm{Isom}^0(M)$. By post-composing with an element of $S$, we may assume that $f$ fixes the coset of the identity. As in the example above, covering space theory gives a lift $\tilde{f}$ of $f$ fixing the identity and satisfying $\tilde{f}\gamma = \gamma\tilde{f}$ for all $\gamma \in \Gamma$; then $\tilde{f}$ fixes all the points $\gamma \in \Gamma \subset S$. The exponential map of $S$ is a diffeomorphism when restricted to $T_1H \subset T_1S$ (see [Kn]), and $\exp^{-1}(\Gamma)$ contains the generators $X, Y$, and $Z$. Then $\tilde{f}_{*1}$ fixes all the integral matrices in $T_1H$, so $\tilde{f}_{*1}$ is trivial on the codimension-one subspace $T_1H$. Because $\tilde{f}_{*1}$ is an orientation-preserving linear isometry of $T_1S$, it must also fix the vector $W$ transverse to $T_1H$—that is, $\tilde{f}_{*1} = Id$. Then $\tilde{f}$ is trivial in a neighborhood of 1, and we conclude that $\tilde{f}$ is trivial.

There are generalizations of this example, where $\mathfrak{h}$ is replaced by $\mathfrak{h}_n$, the $(2n+1)$-*dimensional Heisenberg Lie algebra*. This Lie algebra is generated by

$$X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z$$

with $[X_j, Y_j] = Z$ for $j = 1, \ldots, n$. In the most general examples, the adjoint of $e^{i\theta} \in S\backslash H$ acts by rotation by $r_j\theta$, where $r_j$ is a rational multiple of $2\pi$, on $span\{X_j, Y_j\}$. See [AS1] and [Ze1] for more details.

# 6 Proper actions

Isometry groups of Riemannian manifolds always act properly, but Lorentzian isometry groups may not. Nonproperness is the sole dynamical assumption on the Lorentz-isometric actions in Kowalsky's theorem.

## 6.1 Definition and first consequences

**Definition 6.1** *A continuous action of a topological group $G$ on a Hausdorff, locally compact space $X$ is* proper*, if, for any compact subsets $A, B \subseteq X$, the set*

$$G_{A,B} = \{g \in G \ : \ gA \cap B \neq \emptyset\}$$

*is compact in $G$.*

When $G$ is a discrete group, then a proper $G$-action is usually called *properly discontinuous.*

**Proposition 6.2** *If $G$ acts properly on $X$, then*

- *The stabilizer $G(x)$ of any $x \in X$ is compact.*
- *The orbit $Gx$ of any $x \in X$ is closed.*
- *The quotient space $G\backslash X$ is Hausdorff in the quotient topology.*

**Proof:**

- Since any point $x \in X$ is a compact subset, compactness of $G(x) = G_{\{x\},\{x\}}$ follows directly from the definition.
- Suppose that $g_n \in G$ are such that $g_n x \to y \in X$. Let $A$ be a compact neighborhood of $y$ in $X$. For all $n$ sufficiently large, $g_n \in G_{\{x\},A}$. Because $G_{\{x\},A}$ is compact, there is a convergent subsequence $g_n \to h$. Now by continuity of the action, $g_n x \to hx$. The limit $hx$ must equal $y$, so $y \in Gx$.
- Let $x, y \in X$ be such that $Gx \neq Gy$, so they project to distinct points in the quotient $G\backslash X$. We must find $G$-invariant open sets $U$ containing $x$ and $V$ containing $y$ such that $U \cap V = \emptyset$.

  First choose an open neighborhood $A$ of $x$ such that the closure $\overline{A}$ is compact. Because $G_{\{y\},\overline{A}}$ is compact, the intersection $Gy \cap \overline{A}$ is compact; it does not contain $x$ because $Gx \neq Gy$. Therefore, $x$ has a neighborhood in $A$ not meeting $Gy$. Now replace $A$ by a neighborhood of $x$ with compact closure contained in $A\backslash(Gy \cap A)$.

  Next, let $B$ be a neighborhood of $y$ with $\overline{B}$ compact. Since $G_{\overline{A},\overline{B}}$ is compact, $G\overline{A} \cap \overline{B}$ is compact; by the choice of $A$, this intersection does not contain $y$. Therefore, $y$ has a neighborhood in $B$ not meeting $GA$;

replace $B$ by this neighborhood. Then $U = GA$ and $V = GB$ are disjoint neighborhoods of $Gx$ and $Gy$, as desired.

$\diamondsuit$

## 6.2 Counter-examples

**Example.** The isometry groups of $\mathrm{Min}^n$, $\mathrm{dS}^n$, and $\mathrm{AdS}^n$ do not act properly, because they have noncompact stabilizers. The same is true for the compact homogeneous examples of Section 5. Note also that an action of a noncompact group on a compact space is always nonproper.

**Example.** Consider the action of $\mathbf{R}^*$ on $\mathbf{R}^2\backslash\{0\}$ via the group of matrices

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \qquad \lambda \in \mathbf{R}^*$$

This action is free, and every orbit is closed. But it is not proper. Indeed, let $K$ be the unit circle in $\mathbf{R}^2\backslash\{0\}$. Every $g \in \mathbf{R}^*$ maps $K$ to an ellipse centered at the origin, so every $gK \cap K$ contains at least 4 points.

**Exercise 6.3** *Find two orbits in $\mathbf{R}^2\backslash\{0\}$ with no disjoint invariant neighborhoods. Then conclude by a different argument from that above that this action is not proper.*

The following example is due to Scot Adams.

**Example.** Let

$$\begin{aligned} U &= \{(x,y) \in \mathbf{R}^2 \ : \ y > 0, \ x \geq 1/y\} \\ U' &= \{(x,y) \in \mathbf{R}^2 \ : \ y > 0, \ x < 1/y\} \\ V &= \{(x,y) \in \mathbf{R}^2 \ : \ y < 0, \ x \geq 1/y\} \\ X &= \{(x,y) \in \mathbf{R}^2 \ : \ y = 0\} \end{aligned}$$

Define

$$\begin{aligned} f &: U \to V \\ f &: (x,y) \mapsto (x - 2/y, -y) \end{aligned}$$

Now define $T$ from $U' \cup X \cup V$ to itself by

$$T(x,y) = \begin{cases} (x+1,y), & (x,y) \in V \cup X \\ (x+1,y), & (x,y) \in U' \text{ and } (x+1,y) \in U' \\ f(x+1,y), & (x,y) \in U' \text{ and } (x+1,y) \in U \end{cases}$$

The group $\langle T\rangle$ generated by $T$ acts freely, with closed orbits, and the quotient $(U'\cup X\cup V)/\langle T\rangle$ is Hausdorff. The action is not, however, proper, because, for every positive integer $j$,

$$T^{2j}(0,1/j)=(0,-1/j)$$

## 6.3 Properness of Riemannian isometries

A Riemannian manifold is analogous to a Lorentz manifold, with a smooth choice of positive-definite, rather than type-$(1,n-1)$, inner product on each tangent space. Unlike a Lorentz metric, a Riemannian metric on a manifold $M$ defines a distance, making $M$ into a proper metric space. A *proper* metric space is one in which balls $B(x,\delta)$ have compact closure. There results a crucial difference between Riemannian and Lorentzian isometries: Riemannian isometry groups act properly.

**Proposition 6.4** *Suppose $G$ is a closed subgroup of Isom$(M)$, for $M$ a separable, $\sigma$-finite, proper metric space:*

$$d(x,y)=d(g(x),g(y))\qquad \forall\ x,y\in M,\ g\in G$$

*Then $G$ acts properly on $M$.*

**Proof:** Let $A,B$ be compact subsets of $M$. Let

$$a=\sup\{d(x,y)\ :\ x,y\in A\}$$

the *diameter* of $A$. Compactness of $A$ implies that $a$ is finite. Let

$$D(B,a)=\{x\in M\ :\ d(x,b)\le a \text{ for some } b\in B\}$$

This set is compact. For any $g\in G_{A,B}$, the image $gA\subset D(B,a)$.

Suppose that $g_n$ is a sequence in $G_{A,B}$. Let $A'$ be a countable dense subset of $A$. For each $a\in A'$, the sequence $g_n(a)$ has a convergent subsequence. By diagonalization, we may assume that $g_n$ has a subsequence for which $g_n(a)$ converges for all $a\in A'$. Denote the limit $g(a)$. For any $a,a'\in A'$, the distance

$$d(g(a),g(a'))=\lim d(g_n(a),g_n(a'))=d(a,a')$$

so $g_n$ converges to $g$ uniformly on $A'$, and $g$ preserves distances in $A'$.

Now for any $a \in A$, the sequence $g_n(a)$ is Cauchy in $D(B, a)$; let $g(a)$ be the limit. The sequence $g_n$ converges to $g$ uniformly on $A$. For $a, a' \in A$,

$$d(g(a), g(a')) = \lim d(g_n(a), g_n(a')) = d(a, a')$$

so $g$ preserves distances on $A$.

For any $r > 0$, the set

$$D(A, r) = \{x \in M \ : \ d(x, a) \leq r \text{ for some } a \in A\}$$

is compact, and is mapped by $G_{A,B}$ into the compact $D(B, a + r)$. We can repeat the process above to find a subsequence of $g_n$ converging uniformly to an isometry defined on $D(A, r)$. The entire space $M$ is exhausted by a countable union of compact sets of the form $D(A, r)$. By diagonalization again, we obtain a subsequence of $g_n$ that converges to an isometry $g$ of $M$, uniformly on all compact sets. $\diamondsuit$

There is a converse to this theorem for Riemannian manifolds: a group acting properly and smoothly on a manifold preserves some Riemannian metric. See [Kos] for the proof of this fact and other material on proper actions.

# 7 Structure of simple Lie groups

Let $G$ be a connected Lie group and $\mathfrak{g}$ its Lie algebra. The group $G$ acts by Lie algebra automorphisms on $\mathfrak{g}$ via the adjoint representation. There is a *Lie algebra representation*, also called the adjoint representation, of $\mathfrak{g}$ on itself, that corresponds under the exponential map to Ad.

**Definition 7.1** *For $\mathfrak{g}$ a Lie algebra, the* adjoint representation *ad of $\mathfrak{g}$ on $\mathfrak{g}$ is*

$$ad(X)(Y) = [X, Y]$$

The Lie algebra adjoint consists of infinitesimal Lie algebra automorphisms:

$$[\mathrm{ad}(X)(Y), Z] + [Y, \mathrm{ad}(X)(Z)] = \mathrm{ad}(X)([Y, Z])$$

This follows immediately from the Jacobi identity. Recall that the bracket can be defined in terms of the derivative of the one-parameter subgroup $\mathrm{Ad}(e^{tX})$ of $\mathrm{Aut}(\mathfrak{g})$:

$$\mathrm{ad}(X)(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Ad}(e^{tX})(Y)$$

from which follows the relation

$$\mathrm{Ad}(e^{tX}) = e^{t\mathrm{ad}(X)}$$

where the exponential map on the right is in the group $\mathrm{Aut}(\mathfrak{g})$.

**Definition 7.2** *A Lie algebra* $\mathfrak{g}$ *is* simple *if dim* $\mathfrak{g} > 1$ *and ad*$\mathfrak{g}$ *is irreducible—that is, the only invariant subspaces are* 0 *and* $\mathfrak{g}$. *A Lie group* $G$ *is* simple *if* $\mathfrak{g}$ *is.*

**Exercise 7.3** *Show that definitions 2.9 and 7.2 are equivalent for algebraic Lie groups and Lie algebras. (*Hint*: The identity component* $G^0$ *is Zariski dense in* $G$*.)*

**Definition 7.4** *An element* $X \in \mathfrak{g}$ *is* **R**-split *if ad*$(X)$ *is diagonalizable over* **R**. *It is* nilpotent *if ad*$(X)$ *is nilpotent. An element* $a \in G$ *is* **R**-split *if Ad*$(a)$ *is diagonalizable over* **R**.

Note that if $X$ is **R**-split, then $\mathrm{Ad}(e^X) = e^{\mathrm{ad}(X)}$ is, too, because the exponential of a real diagonal matrix is real diagonal.

**Example.** A Lie algebra may not have any **R**-split elements. Consider the Lie algebra $\mathfrak{o}(n)$ of all $n \times n$ skew-symmetric matrices. The inner product

$$\langle X, Y\rangle = \mathrm{tr} XY$$

is negative definite and $\mathrm{Ad}(O(n))$-invariant. Suppose there were an **R**-split element $g \in O(n)$. Then there are nonzero $\alpha \in \mathbf{R}$ and $X \in \mathfrak{g}$ such that $\mathrm{Ad}(g)(X) = \alpha X$. But then we have a contradiction:

$$0 \neq \langle X, X\rangle = \langle \mathrm{Ad}(g)(X), \mathrm{Ad}(g)(X)\rangle = \alpha^2\langle X, X\rangle$$

**Definition 7.5** *The* **R***-rank of a connected Lie group* $G$ *is the dimension of a maximal abelian subalgebra of* $\mathfrak{g}$ *consisting of* **R***-split elements.*

Maximal abelian **R**-split subalgebras of $\mathfrak{g}$ correspond under the exponential map to maximal connected, abelian, **R**-split subgroups of $G$. If $G$ is an algebraic Lie group, and $X, Y \in \mathfrak{g}$ are such that $XY = YX$, then it is easy to compute that

$$e^X e^Y = e^{X+Y}$$

This formula holds in a general Lie group for any commuting $X$ and $Y$, as a consequence of the Campbell-Hausdorff formula (see [Ja] V.5). It follows that if $\mathfrak{a}$ is an abelian subalgebra, then the group $A$ generated by all the one-parameter groups $e^{tX}$, where $X \in \mathfrak{a}$, is abelian and equals the image $e^{\mathfrak{a}}$.

Now suppose that $A$ is a maximal abelian $\mathbf{R}$-split subalgebra of $\mathfrak{g}$. Let $A_1, \ldots, A_k$ be a basis for $\mathfrak{a}$. There is an eigenspace decomposition for $\mathrm{ad}(A_1)$

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$$

where $\alpha$ ranges over the eigenvalues of $\mathrm{ad}(A_1)$. Because $A_2$ commutes with $A_1$, the endomorphism $\mathrm{ad}(A_2)$ preserves the eigenspaces for $\mathrm{ad}(A_1)$, so there is a refinement of the decomposition above into simultaneous eigenspaces for both $\mathrm{ad}(A_1)$ and $\mathrm{ad}(A_2)$. Continuing in this way, we obtain a decomposition of $\mathfrak{g}$ into simultaneous eigenspaces for $\mathrm{ad}(A_1), \ldots, \mathrm{ad}(A_k)$. To each such eigenspace correspond $k$ real eigenvalues $\alpha_1, \ldots, \alpha_k$, and there is a linear functional $\alpha : \mathfrak{a} \to \mathbf{R}$ such that $\alpha(A_i) = \alpha_i$. Denote by $\mathfrak{g}_\alpha$ the simultaneous eigenspace with eigenvalues given by $\alpha$. That is, for $A \in \mathfrak{a}$ and $X \in \mathfrak{g}_\alpha$,

$$\mathrm{ad}(A)(X) = \alpha(A)X$$

The functional $\alpha$ is called a *root* and $\mathfrak{g}_\alpha$ is the *root space* for $\alpha$.

Here are few basic facts about the root space decomposition.

**Proposition 7.6** *Let $\mathfrak{a}$ be a maximal $\mathbf{R}$-split subalgebra of $\mathfrak{g}$, and*

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$$

*the root space decomposition.*

- *If $\mathfrak{g}$ is simple, then for each $A \in \mathfrak{a}$, there is a least one nonzero root $\alpha$ such that $\alpha(A) \neq 0$.*
- *For any two roots $\alpha$ and $\beta$ of $\mathfrak{g}$,*

  $$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

  *(Note that $\alpha+\beta$ may not be a root of $\mathfrak{g}$, in which case this bracket is zero.)*
- *If $\mathfrak{g}$ is simple and $\alpha$ is a root, then so is $-\alpha$; further $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$.*

**Proof:**

- If $\alpha(A) = 0$ for all roots $\alpha$, then $\mathrm{ad}(A)$ is zero on each root space $\mathfrak{g}_\alpha$. Because these root spaces exhaust $\mathfrak{g}$, it follows that $A$ is in the center of $\mathfrak{g}$. But a simple group cannot have a one-parameter subgroup in its center.

- Let $A \in \mathfrak{a}$, $X \in \mathfrak{g}_\alpha$, and $Y \in \mathfrak{g}_\beta$. Then

$$\begin{aligned}
\mathrm{ad}(A)([X,Y]) &= [\mathrm{ad}(A)(X), Y] + [X, \mathrm{ad}(A)(Y)] \\
&= \alpha(A)[X,Y] + \beta(A)[X,Y] \\
&= (\alpha+\beta)(A)[X,Y]
\end{aligned}$$

- If $-\alpha$ is not a root, then the sum over all roots $\beta$

$$\bigoplus_\beta \mathfrak{g}_{\alpha+\beta}$$

  is an $\mathrm{ad}(\mathfrak{g})$-invariant subspace properly contained in $\mathfrak{g}$, because it does not meet $\mathfrak{g}_0$. It is not zero because it contains $\mathfrak{g}_\alpha$. This circumstance would contradict simplicity. Similarly, if $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 0$, then the same subspace would be proper, nonzero, and invariant.

$\diamondsuit$

**Example.** The $\mathbf{R}$-rank of $O(1,n)$ is 1. First, replace $B_{1,n}$ by another form of type $(1,n)$ with respect to which a maximal $\mathbf{R}$-split subgroup of $O(1,n)$ is diagonal. Namely, let $B$ be given by

$$B = \begin{bmatrix} 0 & & & & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 1 & & & & 0 \end{bmatrix}$$

With respect $B$, the Lie algebra $\mathfrak{o}(1,n)$ consists of matrices

$$\begin{bmatrix} \lambda & x^t & 0 \\ y & m & -x \\ 0 & -y^t & -\lambda \end{bmatrix} \qquad \lambda \in \mathbf{R},\ x, y \in \mathbf{R}^{n-1},\ m \in \mathfrak{o}(n-1)$$

The elements with $x, y, m = 0$, as $\lambda$ ranges over $\mathbf{R}$, form a one-dimensional $\mathbf{R}$-split subalgebra $\mathfrak{a}$. The centralizer of $\mathfrak{a}$ is $\mathfrak{a} + \mathfrak{m}$, where $\mathfrak{m} \cong \mathfrak{o}(n-1)$ consists of elements with $\lambda, x, y = 0$. Consider the eigenspace consisting of elements with $y, m, \lambda = 0$, as $x$ ranges over $\mathbf{R}^{n-1}$. The adjoint of $\mathfrak{m}$ on here is the standard representation of $\mathfrak{o}(n-1)$ on $\mathbf{R}^{n-1}$; all eigenvalues in this representation are purely imaginary. Therefore, no element of $\mathfrak{m}$ can be $\mathbf{R}$-split, so $\mathfrak{a}$ is a maximal abelian $\mathbf{R}$-split subalgebra.

**Exercise 7.7** *Compute explicity the Lie algebra $\mathfrak{o}(2,n)$, and show that the* $\mathbf{R}$-*rank of $O(2,n)$ is 2.*

# 8 Kowalsky's argument

Suppose that $G$ is a simple Lie group acting nonproperly and isometrically on a Lorentz manifold $M$. The goal of this section is to show that, for some $x \in M$, a sum of root spaces in $\mathfrak{g}$ all generate isotropic flows at $x$. Because the maximal dimension of an isotropic subspace of $T_xM$ is 1, a codimension-1 subspace of this sum of root spaces fixes $x$. The argument is based on the interaction of the nonproper dynamics of $G$ on $M$ and the dynamics of $\mathrm{Ad}(G)$ on $\mathfrak{g}$.

Under the nonproperness assumption on the $G$-action, there are compact subsets $B, C \subset M$ such that $G_{B,C}$ is not compact. By continuity of the action $G \times M \to M$, the set $G_{B,C}$ is closed, so it must be unbounded. Let $g_n \in G_{B,C}$ be an unbounded sequence. The $KAK$ decomposition (see [Kn]) of $G$ gives that every element $g \in G$ can be written as a product $g = kak'$, with $k, k' \in K$ and $a \in A$, where $A$ is a maximal connected, abelian, $\mathbf{R}$-split subgroup of $G$, and $\mathrm{Ad}(K)$ is a maximal compact subgroup of $\mathrm{Ad}(G)$. If $Z(G)$ is finite, then $K$ is a maximal compact subgroup of $G$. Write

$$g_n = k_n a_n k_n'$$

Because $g_n$ is unbounded and $K$ is compact, the sequence $a_n$ must be unbounded in $A$. By passing to a subsequence, we may assume $k_n \to k$ and $k_n' \to k'$. Recall that the exponential map of $G$ maps $\mathfrak{a}$ onto $A$. Let $A_n = \log a_n \in \mathfrak{a}$. With respect to any norm on the vector space $\mathfrak{a}$, the sequence $|A_n| \to \infty$. Again passing to a subsequence, we may assume that $A_n/|A_n| \to A$ for some $A$ in the unit sphere of $\mathfrak{a}$.

Each $X \in \mathfrak{g}$ defines a vector field $X^\dagger$ on $M$ by

$$X^\dagger(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 e^{tX} x$$

Differentiating

$$ge^{tX}g^{-1}gx = ge^{tX}x$$

yields

$$\mathrm{Ad}(g)(X)^{\dagger}(gx) = g_{*x}(X^{\dagger}(x))$$

Now each $x \in M$ gives an inner product $\langle\ ,\ \rangle_x$ on $\mathfrak{g}$ by

$$\langle X, Y\rangle_x = \langle X^{\dagger}(x), Y^{\dagger}(x)\rangle$$

Because $G$ acts isometrically, for any $g \in G$,

$$\begin{aligned}
\langle X, Y\rangle_{gx} &= \langle X^{\dagger}(gx), Y^{\dagger}(gx)\rangle \\
&= \langle g_{*x}(\mathrm{Ad}(g^{-1})(X)^{\dagger}(x)), g_{*x}(\mathrm{Ad}(g^{-1})(Y)^{\dagger}(x))\rangle \\
&= \langle \mathrm{Ad}(g^{-1})(X)^{\dagger}(x), \mathrm{Ad}(g^{-1})(Y)^{\dagger}(x)\rangle \\
&= \langle \mathrm{Ad}(g^{-1})(X), \mathrm{Ad}(g^{-1})(Y)\rangle_x
\end{aligned}$$

Now let $x_n \in B$ be such that $g_n x_n = y_n \in C$. Passing to a subsequence, we may assume $x_n \to x \in B$ and $y_n \to y \in C$. For $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$,

$$\begin{aligned}
\langle X, Y\rangle_{a_n k'_n x_n} &= \langle \mathrm{Ad}(a_n^{-1})(X), \mathrm{Ad}(a_n^{-1})(Y)\rangle_{k'_n x_n} \\
&= e^{-\alpha(A_n)-\beta(A_n)}\langle X, Y\rangle_{k'_n x_n}
\end{aligned}$$

On the other hand,

$$\langle X, Y\rangle_{a_n k'_n x_n} = \langle X, Y\rangle_{k_n^{-1} y_n}$$

Now

$$\alpha(A_n) = |A_n| \cdot \alpha\left(\frac{A_n}{|A_n|}\right)$$

and $\alpha(A_n/|A_n|) \to \alpha(A)$. The same holds for $\beta$. Therefore, if $\alpha(A), \beta(A) > 0$, then

$$e^{-\alpha(A_n)-\beta(A_n)}\langle X, Y\rangle_{k'_n x} \to 0$$

while

$$\langle X, Y\rangle_{k_n^{-1} y_n} \to \langle X, Y\rangle_{k^{-1} y}$$

Therefore, the subspace

$$\bigoplus_{\alpha(A)>0} \mathfrak{g}_\alpha$$

is totally isotropic for $\langle\ ,\ \rangle_{k^{-1}y}$.

The conclusions of this argument are summarized in the following proposition.

**Proposition 8.1** *Let $G$ be a simple Lie group with finite center acting isometrically and nonproperly on a Lorentz manifold $M$. Then there is a point $x \in M$ and a nontrivial $\mathbf{R}$-split element $A \in \mathfrak{g}$ such that*

$$\bigoplus_{\alpha(A)>0} \mathfrak{g}_\alpha$$

*is totally isotropic for $\langle , \rangle$.*

The following is the first corollary of proposition 8.1. (This result had been previously obtained by Zimmer in [Zi2]. He showed that when $M$ is compact, almost every stabilizer is discrete, which sufficed for his argument.)

**Theorem 8.2** *([Kow]) Suppose $G$ is a simple Lie group with finite center acting nonproperly and isometrically on a Lorentz manifold $M$. Suppose further that for all $x \in M$, the stabilizer $G(x)$ is discrete. Then $G$ is locally isomorphic to $SL(2, \mathbf{R})$—that is, $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbf{R})$.*

**Proof:** By the assumptions on $G$ and $M$, the argument above applies, so there is some $x \in M$ and an $\mathbf{R}$-split element $A \in \mathfrak{g}$ such that

$$\bigoplus_{\alpha(A)>0} \mathfrak{g}_\alpha$$

is a totally isotropic subspace for $\langle \ , \ \rangle_x$. The additional assumption that $G(x)$ is discrete implies that the linear map $X \mapsto X^\dagger(x)$ is injective. Because a maximal isotropic subspace of $T_xM$ is 1-dimensional, the subspace $\oplus_{\alpha(A)>0}\mathfrak{g}_\alpha$ is at most 1-dimensional.

By proposition 7.6, there is some root $\alpha$ such that $\alpha(A) > 0$. Let $X_\alpha$ be a generator of $\mathfrak{g}_\alpha$. Note that $\beta(A) \leq 0$ for any root $\beta \neq \alpha$.

Applying Kowalsky's argument to the inverse sequence $g_n^{-1} = (k_n')^{-1} a_n^{-1} k_n^{-1}$ gives $A' \in \mathfrak{a}$ and $x' \in M$ such that

$$\bigoplus_{\alpha(A')>0} \mathfrak{g}_\alpha$$

is isotropic for $\langle \ , \ \rangle_{x'}$. But $\log(a_n^{-1}) = -\log(a_n)$, so

$$A' = \lim \frac{\log(a_n^{-1})}{|\log(a_n^{-1})|} = -A$$

Therefore the sum

$$\bigoplus_{\alpha(A)<0} \mathfrak{g}_\alpha$$

is isotropic for $\langle\ ,\ \rangle_{x'}$. Discreteness of $G(x')$ again implies that this space is 1-dimensional. By proposition 7.6, $-\alpha$ is a root, and obviously $-\alpha(A) < 0$. Let $X_{-\alpha}$ be a generator of $\mathfrak{g}_{-\alpha}$. Note $\beta(A) \geq 0$ for any $\beta \neq -\alpha$. Now $\beta(A) = 0$ unless $\beta = \pm\alpha$.

In order to conclude that $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbf{R})$, it suffices to show that $X_\alpha, A$, and $X_{-\alpha}$ generate an $\mathrm{Ad}(G)$-invariant subalgebra isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. Using the exponential map, it suffices to show they generate an $\mathrm{ad}(\mathfrak{g})$-invariant subalgebra isomorphic to $\mathfrak{sl}(2, \mathbf{R})$.

For any nonzero root $\beta$, if $\beta(A) = 0$, then $\beta \pm \alpha$ cannot be a root. Therefore, $[X_\beta, X_{\pm\alpha}] = 0$. Let $B = [X_\alpha, X_{-\alpha}]$; it is nonzero by 7.6. If $\beta$ is a root different from $\pm\alpha$ and $X_\beta \in \mathfrak{g}_\beta$, then the Jacobi identity gives

$$[B, X_\beta] = -[[X_\beta, X_\alpha], X_{-\alpha}] - [[X_{-\alpha}, X_\beta], X_\alpha]$$

The brackets $[X_\beta, X_{-\alpha}]$ and$[X_{-\alpha}, X_\beta]$ are zero unless $\beta = 0$. In this case, the right hand side is

$$-[\alpha(X_\beta)X_\alpha, X_{-\alpha}] - [\alpha(X_\beta)X_{-\alpha}, X_\alpha] = 0$$

Therefore, $\beta(B) = 0$ unless $\beta = \pm\alpha$. Then all roots vanish on $cA - B$ for some $c \in \mathbf{R}$, which implies, again by proposition 7.6, that $B = cA$. Now $X_\alpha, A, X_{-\alpha}$ generate a subalgebra of $\mathfrak{g}$ that is $\mathrm{ad}(\mathfrak{g})$-invariant and isomorphic to $\mathfrak{sl}(2, \mathbf{R})$, so $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbf{R})$. ♢

The main theorem of Kowalsky's thesis says that nonproper isometric actions of simple groups resemble the constant-curvature models:

**Theorem 8.3** *Suppose that $G$ is a simple Lie group with finite center acting nonproperly on a Lorentz manifold $M$. Then $G$ is locally isomorphic to $O(1, n)$ for some $n \geq 2$ or $O(2, n)$ for some $n \geq 3$.*

Toward this end, we will prove the following theorem of [ADZ] in the simple case.

**Theorem 8.4 ([ADZ] 1.6)** *Suppose that a simple group $G$ with finite center and not locally isomorphic to $SL(2, \mathbf{R})$ acts isometrically on a Lorentz manifold $M$ of dimension at least 3. Suppose that $G$ has an orbit of Lorentz type in $M$ with noncompact stabilizer. Then this orbit has constant curvature, and $G$ is locally isomorphic to $O(1, n)$ or $O(2, n)$ for some $n \geq 3$.*

The results of [ADZ] actually apply to semisimple groups with finite center and no local $SL(2, \mathbf{R})$-factors, and the full conclusions describe the isometry type of

$M$ in a neighborhood of the nonproper Lorentz orbit. A semisimple Lie algebra is a direct sum of simple Lie algebras, and a semisimple Lie group is one with semisimple Lie algebra.

Suppose $G$ is simple with finite center and acts nonproperly on a Lorentz manifold $M$. By Kowalsky's argument, associated to a sequence $g_n \to \infty$ are a point $x \in M$ and an $\mathbf{R}$-split $A \in \mathfrak{g}$ such that

$$\bigoplus_{\alpha(A)>0} \mathfrak{g}_\alpha$$

is an isotropic subspace for the inner product $\langle\ ,\ \rangle_x$ on $\mathfrak{g}$. Note that if $e^{tY}$ is a 1-parameter group in the stabilizer $G(x)$, then $Y^\dagger(x) = 0$, so $Y$ is isotropic. Now a maximal isotropic subspace of $T_xM$ is 1-dimensional, so either the sum of root spaces above intersects the stabilizer Lie algebra $\mathfrak{g}(x)$, or it is 1-dimensional. The same is true for the point $x'$ associated to the sequence $g_n^{-1}$. Then either the argument in the discrete stabilizers case above applies, and $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbf{R})$, or one of $x, x'$ has a nonzero root vector in the stabilizer. Thus if $G$ is not locally isomorphic to $SL(2, \mathbf{R})$, then we may assume there is some nonzero $Y \in \oplus_{\alpha(A)>0}\mathfrak{g}_\alpha$ that is also in $\mathfrak{g}(x)$.

The one-parameter group generated by such a $Y$ is not precompact because $\mathrm{Ad}(Y)$ is unipotent. Then $G(x)$ is noncompact. Because $G$ is simple, it acts faithfully on the orbit $Gx$, unless $x$ is a fixed point. Then $Gx$ is not Riemannian type, so it is Lorentzian, degenerate, or a fixed point. In the degenerate case, the results of [DMZ] give that $G$ has a Lorentzian orbit near $x$ with noncompact stabilizer; the fixed point case is easy, and is discussed in [DMZ]. Thus the results of [ADZ] and [DMZ] together give a proof of Kowalsky's theorem under the assumption of no local $SL(2, \mathbf{R})$-factors; the next section gives some idea about the proofs, which are different from those of [Kow].

# 9 Totally geodesic lightlike hypersurfaces and constant curvature

The main tool in [ADZ] is totally geodesic lightlike hypersurfaces, which can be associated to actions with strong dynamics. These will be defined below. In the current setting, the noncompact stabilizer of $x$ will imply existence of one of these hypersurfaces through $x$.

Given a Lorentz metric on a manifold $M$, there is a unique *torsion-free connection*, called the *Levi-Civita connnection*, on $M$ that is compatible with the metric. A connection is a way to differentiate vector fields. Denote by $\Xi(M)$ the vector space of vector fields on $M$. A connection is a function of two vector fields that depends linearly on the first coordinate and as a derivation on the second:

$$\begin{aligned}
\nabla &: \Xi(M) \times \Xi(M) \to \Xi(M) \\
\nabla &: (X, Y) \mapsto \nabla_X Y \\
\nabla_{fX+Y} Z &= f\nabla_X Z + \nabla_Y Z \\
\nabla_X (fY + Z) &= X(f)Y + f\nabla_X Y + \nabla_X Z
\end{aligned}$$

where $f$ is any function on $M$. The vector field $\nabla_X Y$ is called the *covariant derivative* of $Y$ in the direction of $X$. A connection is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for any $X, Y \in \Xi(M)$. Finally, it is compatible with the metric if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for any $X, Y, Z \in \Xi(M)$. The vector field $\nabla_X Y$ is determined by $\langle \nabla_X Y, Z\rangle$ for $Z$ varying over vector fields that span each tangent space. The sum

$$\begin{aligned}
X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle Y, X\rangle &= \langle \nabla_X Y, Z\rangle + \langle \nabla_Y X, Z\rangle \\
&+ \langle Y, \nabla_X Z - \nabla_Z X\rangle + \langle X, \nabla_Y Z - \nabla_Z Y\rangle
\end{aligned}$$

Using that the connection is torsion free, we obtain an explicit formula for the Levi-Civita connection in terms of the metric and the bracket of vector fields:

$$\begin{aligned}
2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle Y, X\rangle \\
&+ \langle [X, Y], Z\rangle - \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle
\end{aligned}$$

The *geodesics* of a connection are the curves $\gamma$ such that $\nabla_{\gamma'}\gamma' = 0$ along $\gamma$. To define the covariant derivative $\nabla_{\gamma'}\gamma'$, extend $\gamma'$ to a vector field $X$ in a neighborhood of the image of the curve $\gamma$. The values of $\nabla_X X$ along $\gamma$ are independent of the choice of $X$; indeed, if $X(\gamma(t)) = Y(\gamma(t)) = \gamma'(t)$ for all $t \in (-\epsilon, \epsilon)$, then $[X, Y]$ vanishes along $\gamma$. Also, along $\gamma$, we have $\nabla_X(X - Y) = \nabla_{\gamma'}(X - Y) = 0$. It follows that $\nabla_X X = \nabla_X Y = \nabla_Y X = \nabla_Y Y$ along $\gamma$.

The equation $\nabla_{\gamma'}\gamma' = 0$ can be expressed as a second-order ODE, so geodesics with a given initial tangent vector always exist on some time interval $(-\epsilon, \epsilon)$. A submanifold $N \subset M$ is *totally geodesic* if, for any vector fields $X, Y$ tangent to $N$, the covariant derivative $\nabla_X Y$ is also tangent to $N$. This condition is equivalent to the condition that, for any $X \in T_x N$, the geodesic through $x$ with initial vector $X$ stays in $N$ for some positive time.

**Example.** In case $M = \mathrm{Min}^n$, the covariant derivative is simply $\nabla_X Y = XY$; to prove this, it suffices to verify that it has all the properties of a torsion-free connection compatible with the metric.

Now let $\gamma(t)$ be a curve in $\mathrm{Min}^n$. The condition for $\gamma$ to be a geodesic is that

$$\begin{aligned} 0 &= \nabla_{\gamma'}\gamma' \\ &= \gamma'\gamma' \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma'(\gamma(t)) \\ &= \gamma''(t) \end{aligned}$$

Therefore, the geodesics of $\mathrm{Min}^n$ are straight lines, curves of the form $\gamma(t) = u + tv$, for $u, v \in \mathbf{R}^n$.

Next suppose that $N$ is a submanifold of $\mathrm{Min}^n$. The condition for $N$ to be totally geodesic is that for every $x \in N$ and $u \in T_x N$, the geodesic of $\mathrm{Min}^n$ through $x$ in the direction $u$ is contained in $N$ for some time. Then $N$ locally contains every line tangent to it, and so $N$ must be an open subset of an affine subspace of $\mathrm{Min}^n$.

The *curvature tensor* $R$ of a connection is the operator

$$R(X,Y) = \nabla_{[X,Y]} + \nabla_Y \nabla_X - \nabla_X \nabla_Y$$

It is skew-symmetric, and $R(X,Y)_x$ depends only on $X(x), Y(x)$. Given a vector field $Z$ and a function $f$, the vector field

$$R(X,Y)(fZ) = fR(X,Y)Z$$

so $R(X,Y)$ is a linear endomorphism of $T_x M$. The tensor $\langle R(X,Y)Z, T\rangle$ has the following symmetries (see [KN]):

$$\begin{aligned} \langle R(X,Y)Z,T\rangle + \langle R(Y,Z)X,T\rangle + R(Z,X)Y,T\rangle &= 0 \qquad \text{(Bianchi identity)} \\ \langle R(X,Y)Z,T\rangle &= -\langle R(Y,X)Z,T\rangle \\ \langle R(X,Y)Z,T\rangle &= -\langle R(X,Y)T,Z\rangle \\ \langle R(X,Y)Z,T\rangle &= \langle R(Z,T)X,T\rangle \end{aligned}$$

Given a nondegenerate plane $P = span\{X, Y\} \subset T_xM$, the *sectional curvature* along $P$ is

$$S(X,Y) = \frac{\langle R(X,Y)X, Y\rangle}{|X \wedge Y|}$$

where $|X \wedge Y| = \langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2$, as above. The sectional curvature only depends on $P$, and not on the choice of basis $X, Y$.

**Definition 9.1** *A* totally geodesic lightlike (tgl) *hypersurface in an $n$-dimensional Lorentz manifold $M$ is an $(n-1)$-dimensional totally geodesic submanifold $N \subset M$ such that $T_xN^\perp \cap T_xN \neq 0$.*

If $N$ is a tgl hypersurface, then for each $x \in N$ there is an isotropic vector $v \in T_xN$ such that $T_xN = v^\perp$. If there are many tgl hypersurfaces through every point of a Lorentz manifold $M$, then $M$ has constant sectional curvature.

**Proposition 9.2 ([Ze3] 3)** *Suppose that $M$ is a Lorentz manifold of dimension at least 3 such that, for each $x \in M$, there are tgl hypersurfaces $H_1, \ldots, H_n$ through $x$ such that $v_1 \in H_1^\perp, \ldots, v_n \in H_n^\perp$ span $T_xM$. Then $M$ has constant sectional curvature.*

**Proof:** For $u \in T_xM$, let $A_u(v) = R(u, v)u$. Because $H_i$ is totally geodesic, $A_u(H_i) \subset H_i$ whenever $u \in H_i$. If $u, w \in H_i = v_i^\perp$, then

$$\begin{aligned}
\langle A_u(v_i), w\rangle &= \langle R(u, v_i)u, w\rangle \\
&= \langle R(u, w)u, v_i\rangle \\
&= 0
\end{aligned}$$

Therefore, whenever $u \in v_i^\perp$, then $A_u(v_i) = \lambda v_i$ for some $\lambda$. Note $\langle v_i, v_k\rangle = 0$ if and only if $i = k$ because the maximal dimension of a totally isotropic subspace is 1. Choose $e_i \in \cap_{i\neq j} H_j$ of unit norm. Let $\lambda_{ij}$ be such that $A_{e_i}(v_j) = \lambda_{ij}v_j$. Because

$$\langle A_{e_i}(v_j), v_k\rangle = \langle A_{e_i}(v_k), v_j\rangle$$

$\lambda_{ij} = \lambda_{ik} = \lambda_i$ whenever $j, k \neq i$ and $j \neq k$. Because the $v_i$ span $T_xM$, the sectional curvature of any nondegenerate plane containing $e_i$ equals $\lambda_i$. Then $\lambda_i = \lambda$ for all $i$.

Next we will show the curvature tensor $R_x$ of $M$ at $x$ equals $R_\lambda$, the curvature tensor of a manifold of constant sectional curvature $\lambda = \lambda(x)$. The difference

$R_x - R_\lambda$ is again multilinear and has the same symmetries. Now we will assume $\lambda(x) = 0$ and show that $R_x$ is zero.

First note that because each $H_i$ is totally geodesic, the vector

$$R(e_i, e_j)e_i \in \cap_{i,j\neq k} H_k = span\{e_i, e_j\}$$

Now, because

$$\langle R(e_i, e_j)e_i, e_j\rangle = 0 = \langle R(e_i, e_j)e_i, e_i\rangle$$

the vector $R(e_i, e_j)e_i = 0$ for all $i, j$. Then $R(e_i + e_j, e_i)(e_i + e_j) = 0$, and

$$\langle R(e_i + e_j, e_k)(e_i + e_j), e_i\rangle = \langle R(e_i + e_j, e_i)(e_i + e_j), e_k\rangle = 0$$

Similarly, $\langle R(e_i + e_j, e_k)(e_i + e_j), e_j\rangle = 0$. Since also

$$\langle R(e_i + e_j, e_k)(e_i + e_j), e_k\rangle = \langle R(e_k, e_i + e_j)e_k, e_i + e_j\rangle = 0$$

we may conclude that $R(e_i + e_j, e_k)(e_i + e_j) = 0$ for all $i, j, k$, using that it belongs to $\cap_{l\neq i,j,k} H_l = span\{e_i, e_j, e_k\}$. Since

$$R(e_i + e_j, e_k)(e_i + e_j) = R(e_i, e_k)e_j + R(e_j, e_k)e_i = 0$$

we get $R(e_i, e_k)e_j = -R(e_j, e_k)e_i$ for all $i, j, k$. The Bianchi identity gives

$$\begin{aligned}
0 &= \langle R(e_i, e_j)e_k, e_l\rangle + \langle R(e_k, e_i)e_j, e_l\rangle + \langle R(e_j, e_k)e_i, e_l\rangle \\
&= \langle -R(e_k, e_j)e_i, e_l\rangle + \langle R(e_k, e_i)e_j, e_l\rangle + \langle R(e_j, e_k)e_i, e_l\rangle \\
&= 2\langle R(e_j, e_k)e_i, e_l\rangle + \langle R(e_k, e_i)e_j, e_l\rangle \\
&= 2\langle R(e_j, e_k)e_i, e_l\rangle + \langle -R(e_i, e_k)e_j, e_l\rangle \\
&= 2\langle R(e_j, e_k)e_i, e_l\rangle + \langle R(e_j, e_k)e_i, e_l\rangle \\
&= 3\langle R(e_j, e_k)e_i, e_l\rangle
\end{aligned}$$

so $R_x = 0$, as desired. Then the original curvature tensor at $x$ was equal to that of a manifold with constant sectional curvature $\lambda(x)$. Finally, using the following result, known as Schur's lemma, we can conclude that $\lambda(x) = \lambda(y)$ for all $y, x \in M$.

**Lemma 9.3 (see [KN])** *Let $M$ be a Lorentz manifold such that, for each $x \in M$, all nondegenerate planes in $T_xM$ have the same sectional curvature. Then $M$ has constant sectional curvature.*

$\diamondsuit$

**Proposition 9.4 ([Ze3] 6)** *Let $M$ be a Lorentz manifold with isometry group $G$. Suppose that $G(x)$ is noncompact for some $x \in M$. Then there is a tgl hypersurface in $M$ through $x$.*

**Proof:** Denote by $\mu$ the metric on $M$.

Let $f_n \to \infty$ in $G(x)$. The graphs $\Gamma(f_n)$ of $f_n$ near $x$ are $n$-dimensional submanifolds of $M \times M$, and they are totally geodesic with respect to the natural connection on the product. They are also isotropic with respect to the type $(n, n)$ pseudo-Riemannian metric $\mu \oplus (-\mu)$ on $M \times M$. After passing to a subsequence, the limit $\Gamma$ of $\Gamma(f_n)$ is an $n$-dimensional, isotropic, totally geodesic submanifold near $x$. It cannot surject onto the first factor near $x$, for then it would again be the graph of some isometry near $x$, contradicting the assumption that $f_n$ does not have any convergent subsequence.

The sequence $f_n^{-1} \to \infty$, and the limit $\Gamma^{-1} = \lim \Gamma(f_n^{-1})$ is obtained from $\Gamma$ by switching the first and second coordinates. Since the projection of $\Gamma^{-1}$ on the first factor is not surjective near $x$, the intersection $\Gamma^{-1} \cap (\{x\} \times M)$ is positive-dimensional; it is also isotropic in $M$, so it has dimension exactly 1. Thus the intersection $\Gamma \cap (M \times \{x\})$ is isotropic, geodesic, and 1-dimensional. It follows that the projection of $\Gamma$ on the first factor is lightlike, geodesic, and $(n-1)$-dimensional. $\diamondsuit$

# 10 Proof of theorem 8.4

Recall that we have a simple Lie group $G$ with a Lorentz orbit in $M$ with noncompact stabilizer. Let $N$ be the Lorentz orbit, and $x \in N$. Then by the proposition above, there is a tgl hypersurface $H$ through $x$. Let $0 \neq v \in H^\perp$. If $g \in G(x)$ and $g_{*x}v = w$, then $w^\perp$ is tangent to another tgl hypersurface $gH$ through $x$.

We will first show that the orbit of $v$ by the isotropy $G(x)$ is infinite, so there are infinitely-many tgl hypersurfaces through $x$. First, the orbit $N$ cannot be contained in $H$, by the assumption that $N$ is of Lorentz type. Suppose that the orbit of $v$ by the isotropy $G(x)$ is a finite set $\{v_1, \ldots, v_k\}$. Then the orbits $g_{*x}v_i$ as $g$ ranges over $G$ give finitely-many isotropic vector fields on $N$ that are permuted by the $G$-action. By passing to a finite-index subgroup of $G$, we may assume that $G$ preserves one of these vector fields. Then $G(x)$ preserves $H$, and the orbit of $H$ by $G$ gives a foliation of $M$ near $x$ by tgl hypersurfaces. The stabilizer of $H$ in $G$ is a codimension-one subgroup. But any simple group with a codimension-one subgroup is locally isomorphic to $SL(2, \mathbf{R})$, a contradiction.

Therefore, we may assume that the isotropy orbit of $v$ is infinite. Let $E$ be the $G(x)$-irreducible subspace of $T_xM$ containing $v$. The restriction of the metric to $E$ is of Lorentz type, because $E$ contains infinitely-many isotropic vectors. But then, by the theorem of [BZ], $G(x)$ contains a subgroup $L$ isomorphic to the identity component $O^0(1, k)$, where $\dim(E) = k + 1$.

The metric on $E^\perp$ is positive-definite, and $L$ acts trivially on it. The translates $g_{*x}(E)$ give a $G$-invariant distribution $\mathcal{E}$ along $N$. This distribution is tangent to a submanifold if and only if, for any two vector fields $X, Y$ along $\mathcal{E}$, the bracket $[X, Y] \in \mathcal{E}$. Let $X, Y$ be two such vector fields near $x$, and denote by $\pi$ the projection from $T_xM$ to $E^\perp$. For any $g \in L$, we have $\pi[g_*X, g_*Y] = g_*\pi[X, Y] = \pi[X, Y]$. Because $L$ acts irreducibly on $E$, the projection $\pi[X, Y]$ must be trivial.

Therefore, there is a submanifold $N'$ tangent to $\mathcal{E}$. This submanifold has consant curvature because each tangent space is spanned by isotropic vectors that are normal to tgl hypersurfaces. Because $N'$ contains $x$ and is $G$-invariant, it contains $N = Gx$; on the other hand $T_xN \subseteq E$ is $G(x)$-invariant, so $T_xN = E$, and $N = N'$. Finally, the irreducibility of $G(x)$ on $E$ means that $G$ is the full isometry group of $N$, so $G = O(1, n)$ or $O(2, n)$ for some $n \geq 3$.

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