The oblique derivative problem for nonlinear elliptic complex equations of second order in multiply connected unbounded domains

Guo Chun Wen

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China

In this article, we discuss that an oblique derivative boundary value problem for nonlinear uniformly elliptic complex equation of second order

$$w_{z\bar{z}}=F(z,w,w_z,\overline{w}_z,w_{zz},\overline{w}_{zz})+G(z,w,,w_z,\overline{w}_z)\ \text{ in }\ D, \tag{0.1}$$

with the boundary conditions

$$\begin{array}{l}\mathrm{Re}[\overline{\lambda_j(t)}w_t+\varepsilon\beta_1(t)w(t)+\tau_1(t)]=0,\\ \mathrm{Re}[\overline{\lambda_2(t)}\overline{w}_t+\varepsilon\beta_2(t)w(t)+\tau_2(t)]=0,\end{array}\quad t\in\Gamma, \tag{0.2}$$

in a multiply connected unbounded domain D, the above boundary value problem will be called Problem P. Under certain conditions, by using the priori estimates of solutions and Leray-Schauder fixed point theorem, we can obtain some results of the solvability for the above boundary value problem (0.1) and (0.2).

Key Words: Oblique derivative problem, nonlinear elliptic complex equations, multiply connected unbounded domains

AMS Mathematics Subject Classification: 35J65, 35J25, 35J15

1. Formulation of oblique derivative problems of second order complex equations and statement of main theorem

In this article, we consider the nonlinear uniformly elliptic complex equation of second order

$$\begin{cases} w_{z\bar{z}}=F(z,w,w_z,\overline{w}_z,w_{zz},\overline{w}_{zz})+G(z,w,w_z,\overline{w}_z),\ F=Q_1w_{zz}+Q_2\overline{w}_{zz}\\ +A_1w_z+A_2\overline{w}_z+A_3w+A_4,\ G=G(z,w,w_z,\overline{w}_z),\ Q_j=Q_j(z,w,w_z,\\ \overline{w}_z,w_{zz},\overline{w}_{zz}),\ j=1,2,\ A_j=A_j(z,w,w_z,\overline{w}_z),\ j=1,...,4,\end{cases} \tag{1.1}$$

in an $N+1-$connected domain D. Denote by $\mathit{\Gamma}=\cup_{j=0}^{N}\mathit{\Gamma}_j$ the boundary contours of the domain D and let $\mathit{\Gamma}\in C^2_\mu(0<\mu<1)$. Without loss of generality, we assume that D is a circular domain in $|z|>1$, bounded by the $(N+1)$-circles $\mathit{\Gamma}_j:|z-z_j|=r_j,\ j=0,1,...,N$ and $\mathit{\Gamma}_0=\mathit{\Gamma}_{N+1}:|z|=1,\ z=\infty\in D$. In this article, the notations are as the same in References [1-6]. Suppose that (1.1) satisfies the following conditions.

Condition C 1) $Q_j(z,w,w_z,\overline{w}_z,S,T)(j=1,2), A_j(z,w,w_z,\overline{w}_z)(j=1,...,4)$ are measurable in $z\in D$ for all continuously differentiable functions $w(z)$ in $\overline{D}$ and any measurable functions $S(z), T(z)$ in D, and satisfy

$$L_{p,2}[A_j(z,w,w_z,\overline{w}_z),\overline{D}] \le k_{j-1},\ j=1,...,4, \tag{1.2}$$

in which $p_0, p\,(2<p_0\le p), k_j(j=0,1,2,3)$ are non-negative constants.

2) The above functions are continuous in $w, w_z, \overline{w}_z \in \mathbb{C}$ for almost every point $z\in D$, $S, T\in\mathbb{C}$, and $Q_j=0(j=1,2)$, $A_j=0\,(j=1,...,4)$ for $z\notin D$.

3) The complex equation (1.1) satisfies the following uniform ellipticity condition, namely for any functions $w(z)\in C^1(\overline{D})$ and $S^j, T^j\in\mathbb{C}\,(j=1,2)$, the inequality

$$\begin{gathered}|F(z,w,w_z,\overline{w}_z,S^1,T^1)-F(z,w,w_z,\overline{w}_z,S^2,T^2)|\\ \le q_1|S^1-S^2|+q_2|T^1-T^2|,\end{gathered} \tag{1.3}$$

holds for almost every point $z\in D$, where $q_1+q_2\le q_0<1$, $q_j\,(j=0,1,2)$ are all non-negative constants.

4) For any function $w(z)\in C^1(\bar{D})$, $G(z,w,w_z,\bar{w}_z)$ satisfies

$$|G(z,w,w_z,\bar{w}_z)|\le A_5|w_z|^\sigma+A_6(\bar{w}_z|^\tau+A_7|w|^\eta,\ 0<\sigma,\tau,\eta<\infty, \tag{1.4}$$

where $A_j=A_j(z)$ satisfying the conditions $L_{p,2}(A_j,\bar{D})\le k_0<\infty(j=5,6,7)$, $p\,(>2), k_0, \sigma, \tau$ and η are positive constants.

The oblique derivative boundary value problem for the complex equation (1.1) may be formulated as follows.

Problem P Find a continuously differentiable solution $w(z)$ of complex equation (1.1) in $\overline{D}$ satisfying the boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda_1(t)}w_t+\varepsilon\beta_1(t)w(t)+\tau_1(t)]=0,\\ &\mathrm{Re}[\overline{\lambda_2(t)}\overline{w}_t+\varepsilon\beta_2(t)w(t)+\tau_2(t)]=0,\end{aligned}\quad t\in\Gamma, \tag{1.5}$$

where $|\lambda_l(z)|=1$ on Γ, $\lambda_l(z)$, $\beta_l(z)$ and $\tau_l(z)(l=1,2)$ satisfy the conditions

$$C_\alpha[\lambda_l,\Gamma]\le k_0,\ C_\alpha[\beta_l(z),\Gamma]\le k_0, C_\alpha[\tau_l(z),\Gamma]\le k_4,\ l=1,2, \tag{1.6}$$

in which $\alpha\,(1/2<\alpha<1)$, $k_j(j=0,4)$ are non-negative constants. Denote

$$K_l=\frac{1}{2\pi}\Delta_\Gamma\arg\lambda_l(z),\quad l=1,2. \tag{1.7}$$

$K=(K_1,K_2)$ is called the index of Problem P. In general, Problem P may not be solvable. Hence we consider its modified well posed-ness shown below.

Problem Q Find a system of continuous solutions $(U(z),V(z),w(z))\,(w(z)\in C^1(\overline{D}), U(z),V(z)\in W^1_{p_0,2}(\overline{D})$ $(2<p_0<p)$ of the first order system of complex equations

$$\begin{aligned}&U_{\overline{z}}=F(z,w,U,V,U_z,V_z)+G(z,w,U,V),\ F=Q_1U_z\\ &+Q_2\overline{V}_{\overline{z}}+A_1U+A_2\overline{V}+A_3w+A_4\overline{w}+A_5,\ V_{\overline{z}}=\overline{U}_z=\overline{\rho(z)},\end{aligned} \tag{1.8}$$

satisfying the boundary conditions

$$
\begin{aligned}
&\mathrm{Re}[\overline{\lambda_1(t)}U(t)]=r_1(t)+h_1(t),\mathrm{Re}[\overline{\lambda_2(t)}V(t)]=r_2(t)+h_2(t),\\
&r_l(t)=-\varepsilon\mathrm{Re}[\beta_l(t)w(t)]+\tau_l(t),\ t\in\Gamma,\ l=1,2.\\
&\mathrm{Im}[\overline{\lambda_1(a_j)}U(a_j)+\varepsilon\beta_1(a_j)w(a_j)]=b_{lj},\\
&\mathrm{Im}[\overline{\lambda_2(a_j)}V(a_j)+\varepsilon\beta_2(a_j)w(a_j)]=b_{2j},\\
&j\in J_l=\begin{cases}1,...,2K_l-N+1,\ K_l\geq N,\\ N-K_l+1,...,N+1,0\leq K_l<N,\end{cases}\quad l=1,2,
\end{aligned}
\tag{1.9}
$$

in which ε is a sufficiently small positive number, and

$$
h_l(z)=\begin{cases}
0,\ z\in\Gamma, & \text{if } K_l\geq N,\\
\left.\begin{array}{l}h_{lj},\ z\in\Gamma_j,\ k=1,...,N-K_l,\\ 0,\ z\in\Gamma_j, j=N-K_l+1,...,N+1\end{array}\right\} & \text{if } 0\leq K_l<N,\\
\left.\begin{array}{l}h_{lj},\ z\in\Gamma_j,\ j=1,...,N,\\ h_{l0}+\mathrm{Re}\displaystyle\sum_{m=1}^{-K_l-1}(h^+_{lm}+ih^-_{lm})z^m, z\in\Gamma_0\end{array}\right\} & \text{if } K_l<0, l=1,2,
\end{cases}
\tag{1.10}
$$

where $h_{lj}\,(j=0,1,...,N)$, $h^{\pm}_{lm}\,(m=1,...,-K_l-1, K_l<0,\ l=1,2)$ are unknown real constants to be determined appropriately, and the relation

$$
w(z)=w_0-\int_1^z[\frac{U(z)}{z^2}dz-\sum_{m=1}^{N}\frac{d_mz_m}{z(z-z_m)}dz]+\frac{\overline{V(z)}}{\bar{z}^2}d\overline{z},
\tag{1.11}
$$

in which $Q_j=Q_j(z,w,U,V,U_z,V_z), j=1,...,4, A_j=A_j(z,w,V,V), j=1,...,7,$ $d_m(m=1,...,N)$ are undetermined complex constants, $|\lambda_l(t)|=1$, and $K_l=\frac{1}{2\pi}\Delta_\Gamma\lambda_l(t)\,(l=1,2)$, $K=(K_1,K_2)$ is called the index of Problem P. We assume that

$$
|b_{lj}|\leq k_4,\ j\in J_l,\ l=1,2,
\tag{1.12}
$$

where k_5 is a real constant as before.

In this article, we first discuss the modified boundary value problem (Problem Q) for a system of first order complex equations, which corresponds to Problem P for the complex equation (1.1). We establish then the integral expression and a priori estimates of solutions for Problem Q. By the estimates and the Leray-Schauder theorem and the Schauder fixed point theorem, we can prove the existence of a solution for Problem Q, and so derive the results of the solvability for Problem P for the system (1.1) with some conditions as follows.

Theorem 1.1 (The Main Theorem) *Suppose that the second order nonlinear system* (1.1) *satisfy Condition* C. *If the constants* q_2,ε,k_1,k_2 *in* (1.2), (1.3), (1.5) *are all sufficiently small, and when* $0<\sigma,\tau,\eta<1$, *or when* $\min(\sigma,\tau,\eta)>1$ *and* $k_3+k_4+k_5$ *is small enough, then Problem* P *for* (1.1) *possesses the following results on solvability:*

(1) *When the indices* $K_j=\frac{1}{2\pi}\Delta_\Gamma\arg\lambda_j(t)\geq N\,(j=1,2)$, *Problem* P *for* (1.1) *has* $2N$ *solvability conditions, and the solution depends on* $2(K_1+K_2-N+2)$ *arbitrarily real constants.*

(2) *When the indices* $0\leq K_j<N\,(j=1,2)$, *the total number of the solvability conditions for Problem* P *is not greater than* $4N-[K_1+1/2]-[K_2+1/2]$ *and the solution depends on* $[K_1]+[K_2]+4$ *arbitrarily real constants.*

(3) *When* $0\leq K_1<N, K_2\geq N$ (*or* $K_1\geq N, 0\leq K_2<N$), *the total number of the solvability conditions for Problem* P *is not greater than* $3N-[K_1+1/2]$ (*or* $3N-[K_2+1/2]$) *and the solution depends on* $[K_1]+2K_2-N+4$ (*or* $2K_1+[K_2]-N+4$) *arbitrarily real constants.*

(4) *When* $K_1<0, K_2\geq N$ (*or* $K_1\geq N, K_2<0$), *Problem* P *has* $3N-2K_1-1$ (*or* $3N-2K_2-1$) *solvability conditions, and the solution depends on* $2K_2-N+3$ (*or* $2K_1-N+3$) *arbitrarily real constants.*

(5) *When* $K_1<0, 0\leq K_2<N$ (*or* $0\leq K_1<N, K_2<0$), *Problem* P *has* $4N-2K_1-[K_2+1/2]-1$ (*or* $4N-[K_1+1/2]-2K_2-1$) *solvability conditions, and the solution depends on* $[K_2]+3$ (*or* $[K_1]+3$) *arbitrarily real constants.*

(6) *When* $K_1<0, K_2<0$, *Problem* P *has* $4N-2K_1-2K_2-2$ *solvability conditions, and the solution depends on two arbitrarily real constants.*

2 A priori estimates of solutions of oblique derivative problem for elliptic complex equations of second order

In this section, we first develop some estimates of solutions of Problem Q for elliptic complex systems (1.8).

Theorem 2.1 *Suppose that Condition* C *holds and the four constants* q_2,ε,k_1, k_2 *in* (1.2), (1.3), (1.5) *are small enough. Then any solution* $[U(z),V(z),\,w(z)]$ *of Problem* Q *for* (1.8) *with* $G(z,w,w_z,\overline{w}_z)=0$ *satisfies the estimates*

$$L_1=L(U)=C_\beta[U(z),\bar{D}]+L_{p_0,2}[|U_{\bar{z}}|+|U_z|,\overline{D}],L_2=L(V)\leq M_1,\tag{2.1}$$

$$S=S(w)=C^1_\beta[w(z),\overline{D}]+L_{p_0,2}[|w_{z\bar{z}}|+|w_{zz}|+|\overline{w}_{zz}|,\overline{D}]\leq M_2,\tag{2.2}$$

where $\beta=\min(\alpha,1-2/p_0)$, $p_0(2<p_0\leq p)$, M_1 *and* M_2 *are non-negative constants,* $M_j=M_j(q_0,p_0,\alpha,k^*,K,D), j=1,2$, $k^*=(k_0,k_3,k_4)$, $K=(K_1,K_2)$, *and* q_0, p_0 *are non-negative constants as stated in Condition* C.

Proof Let the solution $[w(z),U(z),V(z)]$ of Problem Q be substituted into the system (1.8), the boundary conditions (1.9), and the relation (1.11). It is clear that (1.8) and (1.9) can be rewritten in the form

$$U_{\bar{z}}-Q_1U_z-A_1U=A, A=Q_2V_z+A_2V+A_3w+A_4,\ V_{\bar{z}}=\overline{U}_z,\tag{2.3}$$

$$\begin{gathered}\mathrm{Re}[\overline{\lambda_1(z)}U(z)]=r_1(z)+h_1(z),\ \mathrm{Re}[\overline{\lambda_2(z)}V(z)]=r_2(z)+h_2(z),\\ r_l(z)=\tau_l(z)-\varepsilon\mathrm{Re}[\beta_l(z)w(z)],\ z\in\Gamma,\ l=1,2,\end{gathered}\tag{2.4}$$

where A and $r_l(l=1,2)$ satisfy the inequalities

$$L_{p_0,2}[A,\overline{D}]\le q_2L_{p_0,2}[V_z,\overline{D}]+L_{p_0}[A_2,\overline{D}]C[V,\overline{D}]$$
$$+L_{p_0,2}[A_3,\overline{D}]C[w,\overline{D}]+L_{p_0,2}[A_4,\overline{D}]\le q_2L_2+k_1L_2+k_2S_1+k_3, \tag{2.5}$$

$$C_\alpha[r_l,\Gamma]\le\varepsilon C_\alpha[\sigma_l,\Gamma]C[w,\Gamma]+C_\alpha[\tau_l,\Gamma]\le\varepsilon k_0S_1+k_4,\ l=1,2, \tag{2.6}$$

in which $S_1=C[w,\overline{D}]$. Moreover from (2.3) and (2.4), we can obtain

$$L_1\le M_3[(q_2+k_1)L_2+k_2S_1+k_3+\varepsilon k_0S_1+2k_4$$
$$=M_3[(q_2+k_1)L_2+(k_2+\varepsilon k_0)S_1+k_3+2k_4], \tag{2.7}$$

where $M_3=M_3(q_0,p_0,\alpha,k_0,K,D)$. Noting that $V(z)$ is a solution of the modified problem for $V_{\bar z}=\overline{U}_z$, we have

$$L_2\le M_3[L_1+\varepsilon k_0S_1+2k_4]. \tag{2.8}$$

In addition, from (1.11), we can derive that

$$S_1=C[w,\overline{D}]\le k_4+M_4[C(U,\overline{D})+C(V,\overline{D})]\le k_4+M_4(L_1+L_2), \tag{2.9}$$

where $M_4=M_4(D)$. Combining (2.7)-(2.9), we can derive that

$$L_2\le M_3\{M_3[(q_2+k_1)L_2+(k_2+\varepsilon k_0)(k_4+M_4(L_1+L_2))$$
$$+k_3+2k_4]+\varepsilon k_0(k_4+M_4(L_1+L_2))+2k_4\}$$
$$\le M_3\{(q_2+k_1)M_3L_2+(k_2+\varepsilon k_0)(1+M_3)M_4(L_1+L_2)$$
$$+k_4(k_2+\varepsilon k_0)(1+M_3)+(k_3+2k_4)(1+M_3)\}. \tag{2.10}$$

Provided that the constants q_2,ε,k_1,k_2 are sufficiently small, for instance, $M_3[(q_2+k_1)M_3+(k_2+\varepsilon k_0)(1+M_3)M_4]<1/2$, we must have

$$L_2\le 2M_3[(k_2+\varepsilon k_0)(1+M_3)M_4L_1+k_4(k_2+\varepsilon k_0)(1+M_3)$$
$$+(k_3+2k_4)(1+M_3)]=M_5L_1+M_6, \tag{2.11}$$

where $M_5=2M_3(k_2+\varepsilon K_0)(1+M_3)M_4$, $M_6=2M_3[k_4(k_2+\varepsilon k_0)(1+M_3)+(k_3+2k_4)(1+M_3)]$. Letting (2.11) and (2.9) be substituted into (2.7), we can obtain

$$L_1\le M_3[(q_2+k_1)(M_5L_1+M_6)+(k_2+\varepsilon k_0)M_4(L_1+L_2)+k_4(k_2+\varepsilon k_0)$$
$$+k_3+2k_4]\le M_3\{[(q_2+k_1)M_5+(k_2+\varepsilon k_0)M_4(1+M_5)]L_1$$
$$+(q_2+k_1)M_6+(k_2+\varepsilon k_0)M_4M_6+k_4(k_2+\varepsilon k_0)+k_3+2k_4\}. \tag{2.12}$$

Moreover if q_2,ε,k_1,k_2 are small enough such that $M_3[(q_2+k_1)M_5+(k_2+\varepsilon k_0)(1+M_5)M_4]<1/2$, then the estimates

$$L_1\le 2M_3[(q_2+k_1)M_6+(k_2+\varepsilon k_0)M_4M_6+k_4(k_2+\varepsilon k_0)+k_3+2k_4]=M_7 \tag{2.13}$$

is concluded, and

$$L_2\le M_5M_7+M_6\le M_1=\max(M_7,M_5M_7+M_6). \tag{2.14}$$

Furthermore, from (1.11) it follows that (2.2) holds.

From Theorem 2.1, we can derive the following result.

Theorem 2.2 *Under the same conditions in Theorem* 2.1, *any solution* $[U(z), V(z), w(z)]$ *of Problem* Q *for* (1.8) *with the condition* $0<\sigma,\tau,\eta<1$ *satisfies the estimates*

$$L_1=L(U)\le M_8k,\ L_2=L(V)\le M_8k, \tag{2.15}$$

$$S=S(w)\le M_9k, \tag{2.16}$$

where $M_j=M_j(q_0,p_0,\alpha,k_0,K,D), j=8,9,$ *and* $k=k_*+k_5,\ k_*=k_3+2k_4,\ k_5=k_0(M_{10}^\sigma+M_{10}^\tau+M_{10}^\eta)$, *herein* M_{10} *is a solution of the following equation* (2.19) *below.*

Proof We substitute the solution $[U(z), V(z,w(z)]$ of Problem Q into the system (1.8), the boundary conditions (1.9) and the relation (1.11). Similarly to the proof of Theorem 2.1, we can obtain the results as in (2.1) and (2.2), namely

$$\begin{aligned}L_1=L(U)\le M_8[k+k_0(t^\sigma+t^\tau+t^\eta)],\\ L_2=L(V)\le M_8[k+k_0(t^\sigma+t^\tau+t^\eta)],\end{aligned} \tag{2.17}$$

$$S=S(w)\le M_9[k+k_0(t^\sigma+t^\tau+t^\eta)], \tag{2.18}$$

in which $k=k_3+2k_4,\ M_j=M_j(q_0,p_0,\alpha,k_0,K,D), j=8,9$. Consider the algebraic equation for $t:$

$$M_9[k_3+k_0(t^\sigma+t^\tau+t^\eta)+2k_4]=t. \tag{2.19}$$

Because $0<\max(\sigma,\tau,\eta)<1$, the equation (2.19) has a solution $t=M_{10}>0$, which is also the maximum of t in $(0,+\infty)$. Thus we have

$$\begin{aligned}L_1=&L(U)\le M_8[k_*+k_0(t^\sigma+t^\tau+t^\eta)]\le M_{10},\\ L_2=&L(V)\le M_8[k_*+k_0(t^\sigma+t^\tau+t^\eta)]\le M_{10},\\ S=&S(w)\le M_9[k_*+k_0(t^\sigma+t^\tau+t^\eta)]\le M_{10}.\end{aligned} \tag{2.20}$$

In order to prove the uniqueness of solutions of Problem Q for (1.8), we need to add the following condition: For any continuously differentiable functions $w_j(z)(j=1,2)$ on $\overline{D}$ and any continuous functions $U(z),V(z)\in W^1_{p_0,2}(D)(2<p_0\le p)$, there is

$$\begin{aligned}&F(z,w_1,w_{1z},\overline{w}_{1z},U_z,V_z)-F(z,w_2,w_{2z},\overline{w}_{2z}U_z,V_z)\\ &=\tilde{Q}_1U_z+\tilde{Q}_2V_z+\tilde{A}_1(w_{1z}-w_{2z})+\tilde{A}_2(\overline{w}_{1z}-\overline{w}_{2z})+\tilde{A}_3(w_1-w_2),\end{aligned} \tag{2.21}$$

where $|\tilde{Q}_j|\le q_j,\ j=1,2,\ \tilde{A}_j\in L_{p_0,2}(\overline{D}),\ j=1,2,3.$

Theorem 2.3 *If Condition* C *and* q_2,ε,k_1,k_2 *in* (1.2), (1.3), (1.5) *are small enough, then the solution* $[w(z), U(z), V(z)]$ *of Problem* Q *for* (1.8) *with* $G(z,w,U,V)=0$ *is unique,*

Proof Denote by $[w_j(z),U_j(z),V_j(z)](j=1,2)$ two solutions of Problem Q for (1.8), and substitute them into (1.8),(1.9) and (1.11), we see that $[w,U,V]=[w_1(z)-w_2(z),U_1(z)-U_2(z),V_1(z)-V_2(z)]$ is a solution of the following homogeneous boundary value problem

$$U_{\bar z}=\tilde{Q}_1U_z+\tilde{Q}_2V_z+\tilde{A}_1U+\tilde{A}_2V+\tilde{A}w,\ V_{\bar z}=U_z,\ z\in D, \tag{2.22}$$

$$\begin{cases} \mathrm{Re}[\overline{\lambda_1(z)}U(z)+\sigma_1(z)w(z)]=h_1(z), \\ \mathrm{Re}[\overline{\lambda_2(z)}V(z)+\sigma_2(z)w(z)]=h_2(z), \end{cases} z\in\Gamma, \tag{2.23}$$

$$\begin{cases} \mathrm{Im}[\overline{\lambda_1(z)}U(z)+\sigma_1(z)w(z)]|_{z=a_j}=0,\ j\in J_1, \\ \mathrm{Im}[\overline{\lambda_2(z)}V(z)+\sigma_2(z)w(z)]|_{z=a_j}=0,\ j\in J_2, \end{cases} \tag{2.24}$$

$$w(z)=w_0-\int_1^z[\frac{U(z)}{z^2}dz-\sum_{m=1}^N\frac{d_m z_m}{z(z-z_m)}dz]+\frac{\overline{V(z)}}{\bar z^2}d\bar z \text{ in } D, \tag{2.25}$$

the coefficients of which satisfy same conditions of (1.8),(1.9) and (1.11), but $k_3=k_4=0$. On the basis of Theorem 2.2, provided q_2,k_1,k_2 and ε are sufficiently small, we can derive that $w(z)=U(z)=V(z)=0$ on $\overline{D}$, i.e. $w_1(z)=w_2(z), U_1(z)=U_2(z), V_1(z)=V_2(z)$ in $\overline{D}$.

3. Solvability of oblique derivative problem for nonlinear elliptic complex equations of second order I

In the following, we use the foregoing estimates of solutions and the Leray-Schauder theorem to prove the solvability of Problem Q for the nonlinear elliptic complex system (1.8).

Theorem 3.1 *Suppose that Problem* Q *for* (1.8) *with* $G(z,w,w_z,\overline{w}_z)$ $(0<\sigma,\tau,\eta<1)$ *satisfy the same conditions in Theorem* 2.2. *Then Problem* Q *is solvable.*

Proof First of all, we assume that $F(z,w,U,V,U_z,V_z), G(z,w,U,V)$ of (1.8) equal to 0 in the neighborhood D^* of the boundary Γ. The equation is denoted by

$$U_{\bar z}=F^*(z,w,U,V,U_z,V_z)+G^*(z,w,U,V),\ V_{\bar z}=\overline{U}_z \text{ in } D. \tag{3.1}$$

Then we consider the system of first order equations with the parameter $t\in[0,1]$, namely

$$U^*_{\bar z}=t[F^*(z,w,U,V,U^*_z,V^*_z)+G^*(z,w,U,V)],\ V^*_{\bar z}=t\overline{U^*}_z. \tag{3.2}$$

Moreover we introduce the Banach space $B=W^1_{p_0,2}(D)\times W^1_{p_0,2}(D)\times C^1(\overline{D})(2<p_0\le p)$. Denote by B_M the set of systems of continuous functions: $\omega=[U(z),V(z),w(z)]$ satisfying the inequalities:

$$\begin{gathered} L(U)=C_\beta[U,\overline{D}]+L_{p_0,2}[|U_{\bar z}|+|U_z|,\overline{D}]<M_{11}, \\ L(V)<M_{11},\ C^1[w(z),\overline{D}]<M_{11}, \end{gathered} \tag{3.3}$$

in which $M_{11}=\max[M_2,M_{10}]+1$, β, M_2, M_{10} are non-negative constants as stated in (2.2) and (2.20). It is evident that B_M is a bounded open set in B.

Next, we only discuss Problem Q for (3.2) and arbitrarily select a system of functions: $\omega=[U(z),V(z),w(z)]\in B_M$. Substitute it into the appropriate positions of (3.2),(1.9) and (1.11), and then consider the boundary value problem (Problem Q) with the parameter $t\in[0,1]$:

$$U^*_{\bar z}=t[F^*(z,w,U,V,U_z,V_z)+G^*(z,w,U,V)],\ V^*_{\bar z}=t\overline{U}_z,\ z\in D, \tag{3.4}$$

$$\begin{cases} \mathrm{Re}[\overline{\lambda_1(z)}U^*(z)+t\varepsilon\beta_1(z)w(z)]=\tau_1(z)+h_1(z), \\ \mathrm{Re}[\overline{\lambda_2(z)}V^*(z)+t\varepsilon\beta_2(z)w(z)]=\tau_2(z)+h_2(z), \end{cases} z\in\Gamma, \tag{3.5}$$

$$\begin{cases} \mathrm{Im}[\overline{\lambda_1(a_j)}U^*(a_j)+t\varepsilon\beta_1(a_j)w(a_j)]=b_{lj},\ j\in J_1, \\ \mathrm{Im}[\overline{\lambda_2(a_j)}V^*(a_j)+t\varepsilon\beta_2(a_j)w(a_j)]=b_{2j},\ j\in J_2, \end{cases} \tag{3.6}$$

$$w^*(z)=w_0-\int_1^z[\frac{U^*(z)}{z^2}-\sum_{m=1}^{N}\frac{d_m z_m}{z(z-z_m)}]dz+\frac{\overline{V^*(z)}}{\overline{z}^2}d\overline{z},\ z\in D, \tag{3.7}$$

where $U(z),V(z),w(z)$ are known functions as stated before. Noting that Problem Q consists of two modified Riemann-Hilbert problems for elliptic complex equations of first order and applying Theorem 2.2.3, Chapter II, [5], we see that there exist the solutions $U^*(z),V^*(z)\in W^1_{p_0}(D)(2<p_0\le p)$. From (3.7), the single-valued function $w^*(z)$ in $\overline{D}$ is determined. Denote by $\omega^*=[U^*(z),V^*(z),w^*(z)]=T(\omega,t)(0\le t\le 1)$ the mapping from ω onto ω^*. According to Theorem 2.2, if $\omega=[U(z),V(z),w(z)]=T(\omega,t)(0\le t\le 1)$, then $\omega=[U(z),V(z),w(z)]$ satisfies the estimates in (2.20), consequently $\omega\in B_M$. Setting $B_0=B_M\times[0,1]$, we shall verify that the mapping $\omega^*=T(\omega,t)(0\le t\le 1)$ satisfies the three conditions of the Leray-Schauder theorem:

(1) When $t=0$, by Theorem 2.2, it is evident that $\omega^*=T(\omega,0)\in B_M$.

(2) As stated before, the solution $\omega=[U(z),V(z),w(z)]$ of the functional equation $\omega=T(\omega,t)(0\le t\le 1)$ satisfies the estimates in (2.20), which shows that $\omega=T(\omega,t)(0\le t\le 1)$ does not have any solution $\omega=[U(z),V(z),w(z)]$ on the boundary $\partial B_M=\overline{B_M}\backslash B_M$.

(3) For every $t\in[0,1]$, $\omega^*=T(\omega,t)$ continuously maps the Banach space B into itself, and is completely continuous in B_M. Besides, for $\omega\in\overline{B_M},T(\omega,t)$ is uniformly continuous with respect to $t\in[0,1]$.

In fact, let us choose any sequence $\omega_n=[U_n(z),V_n(z),w_n(z)](n=1,2,...)$, which belongs to $\overline{B_M}$. By Theorem 2.1, it is not difficult to see that $\omega_n^*=[U_n^*,V_n^*,w_n^*]=T(\omega_n,t)(0\le t\le 1)$ satisfies the estimates

$$L(U_n^*)\le M_{12},\ L(V_n^*)\le M_{12},\ S(w_n^*)\le M_{13}, \tag{3.8}$$

in which $M_j=M_j(q_0,p_0,\alpha,k_0,K,D,M)$, $j=12,13$, $n=1,2,....$ We can select subsequences of $\{U_n^*(z)\},\{V_n^*(z)\}$, $\{w_n^*(z)\}$, which uniformly converge to $U_0^*(z)$, $V_0^*(z),w_0^*(z)$ in $\overline{D}$, and $\{U_{nz}^*\},\{U_{n\overline{z}}^*\}$, $\{V_{nz}^*\},\{V_{n\overline{z}}^*\}$ in D weakly converge to U_{0z}^*, $U_{0\overline{z}}^*$, $V_{0z}^*,V_{0\overline{z}}^*$, respectively. For convenience, the same notations will be used to denote the subsequences. From $\omega_n^*=T(\omega_n,t)$ and $\omega_0^*=T(\omega_0,t)(0\le t\le 1)$, we obtain

$$\begin{aligned} &U_{n\overline{z}}^*-U_{0\overline{z}}^*=t[F(z,w_n,U_n,V_n,U_{nz}^*,V_{nz}^*)-F(z,w_n,U_n,V_n,U_{0z}^*,V_{0z}^*)+c_n],\\ &c_n=F(z,w_n,U_n,V_n,U_{0z}^*,V_{0z}^*)+G(z,w_n,U_n,V_n)-F(z,w_0,U_0,V_0,\\ &U_{0z}^*,V_{0z}^*)-G(z,w_0,U_0,V_0),\ V_{n\overline{z}}^*-V_{0\overline{z}}^*=t[\overline{U^*}_{nz}-\overline{U^*}_{0z}],\ z\in D, \end{aligned} \tag{3.9}$$

$$\begin{cases} \mathrm{Re}[\overline{\lambda_1(z)}(U_n^*-U_0^*)+t\varepsilon\beta_1(z)(w_n-w_0)]=h_1(z), \\ \mathrm{Re}[\overline{\lambda_2(z)}(V_n^*-V_0^*)+t\varepsilon\beta_2(z)(w_n-w_0)]=h_2(z), \end{cases} z\in\Gamma, \tag{3.10}$$

$$\begin{cases} \mathrm{Im}[\overline{\lambda_1(a_j)}[U_n^*(a_j)-U_0^*(a_j)]+t\varepsilon\beta_1(a_j)[w_n(a_j)-w_0(a_j)]=0,\ j\in J_1, \\ \mathrm{Im}[\overline{\lambda_2(a_j)}[V_n^*(a_j)-V_0^*(a_j)]+t\varepsilon\beta_2(a_j)[w_n(a_j)-w_0(a_j)]]=0,\ j\in J_2, \end{cases} \tag{3.11}$$

$$w_n^*(z)-w_0^*(z)=-\int_1^z[\frac{U_n^*(z)-U_0^*(z)}{z^2}-\sum_{m=1}^N\frac{d_m z_m}{z(z-z_m)}]dz+\frac{\overline{V_n^*(z)}-\overline{V_0^*(z)}}{\overline{z}^2}d\bar{z}. \tag{3.12}$$

By using the way in (2.4.18), Chapter II, [6], we can prove that $L_{p_0}[c_n,\overline{D}]\to 0$ for $n\to\infty$, since when $n\to\infty$, $\{c_n\}$ converges to 0 for almost every point $z\in D$. Because of the completeness of the Banach space B, there exists a system of functions $\omega_0=[U_0(z),V_0(z),w_0(z)]\in B$, such that $L(U_n-U_0)\to 0$, $L(V_n-V_0)\to 0$ and $S(w_n-w_0)\to 0$ as $m\to\infty$. This shows the complete continuity of $\omega^*=T(\omega,t)(0\le t\le 1)$ on $\overline{B_M}$. By a similar method, we can also prove that $\omega^*=T(\omega,t)(0\le t\le 1)$ continuously maps $\overline{B_M}$ into B, and $T(\omega,t)$ is uniformly continuous with respect to $t\in[0,1]$ for $\omega\in\overline{B_M}$.

Hence by the Leray-Schauder theorem, we see that the functional equation $\omega=T(\omega,t)(0\le t\le 1)$ with $t=1$, i.e. Problem Q for (1.8) has a solution.

Finally we can cancel the assumption that $F(z,w,U,V,U_z,V_z),G(z,w,U,V)$ of (1.8) equal to 0 in the neighborhood D^* of the boundary Γ by the method as stated in the proof of Theorem 4.7, Chapter II, [3].

4. Solvability of oblique derivative problem for nonlinear elliptic complex equations of second order II

Theorem 4.1 *Let the complex equation* (1.1) *satisfy Condition* C *and the constants* q_2,ε,k_1,k_2 *be small enough. Then when* $\min(\sigma,\tau,\eta)>1$, *Problem* Q *for* (1.8) *has a continuous solution* $[U(z),V(z),w(z)]$, *provided that*

$$M_{14}=L_{p_0,2}[A_4,\bar{D}]+\sum_{l=1}^2 C_\alpha[\tau_l,\Gamma]+\sum_{j\in J_l,l=1,2}|b_{lj}| \tag{4.1}$$

is sufficiently small.

Proof We shall use the Schauder fixed-point theorem to prove the solvability of Problem Q. In this case, due to M_{14} in (4.1) is small enough, from

$$M_9[k_3+k_0(t^\sigma+t^\tau+t^\eta)+2k_4]=t, \tag{4.2}$$

a solution $t=M_{15}>0$ can be solved, which is also a maximum. Now, we introduce a closed, bounded and convex subset $\omega=\{w(z)\}$ of the Banach space B_M, whose elements are satisfied the estimate

$$B_M=\{w(z)\,|\,C^1[w,\bar{D}]+L_{p_0,2}[|w_{z\bar{z}}|+|w_{zz}|+|\overline{w}_{zz}|,\overline{D}]\le M_{15}\}, \tag{4.3}$$

in which p_0 is stated as in (2.2). We choose an arbitrary function $W(z)\in B_M$ and substitute it into the proper positions of w in $F(z,w,w_z,\overline{w}_z,w_{zz},\overline{w}_{zz})+G(z,w,w_z,\overline{w}_z)$ and obtain the equation

$$w_{z\bar{z}}=\tilde{F}(z,w,w_z,\overline{w}_z,W,W_z,\overline{W}_z,w_{zz},\overline{w}_{zz})+G(z,W,W_z,\overline{W}_z), \tag{4.4}$$

in which

$$\tilde{F} = \tilde{Q}_1 w_{zz} + \tilde{Q}_2 \overline{w}_{zz} + \tilde{A}_1 w_z + \tilde{A}_2 \overline{w}_z + \tilde{A}_3 w + \tilde{A}_4,$$

$$\tilde{Q}_j = Q_j(z, W, W_z, \overline{W}_z, w_{zz}, \overline{w}_{zz}), j=1,2, \tilde{A}_j = A_j(z, W, W_z, \overline{W}_z), j=1,...,4.$$

Similarly to Theorems 3.1 and 3.2, a solution $w(z) \in B_M$ of Problem Q for the equation (4.4) can be found, and the solution of Problem Q for (4.4) is unique. Denote by $w = S[W(z)]$ the mapping from $W(z)$ to $w(z)$. Moveover, we can derive that

$$\begin{aligned} Sw &\le M_9\{L_{p_0,2}[A_4, \bar{D}] + \sum_{l=1}^{2} C_\alpha[\tau_l, \Gamma] + \sum_{j\in J_l, l=1,2} |b_{lj}| \\ &+ L_{p_0,2}[G, \bar{D}]\} \le M_9\{k_3 + 2k_4 + k_0[C[w_z, \bar{D}]^\sigma + C[\bar{w}_z, \bar{D}]^\tau \\ &+ C[w, \bar{D}]^\eta]\} \le M_9\{k^* + k_0(M_{15}^\sigma + M_{15}^\tau + M_{15}^\eta)\} = M_{15}, \end{aligned} \quad (4.5)$$

in which $k^* = k_3 + 2k_4$. This shows that $w = S(W)$ maps B_M onto a compact subset in itself. Next, we verify that S in B_M is a continuous operator. In fact, arbitrarily select a sequence $\{W_n(z)\}$ in B_M, such that $C^1[W_n - W_0, \bar{D}] \to 0$ as $n \to \infty$. Similarly to Lemma 2.4.2, Chapter II, [6], we can prove that

$$L_{p_0,2}[A_j(z, W_n, W_{nz}, \overline{W}_{nz}) - A_j(z, W_0, W_{0z}, \overline{W}_{0z}), \bar{D}] \to 0 \text{ as } n \to \infty, j=1,...,4. \quad (4.6)$$

Moreover, from $w_n = S[W_n]$, $W_0 = S[W_0]$, it is clear that $w_n - w_0$ is a solution of Problem B_M for the following equation

$$\begin{aligned} &(w_n - w_0)_{z\bar{z}} = \tilde{F}(z, w_n, w_{nz}, \overline{w}_{nz}, W_n, W_{nz}, \overline{W}_{nz}, w_{nzz}, \overline{w}_{nzz}) \\ &-\tilde{F}(z, w_0, w_{0z}, \overline{w}_{0z}, W_0, W_{0z}, \overline{W}_{0z}, w_{0zz}, \overline{w}_{0zz}) \\ &+G(z, W_n, W_{nz}, \overline{W}_{nzz}) - G(z, W_0, W_{0z}, \overline{W}_{0z}), \ \ z \in D, \end{aligned} \quad (4.7)$$

$$\begin{aligned} &\begin{cases} \operatorname{Re}[\overline{\lambda_1(z)}(\overline{w_n(z)} - \overline{w_0(z)})_z] + \varepsilon\beta_1(z)(w_n(z) - w_0(z))] = h_1(z), \\ \operatorname{Re}[\overline{\lambda_2(z)}(w_n(z) - w_0(z))_z] + \varepsilon\beta_2(z)(w_n(z) - w_0(z))] = h_2(z), \end{cases} z \in \Gamma, \\ &\begin{cases} \operatorname{Im}[\overline{\lambda_1(a_j)}(\overline{w_{nz}(a_j)} - \overline{w_{0z}(a_j)}] + \varepsilon\beta_1(a_j)(w_n(a_j) - w_0(a_j))] = 0, j \in J_1, \\ \operatorname{Im}[\overline{\lambda_2(a_j)}(w_{nz}(a_j) - w_{0z}(a_j)) + \varepsilon\beta_2(a_j)(w_n(a_j) - w_0(a_j))] = 0, j \in J_2. \end{cases} \end{aligned} \quad (4.8)$$

By means of the method in the proof of Theorem 2.2, we can obtain the estimate

$$\begin{aligned} S[w_n - w_0] \le M_9 L_{p_0,2}[&|A_4(z, W_n, W_{nz}, \overline{W}_{nz}) - A_4(z, W_0, W_{0z}, \overline{W}_{0z})| \\ &+ |G(z, W_n, W_{nz}, \overline{W}_{nz}) - G(z, W_0, W_{0z}, \overline{W}_{0z})|, \overline{D}]. \end{aligned}$$

Hence $C^1[w_n - w_0, \bar{D}] \to 0$ as $n \to \infty$. On the basis of the Schauder fixed-point theorem, there exists a function $w(z) \in C^1(\bar{D})$ such that $w(z) = S[w(z)]$, and from Theorem 2.2, we can see that $w(z) \in B = C^1(\bar{D}) \cap W^2_{p_0,2}(D)$, and $w(z)$ is a solution of Problem Q for the equation (2.1) with $\min(\sigma, \tau, \eta) > 1$.

From the above theorem, the result in Theorem 1.1 can be derived.

Proof of Theorem 1.1 We first discuss the case: $0 \le K_l < N$ $(l = 1, 2)$. Let the solution $[w(z), U(z), V(z)]$ of Problem Q for the complex system (1.8) be substituted into (1.9)–(1.11). The functions $h_l(z)(l=1,2)$ and the complex constants d_m $(m=1,...,N)$ are then determined. If the functions and the constants are equal to zero, namely the following equalities hold:

$$h_l(z)=h_{lj}=0,\ j=1,...,N-K_l,\ \text{when } 0\le K_l<N, l=1,2, \tag{4.9}$$

and

$$d_m=\mathrm{Re}d_m+i\mathrm{Im}d_m=0,\ m=1,...,N, \tag{4.10}$$

then $w_z=U(z)$, $\overline{w}_z=V(z)$, $w(z)$ is a solution of Problem P for (1.1). Hence when $0\le K_l<N(l=1,2)$, Problem P for (1.1) has $4N-K_1-K_2$ solvability conditions. In addition, the real constants $b_{lj}(j=N-K_l+1,...,N+1, l=1,2)$ in (1.9) and the complex constant w_0 in (1.11) may be arbitrary, this shows that the general solution of Problem P $(0\le K_l<N, l=1,2)$ is dependent on K_1+K_2+4 arbitrary real constants. Thus (2) is proved.

Similarly, other cases can be obtained.

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