Initial-oblique derivative boundary value problem for nonlinear parabolic equations of second order

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Abstract. In this article, we discuss that an initial-oblique derivative boundary value problem for nonlinear uniformly parabolic complex equation of second order

$$A_0u_{z\bar{z}}-\mathrm{Re}[Qu_{zz}+A_1u_z]-\hat{A}_2u-u_t=A_3+G(z,t,u,u_z)\ \text{ in } G,$$

in a multiply connected domain, the above boundary value problem will be called Problem O. If the above complex equation satisfies the conditions similar to Condition C′ and (1.12), and the boundary conditions satisfy the conditions similar to (1.4)-(1.7) and (1.11), then we can obtain some solvability results of Problem O in G.

Key Words: Initial-oblique derivative problem, nonlinear parabolic complex equations, multiply connected domains

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1. Formulation of initial-oblique derivative problems for second order parabolic complex equations

Let D be an $(N+1)$-connected bounded domain in the $z = x + \mathrm{i}y$ plane $\mathbb{C}$ with the boundary $\Gamma = \sum_{j=0}^{N} \Gamma_j \in C^2_\mu(0 < \mu < 1)$. Without loss of generality, we may consider that D is a circular domain in $|z| < 1$ with the boundary $\Gamma = \sum_{j=0}^{N} \Gamma_j$, where $\Gamma_j = \{|z - z_j| = \gamma_j\}$, $j = 0, 1, \cdots, N$, $\Gamma_0 = \Gamma_{N+1} = \{|z| = 1\}$ and $z = 0 \in D$. Denote $G = D \times I$, in which $I = \{0 < t \le T\}$. Here T is a positive constant, and $\partial G = \partial G_1 \cup \partial G_2$ is the parabolic boundary of G, where $\partial G_1, \partial G_2$ are the bottom $\{z \in D, t = 0\}$ and the lateral boundary $\{z \in \Gamma, t \in \bar{I}\}$ of the domain G respectively.

We consider the nonlinear nondivergent parabolic equation of second order

$$\Phi(x, y, t, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) - u_t = 0 \text{ in } G, \tag{1.1}$$

where Φ is a real-valued function of $x, y, t\,(\in G), u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}(\in \mathbb{R})$. Under certain conditions, the equation (1.1) can be reduced to the complex form

$$A_0u_{z\bar{z}} - \mathrm{Re}[Qu_{zz} + A_1u_z] - \hat{A}_2u - u_t = A_3, \tag{1.2}$$

where $z = x + \mathrm{i}y$, $\Phi = \Psi(z, t, u, u_z, u_{zz}, u_{z\bar{z}})$, and

$$\begin{aligned}
&A_0=\int_0^1 \Psi_{\tau u_{z\bar z}}(z,t,u,u_z,\tau u_{zz},\tau u_{z\bar z})\mathrm{d}\tau = A_0(z,t,u,u_z,u_{zz},u_{z\bar z}),\\
&Q=-2\int_0^1 \Psi_{\tau u_{zz}}(z,t,u,u_z,\tau u_{zz},\tau u_{z\bar z})\mathrm{d}\tau = Q(z,t,u,u_z,u_{zz},u_{z\bar z}),\\
&A_1=-2\int_0^1 \Psi_{\tau u_z}(z,t,u,\tau u_z,0,0)\mathrm{d}\tau = A_1(z,t,u,u_z),\\
&\hat{A}_2=-\int_0^1 \Psi_{\tau u}(z,t,\tau u,0,0,0)\mathrm{d}\tau = A_2(z,t,u)+|u|^\sigma,\\
&A_3=-\Psi(z,t,0,0,0,0)=A_3(z,t),
\end{aligned}\tag{1.3}$$

where σ is a positive constant (see [4]).

Suppose that the equation (1.2) satisfies the following conditions, namely

Condition C. (1) $A_0(z,t,u,u_z,u_{zz},u_{z\bar z})$, $Q(z,t,u,u_z,u_{zz},u_{z\bar z}), A_1(z,t,u,u_z)$, $A_2(z,t,u)$, $A_3(z,t)$ are measurable for any continuously differentiable function $u(z,t)\in C^{1,0}(\overline{G})$ and measurable functions $u_{zz},u_{z\bar z}\in L_2(G^*)$ and satisfy the conditions

$$0<\delta\le A_0\le \delta^{-1}, \tag{1.4}$$

$$|A_j|\le k_0,\ j=1,2,\ L_p[A_3,\overline{G}]\le k_1,\ p>4, \tag{1.5}$$

where G^* is any closed subset in the domain G.

(2) The above functions with respect to $u\in\mathbb{R}, u_z\in\mathbb{C}$ are continuous for almost every point $(z,t)\in G$ and $u_{zz}\in\mathbb{C}$, $u_{z\bar z}\in\mathbb{R}$.

(3) For almost every point $(z,t)\in G$ and $u\in\mathbb{R}, u_z, U^j\in\mathbb{C}, V^j\in\mathbb{R}, j=1,2,$ there is

$$\begin{aligned}
&\Psi(z,t,u,u_z,U^1,V^1)-\Psi(z,t,u,u_z,U^2,V^2)\\
&=\tilde{A}_0(V^1-V^2)-\mathrm{Re}[\tilde{Q}(U^1-U^2)],\ \delta<\tilde{A}_0\le\delta^{-1},
\end{aligned}\tag{1.6}$$

$$\sup_G(\tilde{A}_0^2+|\tilde{Q}|^2)/\inf_G \tilde{A}_0^2\le q<4/3. \tag{1.7}$$

In (1.4)-(1.7), $\delta\,(>0)$, $q\,(\ge 1)$, k_0, k_1, $p\,(>4)$ are non-negative constants. For instance the nonlinear parabolic complex equation

$$\begin{aligned}
&u_{z\bar z}=G(z,t,u,u_z,u_{zz})+(1+|u|^4)u+u_t,\\
&G(z,t,u,u_z,u_{zz})=\begin{cases} u_{zz}^2/8 \ \text{ for } |u_{zz}|\le 1,\\ u_{zz}^{-2}/8 \ \text{ for } |u_{zz}|>1,\end{cases}
\end{aligned}$$

satisfies Condition C. In this article, the notations are the same as in References [1-8].

Now we explain the derivation of 3/4 in the condition (1.7). Let $\Lambda=r\inf_G A_0^2>0$, thus $\inf_G \tilde{A}^2=\inf_G A_0^2/\Lambda=\inf_G A_0^2/(r\inf_G A_0^2)=1/r$. By the requirement below, we need the inequality

$$\eta=\sup_G[(\tilde{A}_0-1)^2+|\tilde{Q}|^2]<\frac{1}{4},\ \text{ i.e. } \sup_G[\tilde{A}_0^2+|\tilde{Q}|^2-2\tilde{A}_0]<\frac{1}{4}-1,$$

so it is sufficient that

$$\frac{\sup_G[A_0^2+|Q|^2]}{r^2\inf_G A_0^2}<\frac{2}{r}-\frac{3}{4},\ \text{ i.e. } \frac{\sup_G[A_0^2+|Q|^2]}{\inf_G A_0^2}<2r-\frac{3}{4}r^2=f(r).$$

We can find the maximum of the function $f(r)=2r-(3r^2)/4$ on $(0,\infty)$, due to $f'(r)=2-(3r)/2=0$. It is easy to see that $f(r)$ takes its maximum on $(0,\infty)$ at the point $r=4/3$, and then $f(4/3)=2(4/3)-(3/4)(4/3)^2=4/3$, leading to the inequality (1.7). (see [2,4])

In this article, we mainly discuss the nonlinear parabolic equation of second order

$$A_0u_{z\bar{z}}-\mathrm{Re}[Qu_{zz}+A_1u_z]-\hat{A}_2u-u_t=A_3+F(z,t,u,u_z),\tag{1.8}$$

satisfying Condition C′, in which the coefficients $A_j(j=0,1,2,3),Q$ of equation (1.8) satisfy the conditions (1.4)–(1.7) and $F(z,t,u,u_z)$ satisfies the the condition:

(4) $$|F(z,u,u_z)|\le B_1(z)|u_z|^\eta+B_2(z)|u|^\tau,\ |B_j|\le k_0,\ j=1,2,\tag{1.9}$$

for positive constants η,τ,k_0. We can see that $F(z,t,u,u_z)$ implies the nonlinear items.

Problem O. The so-called initial-oblique derivative boundary value problem for the equation (1.8) is to find a continuous solution $u(z,t)\in C^{1,0}(\overline{G})$ of (1.8) in $\overline{G}$ satisfying the initial-boundary conditions

$$\begin{cases} u(z,0)=g(z)\ \text{ on } \partial G_1=D,\\ \dfrac{\partial u}{\partial\nu}+b_1(z,t)u=b_2(z,t)\ \text{ on } \partial G_2,\ \text{ i.e.}\\ 2\mathrm{Re}[\overline{\lambda(z,t)}u_z]+b_1(z,t)u=b_2(z,t)\ \text{ on } \partial G_2,\end{cases}\tag{1.10}$$

where ν is the unit vector at every point on ∂G_2. There is no harm in assuming that ν is parallel to the plane $t=0$. In addition, $g(z),b_j(z,t)(j=1,2)$ and $\lambda(z,t)=\cos(\nu,x)-\mathrm{i}\cos(\nu,y)$ are known functions satisfying the conditions

$$\begin{cases} C_\alpha^2[g,\partial\Gamma_1]\le k_2,\ \dfrac{\partial g}{\partial\nu}+b_1(z,0)g=b_2(z,0)\ \text{ on } \partial G_1\times\{t=0\},\\ C^{1,0}_{\alpha,\alpha/2}[\eta,\partial G_2]=C^{0,0}_{\alpha,\alpha/2}[\eta,\partial G_2]+C^{0,0}_{\alpha,\alpha/2}[\eta_z,\partial G_2]\le k_0,\ \eta=\{b_1,\lambda\},\\ C^{2,1}_{\alpha,\alpha/2}[b_2,\partial G_2]\le k_3,\ b_1(z,t)\ge 0,\ \cos(\nu,n)>0 \text{ on } \partial G_2,\end{cases}\tag{1.11}$$

in which n is the unit outward normal vector at every point on ∂G_2, $\alpha(1/2<\alpha<1),k_0,k_2,k_3$ are non-negative constants. The above initial-boundary value problem is the initial-oblique derivative boundary value problem (Problem O). In particular, Problem O with the condition $\nu=n,a_1(z,t)=1,a_2(z,t)=0$ on ∂G_2 is the so-called initial-Neumann boundary value Problem, which will be called Problem N. Problem O for (1.2) with $A_3(z,t)=0$ and $g(z)=0$, $b_2(z,t)=0$ is called Problem O_0.

In order to discuss the uniqueness of solutions of Problem O for the equation (1.2), we add the condition: For any $u^j \in \mathbb{R}, u^j_z (j=1,2), U \in \mathbb{C}, V \in \mathbb{R}$, there is

$$\begin{aligned}&\Psi(z,t,u^1,u^1_z,U,V) - \Psi(z,t,u^2,u^2_z,U,V)\\ &= \tilde{A}_0(u_1-u_2)_{z\bar{z}} - \mathrm{Re}[\tilde{Q}u_{zz} + \tilde{A}_1(u^1-u^2)_z + \tilde{A}_2(u^1-u^2)] \text{ on } \partial G_2,\end{aligned} \tag{1.12}$$

where $\tilde{A}_)$, $\tilde{Q}$ satisfy (1.7) and $\tilde{A}_j (j=1,2)$ satisfy

$$|\tilde{A}_j| < \infty \text{ in } \bar{G},\ j=1,2. \tag{1.13}$$

Theorem 1.1. *Suppose that the equation* (1.2) *satisfies Condition* C *and* (1.12). *Then the solution* $u(z,t)$ *of Problem* O *for* (1.2) *is unique. Moreover the homogeneous Problem* O *(Problem* O_0*) of equation* (1.2) *with* $A_3 = 0$ *only has the trivial solution.*

Proof. Let u_j $(j=1,2)$ be two solutions of Problem O for (1.2). It is easy to see that $u = u_1(z,t) - u_2(z,t)$ is a solution of the following initial-boundary value problem

$$\tilde{A}_0 u_{z\bar{z}} - \mathrm{Re}[\tilde{Q}u_{zz} + \tilde{A}_2 u_z] - \tilde{A}_3 u - u_t = 0 \text{ in } G, \tag{1.14}$$

$$\begin{cases} u(z,0) = 0 \text{ on } D,\\ \dfrac{\partial u}{\partial \nu} + b_1(z,t)u = 0 \text{ on } \partial G_2,\end{cases} \tag{1.15}$$

where

$$\begin{cases} \tilde{A}_0 = \displaystyle\int_0^1 \Psi_s(z,t,v,p,q,s)\mathrm{d}\tau,\ s = u_{2z\bar{z}} + \tau(u_1-u_2)_{z\bar{z}},\ q = u_{2zz} + \tau(u_1-u_2)_{zz},\\ \tilde{Q} = -2\displaystyle\int_0^1 \Psi_q(z,t,v,p,q,s)\mathrm{d}\tau,\ p = u_{2z} + \tau(u_1-u_2)_z,\ v = u_2 + \tau(u_1-u_2),\\ \tilde{A}_1 = -2\displaystyle\int_0^1 \Psi_p(z,t,v,p,q,s)\mathrm{d}\tau,\ \tilde{A}_2 = -\displaystyle\int_0^1 \Psi_v(z,t,v,p,q,s)\mathrm{d}\tau.\end{cases} \tag{1.16}$$

Introducing a transformation $v = v(z,t) = u\mathrm{e}^{-Bt}$, where B is an undetermined real constant, the complex equation (1.14) and the initial-boundary condition (1.15) can be reduced to the form

$$\tilde{A}_0 v_{z\bar{z}} - \mathrm{Re}[\tilde{Q}V_{zz} + \tilde{A}_1 v_z] - (\tilde{A}_2 + B)v - v_t = 0, \tag{1.17}$$

$$\begin{cases} v(z,0) = 0 \text{ in } D,\\ \dfrac{\partial v}{\partial \nu} + b_1(z,t)v = 0 \text{ on } \partial G_2.\end{cases} \tag{1.18}$$

Let the above equation be multiplied by v, thus an equation of v^2

$$\begin{aligned}&\frac{1}{2}[\tilde{A}_0(v^2)_{z\bar{z}} - \mathrm{Re}[\tilde{Q}(v^2)_{zz} - (v^2)_t]\\ &= \tilde{A}_0|v_z|^2 - \mathrm{Re}[\tilde{Q}(v_z)^2 + \frac{1}{2}\tilde{A}_1\mathrm{Re}(v^2)_z] + (\tilde{A}_2 + B)v^2\end{aligned} \tag{1.19}$$

can be obtained. If the maximum of v^2 occurs at an inner point $P_0 \in G$ with $|v(P_0)|^2 \neq 0$, then in a neighborhood of P_0, the right hand side of (1.19) $\geq [B-k_0]v^2$. Moreover, we choose the constant B such that $B > k_0$. By using the maximum principle (see [3,4]), the function v^2 can not take the positive maximum in G. If v^2 takes the positive maximum at a point $P_0 \in \partial G_2$, then we have

$$\left[\frac{1}{2}\frac{\partial v^2}{\partial \nu}+b_1(z,t)v^2\right]\bigg|_{P=P_0}>0. \tag{1.20}$$

This contradicts (1.18). Hence we derive that $u=0$, i.e. $u_1-u_2=0$ in $\overline{G}$. Similarly we can prove the other part in this theorem.

2. A prior estimate of solutions of the initial-oblique derivative problem of second order parabolic complex equations

Theorem 2.1. *If the equation* (1.2) *satisfies condition* C, *then the solution* $u(z,t)$ *of Problem* O *for* (1.2) *satisfies the estimate*

$$\hat C^{1,0}_{\beta,\beta/2}[u,\overline{G}]=C^{1,0}_{\beta,\beta/2}[|u|^{\sigma+1},\overline{G}]\leq M_1,\ ||u||_{W_2^{2,1}(G)}\leq M_2, \tag{2.1}$$

where $\beta\,(0<\beta\leq\alpha)$, $k=k(k_0,k_1,k_2,k_3)$, $M_j=M_j(\delta,q,p,\beta,k,G)$ $(j=1,2)$ *are non-negative constants only dependent on* δ,q,p,β,k,G.

Proof. We shall prove that the following estimate holds

$$\hat C^{1,0}[u,\bar G]=C^{1,0}[|u|^{\sigma+1},\overline{G}]\leq M_3=M_3(\delta,q,p,\beta,k,G). \tag{2.2}$$

If (2.2) is not true, then there exists a sequence of parabolic equations

$$A_0^m u_{z\bar z}-\mathrm{Re}[Q^m u_{zz}+A_1^m u_z]-\hat A_2^m u-u_t=A_3^m \text{ in } G, \tag{2.3}$$

and a sequence of initial-boundary conditions

$$\begin{cases} u(z,0)=g^m(z) \text{ on } D,\\ \dfrac{\partial u}{\partial \nu}+b_1^m u=b_2^m \text{ on } \partial G_2, \end{cases} \tag{2.4}$$

with $\{A_0^m\}$, $\{Q^m\}$, $\{A_1^m\}$, $\{\hat A_2^m\}$, $\{A_3^m\}$ in G satisfying Condition C and g^m, b_1^m, b_2^m satisfying (1.11), where $\{A_0^m\}$, $\{Q^m\}$, $\{A_1^m\}$, $\{\hat A_2^m\}$, $\{A_3^m\}$ in G weakly converge to A_0^0, Q^0, A_1^0, $\hat A_2^0$, A_3^0 and $\{g^m(z)\}$, $b_1^m(z,t)\}$, $b_2^m(z,t)\}$ in $D,\partial G_2$ uniformly converge to $g^0(z)$, $b_1^0(z,t)$, $b_2^0(z,t)$ respectively, and the initial-boundary value problem (2.3)–(2.4) have the solution $u^m(z,t)\in\hat C^{1,0}(\overline{G})\,(m=1,2,\cdots)$ such that $\hat C^{1,0}[u^m,\overline{G}]=H_m\to\infty$ as $m\to\infty$. There is no harm in assuming that $H_m\geq\max[k_1,k_2,k_3,1]$. Let $U^m=u^m/H_m$, it is easy to see that U^m satisfies the complex equation and initial-boundary conditions

$$\begin{cases} A_0^m U_{z\bar z}^m - \mathrm{Re}[Q^m U_{zz}^m + A_1^m U_z^m] - \hat A_2^m U^m - U_t^m = A_3^m/H_m \text{ in } G, \\ U^m(z,0) = g^m(z)/H_m \text{ on } D, \\ \dfrac{\partial U^m}{\partial \nu} + b_1^m U^m = b_2^m/H_m \text{ on } \partial G_2. \end{cases} \tag{2.5}$$

We can see that the some coefficients in the above equation and boundary conditions satisfy the condition C and

$$|u^{(m)}|^{\sigma+1}/H_m \le 1,\ L_p[A_3^{(m)}/H_m, \overline{G}] \le 1,$$
$$C_\alpha[g^{(m)}(z)/H_m, D] \le 1,\ |b_2^{(m)}/H_m| \le 1.$$

Hence by Theorem 5.3.1, [7], we can obtain the estimates

$$\hat C^{1,0}_{\beta,\beta/2}[u^m, \overline{G}] \le M_4,\ ||u^m||_{W_2^{2,1}(G)} \le M_5, \tag{2.6}$$

in which $\beta\,(0<\beta\le\alpha), M_j = M_j(\delta, q, p, \beta, k, G)\ (j=4,5)$ are non-negative constants. Thus from $\{U^m\}, \{U_z^m\}$ we can select the subsequences $\{U^{m_k}\}, \{U_z^{m_k}\}$, such that they uniformly converge to U^0, U_z^0 in $\overline{G}$ and $\{U_{z\bar z}^{m_k}\}$, $\{U_{zz}^{m_k}\}$, $\{U_t^{m_k}\}$ weakly converge to $U_{z\bar z}^0, U_{zz}^0, U_t^0$ in G respectively, and U^0 is a solution of the following initial-boundary value problem

$$\begin{cases} A_0^0 U_{z\bar z}^0 - \mathrm{Re}[Q^0 U_{zz}^0 + A_1^0 U_{zz}^0 + \hat A_2^0 U^0] - U_t^0 = 0 \text{ in } G, \\ U^0(z,0) = 0 \text{ on } D, \\ \dfrac{\partial U_0}{\partial \nu} + b_1^0 U^0 = 0 \text{ on } \partial G_2. \end{cases} \tag{2.7}$$

From Theorem 1.1, we see that $U^0 = 0$. However, from $\hat C^{1,0}[U^m, \bar G] = 1$, there exists a point $(z^*, t^*) \in \bar G$, such that $|U^0(z^*, t^*)| + |U_z^0(z^*, t^*)| > 0$. This contradiction shows that the estimate (2.2) is true. Moreover, by using the method from (2.2) to (2.6), two estimates in (2.1) can be derived.

Theorem 2.2. *Suppose that Condition* C$'$ *holds. Then any solution $u(z,t)$ of Problem* O *for* (1.8) *satisfies the estimates*

$$\hat C^{1,0}_{\beta,\beta/2}[u, \bar G] = C^{1,0}_{\beta,\beta/2}[|u^{\sigma+1}, \overline{G}] \le M_6 k',\ ||u||_{W_2^{2,1}(G)} \le M_7 k', \tag{2.8}$$

where $\beta\,(0<\beta\le\alpha)$, $k' = k_1 + k_2 + k_3 + k_0(|u_z|^\eta + |u|^\tau)$, $M_j = M_j(\delta, q, p, \beta, k_0, G)$ $(j = 6, 7)$ *are non-negative constants.*

Proof. If $k' = 0$, i.e. $k_0 = k_1 = k_2 = k_3 = 0$, from Theorem 1.1, it follows that $u(z) = 0$ in $\overline{G}$. If $k' > 0$, it is easy to see that $U(z) = u(z)/k'$ satisfies the complex equation and boundary conditions

$$A_0 U_{z\bar z} - \mathrm{Re}[Q U_{zz} + A_1 U_z] - \hat A_2 U - U_t = [A_3 + F(z,t,u,u_z)]/k', \tag{2.9}$$

and

$$\begin{cases} U(z,0)=\dfrac{g(z)}{k^*}, \ \ z\in D, \\ \dfrac{\partial U}{\partial \nu}+b_1(z,t)U=\dfrac{b_2(z,t)}{k^*}, \ (z,t)\in \partial G_2. \end{cases} \tag{2.10}$$

Noting that

$$L_p[A_3(z,t)/k',\overline{G}]\le 1, \ C^1_\alpha[g/k',D]\le 1, \ C^{1,0}_{\alpha,\alpha/2}[b_2/k',\partial G_2]\le 1,$$

and according to the proof of Theorem 2.1, we have

$$\hat{C}^{1,0}_{\beta,\beta/2}[U,\overline{G}]\le M_6, \ ||U||_{W^{2,1}_2(G)}\le M_7. \tag{2.11}$$

From the above estimates, it immediately follows that two estimates in (2.8) hold.

3. Solvability of the initial-oblique derivative problem of second order parabolic complex equations

We consider the complex equation (1.8) namely the equation

$$\begin{aligned} A_0u_{z\bar{z}}-\mathrm{Re}[Qu_{zz}]-u_t=f(z,t,u,u_z), \ f(z,t,u,u_z)= \\ =\mathrm{Re}[Qu_{zz}+A_1u_z]+\hat{A}_2u+A_3+F(z,t,u,u_z) \ \text{ in } G, \end{aligned} \tag{3.1}$$

in which $A_0=A_0(z,t,u,u_z,u_{zz}), Q=Q(z,t,u,u_z,u_{zz}), \ A_1=A_1(z,t,u,u_z), \hat{A}_2=A_2(z,t,u)+|u|^\sigma, A_3=A_3(z,t)$.

Theorem 3.1. *Suppose that equation* (1.8) *satisfies Condition* C$'$ *and* (1.12).

(1) *When* $0<\eta,\tau<1$, *Problem* O *for* (1.8) *has a solution* $u(z,t)\in C^{1,0}(\overline{G})$.

(2) *When* $\min(\eta,\tau)>1$, *Problem* O *for* (1.8) *has a solution* $u(z,t)\in C^{1,0}(\overline{G})$, *provided that*

$$M_8=L_p[A_3,\overline{G}]+C^2_\alpha[g,\overline{D}]+C^{2,1}_{\alpha,\alpha/2}[b_2,\partial G_2] \tag{3.2}$$

is small enough.

(3) When $F(z,t,u,u_z)$ in (1.8) possesses the form

$$F(z,u,u_z)=\mathrm{Re}B_1u_z+B_2|u|^\tau \ \text{ in } D \tag{3.3}$$

in which $0<\tau<\infty$, $L_p[B_j,\overline{D}]\le k_0\,(<\infty, p>4, j=1,2)$ with a positive constant k_0, if $\tau<1$, and if $\tau>1$ and M_8 in (3.2) is small enough, then (1.8) has a solution $u(z,t)\in C^{1,0}(\overline{G})$.

Proof. (1) Consider the algebraic equation for t

$$M_6[k_1+k_0(t^\eta+t^\eta)+k_2+k_3]=t. \tag{3.4}$$

Because $0<\eta,\tau<1$, the the above equation has a solution $t=M_9>0$, which is also the maximum of t in $(0,+\infty)$. Now, we introduce a closed, bounded and convex subset B of the Banach space $C^{1,0}(\overline{G})$, whose elements are of the form $u(z)$ satisfying the condition

$$C^{1,0}[|u(z)|^{n+1},\overline{G}]\le M_9. \tag{3.5}$$

We choose an arbitrary function $u(z) \in B$ and substitute it into the proper positions in the following equation and initial-boundary conditions (Problem O^h) with the parameter $h \in [0,1]$

$$\begin{cases} A_0 u_{z\bar{z}} - \mathrm{Re}[Q u_{zz}] - u_t - h f(z,t,u,u_z) = A(z,t),\ (z,t) \in G, \\ u(z,0) = g(z),\ \ z \in D, \\ \dfrac{\partial u}{\partial \nu} + h b_1(z,t) u = b(z,t),\ (z,t) \in \partial G_2, \end{cases} \tag{3.6}$$

where $A(z,t)$ are any measurable functions with the condition $A(z,t) \in L_p(\overline{G})$, $p > 4$, and $b(z,t)$ is a continuously differentiable function with the condition $b(z,t) \in C^{1,0}_{\beta,\beta/2}(\partial G_2)$. When $h = 0$, according to Theorem 4.3, Chapter IV, [4], we see that there exists a solution $u_0(z,t) \in B = \hat{C}^{1,0}_{\beta,\beta/2}(\bar{G}) \cap W^{2,1}_2(G)$ of Problem O^0. Suppose that when $h = h_0\,(0 \le h_0 < 1)$, Problem O^{h_0} for (3.6) is solvable. We shall prove that there exists a positive constant ε independent of h_0, such that for any $h \in E = \{|h - h_0| \le \varepsilon, 0 \le h \le 1\}$, Problem O^h for (3.6) possesses a solution $u(z,t) \in B$. Let the above problem be rewritten in the form

$$\begin{cases} A_0 u_{z\bar{z}} - \mathrm{Re}[Q u_{zz}] - u_t - h_0 f(z,t,u,u_z) \\ = (h - h_0) f(z,t,u,u_z) + A(z,t) \ \text{ in } G, \\ u(z,0) = g(z) \ \text{ on } D, \\ \dfrac{\partial u}{\partial \nu} + h_0 b_1 u = (h_0 - h) b_1 + b(z,t) \ \text{ on } \partial G_2. \end{cases} \tag{3.7}$$

We arbitrarily choose a function $u^0(z,t) \in B$ and substitute it into the position of u on the right hand side of (3.7). It is easily seen that

$$\begin{aligned} &(h - h_0) f(z,t,u^0,u^0_z) + A(z,t) \in L_p(\overline{G}), \\ &(h_0 - h)\, b_2(z,t) + b(z,t) \in C^{0,0}_{\alpha,\alpha/2}(\partial G_2). \end{aligned} \tag{3.8}$$

By the hypothesis of h_0, there exists a solution $u^1(z,t) \in B$ of Problem O^h corresponding to

$$\begin{cases} A_0 u_{z\bar{z}} - \mathrm{Re}[Q u_{zz}] - u_t - h_0 f(z,t,u,u_z) \\ = (h - h_0) f(z,t,u^0,u^0_z) + A(z,t) \ \text{ in } G, \\ u(z,t) = g(z) \ \text{ in } D, \\ \dfrac{\partial u}{\partial \nu} + h_0 b_1 u = (h_0 - h) u^0 + b(z,t) \ \text{ on } \partial G_2. \end{cases} \tag{3.9}$$

By using the successive iteration, we obtain a sequence of solutions $u^m(z,t)$ $(m = 1, 2, \cdots) \in B$ of Problem O^h, which satisfy

$$\begin{cases} A_0 u_{z\bar{z}}^{m+1} - \mathrm{Re}[Q u_{zz}^{m+1}] - u_t^{m+1} - h_0 f(z,t,u^{m+1}) \\ = (h-h_0) f(z,t,u^m) + A(z,t) \text{ in } G, \\ u^{m+1}(z,0) = g(z) \text{ on } D, \\ \dfrac{\partial u^{m+1}}{\partial \nu} + h_0 b_1 u^{m+1} = (h_0 - h) b_1 u^m + b(z,t) \text{ on } \partial G_2, \\ m = 1, 2, \cdots . \end{cases} \tag{3.10}$$

According to the way in the proof of Theorem 2.2, we can obtain

$$C^{1,0}[u^{m+1}, \bar{G}] = ||u^{m+1}|| \le ||h - h_0|| M_{10} C^{1,0}[u^m, \overline{G}],$$

where $M_{10} = M_{10}(\delta, q, p, \beta, k, G) \ge 0$. Setting $\varepsilon = 1/2(M_{10}+1)$, we have

$$||u^{m+1}|| = C^{1,0}[u^{m+1}, \overline{G}] \le \frac{1}{2} ||u^m|| \text{ for } h \in E.$$

Hence when $n \ge m > N+2 (> 2)$, there are

$$||u^{m+1} - u^m|| \le 2^{-N} ||u^1 - u^0||,$$

$$||u^n - u^m|| \le 2^{-N} \sum_{j=1}^{\infty} 2^{-j} ||u^1 - u^0|| = 2^{-N+1} ||u^1 - u^0||.$$

This shows that $\| u^n - u^m \| \to 0$ as $n, m \to \infty$. By the completeness of the Banach space B, there exists $u^* \in B$, such that $\| u^n - u^* \| \to 0$ as $n \to \infty$ and u^* is the solution of Problem O^h with $h \in E$. Thus from the solvability of Problem O^0, we can derive the solvability of Problem O^1, in particular Problem O^1 with $A = 0$ and $b(z,t) = 0$, i.e. Problem O for (3.1) has a solution. This completes the proof.

(2) For the case $\min(\eta, \tau) < 1$, due to M_8 in (3.2) is small enough, from

$$M_6[k_1 + k_0(t^\eta + t^\tau) + k_2 + k_3] = t,$$

a solution $t = M_{11} > 0$ can be solved, which is also a maximum. Now we consider a subset B_* in the Banach space $C^1(\bar{D})$, i.e.

$$B_* = \{u(z) \,|\, C^{1,0}[u, \overline{G}] \le M_{11}\},$$

and apply a similar method as before. We can prove that there exists a solution $u(z) \in B_* = C^{1,0}(\overline{G})$ of Problem O for (1.8) with the constant $\min(\eta, \tau) > 1$.

(3) By using the similar method as in proofs of (1) and (2), we can verify the solvability of Problem O for (1.8) with the conditions $0 < \tau < 1$ and $1 < \tau < \infty$ as in (3) of the theorem.

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