High Order Moment Closure with Global Hyperbolicity of Vlasov-Maxwell Equations

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Abstract

In this paper, we derive the extended magnetohydrodynamic models based on the moment closure of the Vlasov-Maxwell equation. We adopt the Grad type moment expansion which was firstly proposed in [12] for the Boltzmann equation. A new regularization method [5] for the Grad's moment system of the Boltzmann equation was recently proposed to achieve the globally hyperbolicity so that the local well-posedness of the moment system is attained. For the Vlasov-Maxwell equation, the moment expansion of the convection term is exactly the same as that in the Boltzmann equation, thus the new developed regularization applies. The moment expansion of the electromagnetic force term in the Vlasov-Maxwell equation turns to be a linear source term, which can preserve the properties of the distribution function in the Vlasov-Maxwell equation perfectly.

Keywords: Moment closure; Vlasov-Maxwell equation; Boltzmann equation; Extended magnetohydrodynamics

1 Introduction

The collisionless plasmas has been studied in a wide variety of fields, such as in laboratory plasma physics, space physics, and astrophysics. Evolution of collisionless plasmas and self-consistent electromagnetic fields is fully described by the Vlasov-Maxwell equations. Thanks to recent development in computational technology, self-consistent numerical simulations of collisionless plasmas have been successfully performed from the first-principle Vlasov-Maxwell system of equations. There are two numerical methods to solve the Vlasov equation, including the popular Particle-In-Cell (PIC) method [4] and direct Vlasov simulation.

The PIC method has been used for a wide variety of plasma phenomena, which approximates the plasma by a finite number of macro-particles. The trajectories of the macro-particles are calculated from the equation of motion, which are continuous in space. The electromagnetic fields are calculated on grid points in space. Though the PIC method can often give satisfying results even with a relatively small number of particles, the PIC

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method inherently has the large statistical noise due to an approximation of the distribution function by a finite number of particles. This noise only decreases reciprocal to the number of particles, making it difficult to study such as particle acceleration and thermal transport processes, in which a small number of high energy particles play an important role.

The direct Vlasov simulation is to solve the advection equation by directly discretization on grid points in both spatial space and phase space. It has been widely known that a numerical solution of the advection equation suffers from spurious oscillations and numerical diffusion, while a highly accurate scheme is required to preserve characteristics of the Vlasov equation, saying the Liouville theorem, as much as possible. Though there are a lot of study [17, 18, 20, 24, 28] on the direct simulation, no standard scheme for the Vlasov simulation has been established so far.

On the other hand, traditional approaches to modelling space plasmas use various levels of approximations such as the Euler, the Navior-Stokes, magnetohydrodynamics (MHD), Chew-Goldberger-Low [10] (CGL), and finite Larmor radius (FLR) models. The intrinsic limitations of the models are seldom discussed, due in part to the difficulty in assessing such limits. By studying the manner in which they are derived from kinetic equations, it can be found that the difficulties involve representing nonequilibrium condition, wave-particle interactions, anisotropies and low collisionality. Mathematically, this is translated into the problem of closure of moments, since any finite number of velocity moments of the Boltzmann equation does not constitute a closed set and the equation governing the evolution of the velocity moment of order n contains the moment of order $n + 1$. Considerable efforts have gone into developing methods for closing various sets of moment equations. A variety of quasi-fluid descriptions have recently been introduced to describe intermediate collisional regimes. For example, recent attempts include neoclassical and anomalous transport in electrostatic turbulence [23], a fluid model of Landau damping [14], anisotropic fluid plasma closure [11], fluid descriptions of ion acoustic waves [29], and a MHD Vlasov study of toroidal Alfvén eigenmodes [25]. Actually, the moment closure problem is originated in the Boltzmann equation.

The moment closure method of Boltzmann equation can be tracked back to Grad's work in 1949 [12], where a 13-moment model was given as an extension of the classic Euler equations. In [13], its major deficiencies were found soon, including the appearance of subshocks in the structure of a strong shock wave and the loss of global hyperbolicity. In the later study, a number of regularizations were attempted to solve or alleviate these problems, such as Levermore's work [16]. Jin and Slemrod [15] gave a regularization of the Burnett equations via relaxation, which resulted in a set of equations containing the same variables with Grad's 13-moment theory, and no subshocks appeared in the structure of shock waves. By integrating the moment method with Chapman-Enskog expansion, Struchtrup and Torrilhon [22, 21] regularized Grad's system to give the R13 equations. The R13 system removes the discontinuities in the shock wave and extends the region of hyperbolicity considerably [26].

Due to the complexity of the explicit expressions, systems with large number of moments are not investigated until recently. In [27], Torrilhon and his coworkers developed a software named ET_{XX} [2] which is able to generate moment systems with almost any number of moments in one-dimensional space. And in [3], some numerical results for a shock tube were carried out to show the behavior of characteristic waves in the extended thermodynamics. A numerical method solving large moment systems was proposed in [7], and therein the regularization technique in [22] was applied to general moment systems.

In [8], the order of magnitude method was also integrated into large moment systems. In [6], the authors focused on the one-dimensional velocity space, and the characteristic polynomial of the quasi-linear coefficient matrix was found to be very simple; thus a brand-new regularized model with global hyperbolicity was proposed by the correction of the characteristic speed. Such regularization was extended to the multi-dimensional velocity space in [5].

It is clear that the moment expansion of the spatial convective term in Boltzmann equation can be extended to the Vlasov-Maxwell equation. The resulting convective term in the moment system expanded from the drift term of the Vlasov-Maxwell equation has exactly the same format as that of the Boltzmann equation. Thus the method of the hyperbolic regularization in [5] can be applied to the Vlasov-Maxwell equation to achieve the global hyperbolicity. The major difference of the Vlasov-Maxwell equation from the Boltzmann equation is the acceleration to the particles due to the electromagnetic field. The moment expansion of the distribution function turns the acceleration term into a linear source term, with compact sparse coefficient matrix. In this source term, the coefficient matrix is block diagonal for the moments with the same order, thus the evolutions of the moments with different orders are separated. It is shown that a weighted l_2 norm of the moments of the same order is invariant in time accelerated by the magnetic field alone. As a result, the high order moments due to the source term are not growing at all. This makes that the derived moment system is formulated as a quasi-linear system, plus a linear source term which induces no growth of the high order moments. Since the convection term in the system is guaranteed to be globally hyperbolic by the regularization, the local well-posedness of the system is partially achieved.

The rest of this paper is arranged as follows: in Section 2 we present the elementary formula of Vlasov-Maxwell equation. The moment expansion of Vlasov-Maxwell equation is carried out in Section 3 and the system obtained is closed by truncation of the expansion and regularized using method in [5] in Section 4 to achieve the final hyperbolic moment system. In Section 5, we discuss an exact Vlasov-Maxwell equilibria with sheath-like magnetic field for better understanding of the structure of the derived moment system. Concluding remarks are in the last section.

2 The Vlasov-Maxwell Equations

The evolution of the distribution function f_s of electrically charged particles of type s (electron or ion), each particle having the charge q_s and mass m_s, is described by the nonrelativistic Vlasov equation

$$\frac{\partial f_s}{\partial t} + \boldsymbol{p} \cdot \nabla_{\boldsymbol{x}} f_s + \frac{\boldsymbol{F}_s}{m_s} \cdot \nabla_{\boldsymbol{p}} f_s = 0, \quad \boldsymbol{x} \in \mathbb{R}^3,\ \boldsymbol{p} \in \mathbb{R}^3, \tag{1}$$

where the acceleration term in the phase space is

$$\boldsymbol{F}_s = q_s[\boldsymbol{E} + \boldsymbol{p} \times (\mathbf{B} + \mathbf{B}_{ext})]. \tag{2}$$

The part $q_s\boldsymbol{E}$ is the Column force and the part $q_s\boldsymbol{p} \times (\mathbf{B} + \mathbf{B}_{ext})$ is the Lorentz force. Here, the magnetic field is separated into two parts, where $\mathbf{B}_{ext}$ is an external magnetic field, and $\mathbf{B}$ is the self-consistent part of the magnetic field, created by the plasma. One Vlasov equation is needed for each species of particles.

The particles interact via the electromagnetic field. The charge and current densities, ρ and $\boldsymbol{j}$, act as sources of self-consistent electromagnetic fields according to the Maxwell equations

$$\begin{aligned}
\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla_{\boldsymbol{x}} \cdot \mathbf{B} &= 0 \\
\nabla_{\boldsymbol{x}} \times \boldsymbol{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla_{\boldsymbol{x}} \times \mathbf{B} &= \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}.
\end{aligned} \tag{3}$$

The charge and current densities are related to the particle number densities n_s and mean velocities $\boldsymbol{u}_s$ as

$$\rho = \sum_s q_s n_s \tag{4}$$

and

$$\boldsymbol{j} = \sum_s q_s n_s \boldsymbol{u}_s, \tag{5}$$

respectively, where the particle number densities and mean velocities are obtained as moments of the particle distribution functions, as

$$n_s(t,\boldsymbol{x}) = \int f_s(t,\boldsymbol{x},\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} \tag{6}$$

and

$$\boldsymbol{u}_s(t,\boldsymbol{x}) = \frac{1}{n_s(t,\boldsymbol{x})}\int \boldsymbol{p} f_s(t,\boldsymbol{x},\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}, \tag{7}$$

respectively.

The Vlasov equation (1) together with the Maxwell equations (3) and the constitutive equations form a closed, coupled system of nonlinear partial differential equations and integral equations. The system conserves the energy norm

$$\|f_s\|^2 = \int\int f_s^2 d\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}, \tag{8}$$

the total number of particles

$$N_s = \int\int f_s d\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}, \tag{9}$$

the total linear momentum

$$\boldsymbol{P} = \int \left[\int \boldsymbol{p}\left(m_i f_i + m_e f_e\right)\,\mathrm{d}\boldsymbol{p} + \varepsilon \boldsymbol{E} \times \mathbf{B}\right]\mathrm{d}\boldsymbol{x}, \tag{10}$$

and the total energy

$$W = \int \left[\int \frac{1}{2}\boldsymbol{p}^2\left(m_i f_i + m_e f_e\right)\,\mathrm{d}\boldsymbol{p} + \frac{1}{2}\left(\varepsilon \boldsymbol{E}^2 + \frac{\mathbf{B}^2}{\mu_0}\right)\right]\mathrm{d}\boldsymbol{x}. \tag{11}$$

For example, when the density of electrons are not extremely high, we can assume that the equilibrium distribution is a Maxwellian distribution,

$$f_{\mathrm{eq}}(t,\boldsymbol{x},\boldsymbol{p}) = \frac{n(t,\boldsymbol{x})}{(2\pi k_B T(t,\boldsymbol{x}))^{3/2}}\exp\left(-\frac{(\boldsymbol{p}-\boldsymbol{u}(t,x))^2}{2k_B T(t,\boldsymbol{x})}\right) \tag{12}$$

where k_B is the Boltzmann constant, $T(t,\boldsymbol{x})$ is the particle temperature, which is related with the distribution function as below:

$$3n(t,\boldsymbol{x})k_B T(t,\boldsymbol{x}) = \int |\boldsymbol{p}-\boldsymbol{u}|^2 f(t,\boldsymbol{x},\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}. \tag{13}$$

3 Grad Moment System

In this section, we derive the moment system of the Vlasov equation using the Grad type moment expansion.

3.1 Hermite expansion of the distribution function

Following the method in [7, 8], we expand the distribution function into Hermite series as

$$f(t,\boldsymbol{x},\boldsymbol{p}) = \sum_{\alpha\in\mathbb{N}^3} f_\alpha(t,\boldsymbol{x})\mathcal{H}_{\mathcal{T},\alpha}\left(\frac{\boldsymbol{p}-\boldsymbol{u}(t,\boldsymbol{x})}{\sqrt{\mathcal{T}(t,\boldsymbol{x})}}\right), \tag{14}$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ is a three-dimensional multi-index. The basis functions $\mathcal{H}_{\mathcal{T},\alpha}$ are the 3-dimensional Hermite functions defined by

$$\mathcal{H}_{\mathcal{T},\alpha}(\boldsymbol{\xi}) = \prod_{d=1}^{3}\frac{1}{\sqrt{2\pi}}\mathcal{T}^{-\frac{\alpha_d+1}{2}} He_{\alpha_d}(\xi_d)\exp\left(-\frac{\xi_d^2}{2}\right), \tag{15}$$

where $He_n(x)$ is the Hermite polynomial of order n

$$He_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp\left(-\frac{x^2}{2}\right). \tag{16}$$

For convenience, $He_n(x)$ is taken as zero if $n<0$, thus $\mathcal{H}_{\mathcal{T},\alpha}(\boldsymbol{\xi})$ is zero when any component of α is negative. The parameter $\mathcal{T}$ in the expansion is the scaled local temperature as

$$\mathcal{T}(t,\boldsymbol{x}) = k_B T(t,\boldsymbol{x}). \tag{17}$$

It is clear that the equilibrium distribution f_{eq} is coincidently equal to the first term of expansion, i.e.,

$$f_{\mathrm{eq}}(t,\boldsymbol{x},\boldsymbol{p}) = f_0(t,\boldsymbol{x})\mathcal{H}_{\mathcal{T},0}\left(\frac{\boldsymbol{p}-\boldsymbol{u}(t,\boldsymbol{x})}{\sqrt{\mathcal{T}(t,\boldsymbol{x})}}\right), \tag{18}$$

where $f_0(t,\boldsymbol{x}) = n(t,\boldsymbol{x})$.

The definition of the Hermite function (15) shows that each basis function is an exponentially decaying function multiplied by a multi-dimensional Hermite polynomial shifted by the local macroscopic momentum $\boldsymbol{u}$ and scaled by the square root of the local temperature $\mathcal{T}$.

If one uses arbitrary known function $\boldsymbol{u}'(t,\boldsymbol{x})$ and $\mathcal{T}'(t,\boldsymbol{x})$ in (14) to expand the distribution function $f(t,\boldsymbol{x},\boldsymbol{p})$ as

$$f(t,\boldsymbol{x},\boldsymbol{p}) = \sum_{\alpha\in\mathbb{N}^3} f'_\alpha(t,\boldsymbol{x})\mathcal{H}_{\mathcal{T}',\alpha}\left(\frac{\boldsymbol{p}-\boldsymbol{u}'(t,\boldsymbol{x})}{\sqrt{\mathcal{T}'(t,\boldsymbol{x})}}\right), \tag{19}$$

then the following relations between the macroscopic quantities $\boldsymbol{u}$, $\mathcal{T}$, the coefficients f_α, and $\boldsymbol{u}'$, $\mathcal{T}'$, the coefficients f'_α can be derived as follows,

$$n = f_0 = f'_0, \tag{20a}$$

$$n\boldsymbol{u} = n\boldsymbol{u}' + (f'_{e_d})^T_{d=1,2,3}, \tag{20b}$$

$$n|\boldsymbol{u} - \boldsymbol{u}'|^2 + 3n\mathcal{T} = \sum_{d=1}^{3} (\mathcal{T}' f'_0 + 2 f'_{2e_d}), \tag{20c}$$

where e_d is the unit vector with its d-th entry to be 1. It is clear that the coefficients f_α expanded using parameters $\boldsymbol{u}$ and $\mathcal{T}$ satisfy the following conditions:

$$f_{e_i} = 0, \quad \sum_{d=1}^{3} f_{2e_d} = 0, \quad i = 1, 2, 3. \tag{21}$$

Moreover, if we define the heat flux q_i and the pressure tensor $P = \{p_{ij}\},\ i, j = 1, 2, 3$ with

$$q_i = \frac{1}{2} \int_{\mathbb{R}^3} |\boldsymbol{p} - \boldsymbol{u}|^2 (p_i - u_i) f \,\mathrm{d}\boldsymbol{p}, \tag{22}$$

$$p_{ij} = \int_{\mathbb{R}^3} (p_i - u_i)(p_j - u_j) f \,\mathrm{d}\boldsymbol{p}, \tag{23}$$

then direct calculations give us the relations between them and the coefficients f_α in (14) as

$$q_i = 2 f_{3e_i} + \sum_{d=1}^{3} f_{2e_d + e_i}, \tag{24}$$

$$p_{ij} - \frac{1}{3}\delta_{ij} \sum_{d=1}^{3} p_{dd} = (1 + \delta_{ij}) f_{e_i + e_j}. \tag{25}$$

By the definition of the temperature (13) and (17) and the definition of the tensor pressure (25), the scaled temperature $\mathcal{T}$ is a linear combination of p_{ij} as

$$n\mathcal{T} = \frac{1}{3} \sum_{d=1}^{3} p_{dd}. \tag{26}$$

With the relation (25), we then have

$$p_{ij} = \delta_{ij} n\mathcal{T} + (1 + \delta_{ij}) f_{e_i + e_j}. \tag{27}$$

3.2 Moment expansion of the Vlasov equation

Now we are ready to derive the moment system by taking the moments of the Vlasov equation. The general method to get the moment system is to first multiply the Vlasov equation (1) by polynomials of momentum $\boldsymbol{p}$ of different order and then integrate both sides over momentum $\boldsymbol{p}$ on $\mathbb{R}^3$. One equivalent way is as follows. First, we substitute the expansion of the distribution function (14) into the Vlasov equation (1), then we collect the coefficients of the basis functions of the same order on both sides, and finally we equate them to yield the derived moment system. It should be noted that the Hermite function (15) used in this paper depends also on the time t and position $\boldsymbol{x}$ through $\boldsymbol{u}(t,\boldsymbol{x})$ and $\mathcal{T}(t,\boldsymbol{x})$, which is different from the general expansion using the Hermite functions depending only on the momentum $\boldsymbol{p}$ [19]. For convenience, we list some useful relations of Hermite polynomials as below [1]:

1. Orthogonality: $\displaystyle\int_{\mathbb{R}} He_l(x)He_n(x)\exp(-x^2/2)\,\mathrm{d}x = l!\sqrt{2\pi}\delta_{l,n}$;
2. Recursion relation: $He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x)$;
3. Differential relation: $He_n'(x) = nHe_{n-1}(x)$.

And the following equality can be derived from the last two relations:

$$[He_n(x)\exp(-x^2/2)]' = -He_{n+1}(x)\exp(-x^2/2). \tag{28}$$

Especially, we have

$$\frac{\partial}{\partial p_j}\mathcal{H}_{\mathcal{T},\alpha}\left(\frac{\boldsymbol{p}-\boldsymbol{u}}{\sqrt{\mathcal{T}}}\right) = -\mathcal{H}_{\mathcal{T},\alpha+e_j}\left(\frac{\boldsymbol{p}-\boldsymbol{u}}{\sqrt{\mathcal{T}}}\right). \tag{29}$$

With these relations, the part

$$\frac{\partial f}{\partial t} + \boldsymbol{p}\cdot\nabla_{\boldsymbol{x}} f$$

of (1) is expanded as

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}^3}\Bigg\{&\left(\frac{\partial f_\alpha}{\partial t} + \sum_{d=1}^{3}\frac{\partial u_d}{\partial t}f_{\alpha-e_d} + \frac{1}{2}\frac{\partial\mathcal{T}}{\partial t}\sum_{d=1}^{3}f_{\alpha-2e_d}\right)\\ &+\sum_{j=1}^{3}\Bigg[\left(\mathcal{T}\frac{\partial f_{\alpha-e_j}}{\partial x_j} + u_j\frac{\partial f_\alpha}{\partial x_j} + (\alpha_j+1)\frac{\partial f_{\alpha+e_j}}{\partial x_j}\right)\\ &+\sum_{d=1}^{3}\frac{\partial u_d}{\partial x_j}\left(\mathcal{T}f_{\alpha-e_d-e_j} + u_jf_{\alpha-e_d} + (\alpha_j+1)f_{\alpha-e_d+e_j}\right)\\ &+\frac{1}{2}\frac{\partial\mathcal{T}}{\partial x_j}\sum_{d=1}^{3}\left(\mathcal{T}f_{\alpha-2e_d-e_j} + u_jf_{\alpha-2e_d} + (\alpha_j+1)f_{\alpha-2e_d+e_j}\right)\Bigg]\Bigg\}\mathcal{H}_{\mathcal{T},\alpha}\left(\frac{\boldsymbol{p}-\boldsymbol{u}}{\sqrt{\mathcal{T}}}\right).\end{aligned} \tag{30}$$

The acceleration term $\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}}f$ is expanded as

$$-\sum_{\alpha\in\mathbb{N}^3}\sum_{d=1}^{3}E_df_{\alpha-e_d}\mathcal{H}_{\mathcal{T},\alpha}\left(\frac{\boldsymbol{p}-\boldsymbol{u}}{\sqrt{\mathcal{T}}}\right). \tag{31}$$

The term $\boldsymbol{p}\times\mathbf{B}\cdot\nabla_{\boldsymbol{p}}f$ is expanded as

$$-\sum_{\alpha\in\mathbb{N}^3}\sum_{d,k,m=1}^{3}\varepsilon_{dkm}\Big[u_kB_mf_{\alpha-e_d} + (\alpha_k+1)\,B_mf_{\alpha-e_d+e_k}\Big]\mathcal{H}_{\mathcal{T},\alpha}\left(\frac{\boldsymbol{p}-\boldsymbol{u}}{\sqrt{\mathcal{T}}}\right), \tag{32}$$

where the Levi-Civita symbol ε_{dkm} is defined as

$$\varepsilon_{dkm} = \begin{cases}+1, & \text{if } (d,k,m)\in\{(1,2,3),(3,1,2),(2,3,1)\},\\ -1, & \text{if } (d,k,m)\in\{(1,3,2),(3,2,1),(2,1,3)\},\\ 0, & \text{if } d=k \text{ or } k=m \text{ or } m=d,\end{cases}$$

i.e., ε_{dkm} is 1 if (d,k,m) is an even permutation of $(1,2,3)$, -1 if it is an odd permutation, and 0 if any index is repeated.

Collecting the three terms (30), (31) and (32), we can get the following general moment equations with a slight rearrangement by matching the coefficients of the same weight function:

$$
\begin{aligned}
&\frac{\partial f_\alpha}{\partial t} + \sum_{d=1}^{3}\left[\frac{\partial u_d}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_d}{\partial x_j} - \frac{q}{m}E_d - \frac{q}{m}\sum_{k,m=1}^{3}\varepsilon_{dkm}u_k B_m\right] f_{\alpha-e_d} \\
&\qquad - \sum_{d,k,m=1}^{3}\frac{q}{m}\varepsilon_{dkm}\left(\alpha_k+1\right)B_m f_{\alpha-e_d+e_k} \\
&\qquad + \frac{1}{2}\left(\frac{\partial \mathcal{T}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial \mathcal{T}}{\partial x_j}\right)\sum_{d=1}^{3} f_{\alpha-2e_d} \\
&\qquad + \sum_{j,d=1}^{3}\left[\frac{\partial u_d}{\partial x_j}\left(\mathcal{T} f_{\alpha-e_d-e_j} + (\alpha_j+1) f_{\alpha-e_d+e_j}\right)\right. \\
&\qquad + \left.\frac{1}{2}\frac{\partial \mathcal{T}}{\partial x_j}\left(\mathcal{T} f_{\alpha-2e_d-e_j} + (\alpha_j+1) f_{\alpha-2e_d+e_j}\right)\right] \\
&\qquad + \sum_{j=1}^{3}\left(\mathcal{T}\frac{\partial f_{\alpha-e_j}}{\partial x_j} + u_j\frac{\partial f_\alpha}{\partial x_j} + (\alpha_j+1)\frac{\partial f_{\alpha+e_j}}{\partial x_j}\right) = 0.
\end{aligned}
\tag{33}
$$

By setting $\alpha = 0$ in (33), we deduce the mass conservation

$$
\frac{\partial n}{\partial t} + \sum_{j=1}^{3}\left(u_j\frac{\partial n}{\partial x_j} + n\frac{\partial u_j}{\partial x_j}\right) = 0. \tag{34}
$$

By setting $\alpha = e_d$, with $d = 1,2,3$ and noting that $f_{e_d} = 0$ in (33), we obtain

$$
\begin{aligned}
&n\left[\frac{\partial u_d}{\partial t} + \sum_{j=1}^{3} u_j\frac{\partial u_d}{\partial x_j} - \frac{q}{m}E_d - \frac{q}{m}\sum_{k,m=1}^{3}\varepsilon_{dkm}u_k B_m\right] \\
&\qquad + n\frac{\partial \mathcal{T}}{\partial x_d} + \mathcal{T}\frac{\partial n}{\partial x_d} + \sum_{j=1}^{3}(\delta_{jd}+1)\frac{\partial f_{e_d+e_j}}{\partial x_j} = 0,
\end{aligned}
\tag{35}
$$

which is simplified as

$$
n\left[\frac{\partial u_d}{\partial t} + \sum_{j=1}^{3} u_j\frac{\partial u_d}{\partial x_j} - \frac{q}{m}E_d - \frac{q}{m}\sum_{k,m=1}^{3}\varepsilon_{dkm}u_k B_m\right] + \sum_{j=1}^{3}\frac{\partial p_{jd}}{\partial x_j} = 0. \tag{36}
$$

By setting $\alpha = 2e_d$, with $d = 1,2,3$ and noting that $f_{e_d} = 0$, we obtain

$$
\begin{aligned}
&\frac{\partial f_{2e_d}}{\partial t} + \frac{n}{2}\left(\frac{\partial \mathcal{T}}{\partial t} + \sum_{j=1}^{3} u_j\frac{\partial \mathcal{T}}{\partial x_j}\right) + n\mathcal{T}\frac{\partial u_d}{\partial x_d} + \sum_{j,l}(1+2\delta_{jd}) f_{2e_d-e_l+e_j}\frac{\partial u_l}{\partial x_j} \\
&\qquad - \frac{q}{m}\sum_{k,m=1}^{3}\varepsilon_{dkm}B_m f_{e_d+e_k} + \sum_{j=1}^{3} u_j\frac{\partial f_{2e_d}}{\partial x_j} + (1+2\delta_{jd})\frac{\partial f_{2e_d+e_j}}{\partial x_j} = 0.
\end{aligned}
\tag{37}
$$

Noting that $\sum_{d=1}^{3} f_{2e_d} = 0$, we sum the upper equations over d to get

$$n\left(\frac{\partial \mathcal{T}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial \mathcal{T}}{\partial x_j}\right) + \frac{2}{3}\sum_{j=1}^{3}\left(\frac{\partial q_j}{\partial x_j} + \sum_{d=1}^{3} p_{jd}\frac{\partial u_d}{\partial x_j}\right) = 0. \tag{38}$$

Since $n\mathcal{T} = \frac{1}{3}\sum_{d=1}^{3} p_{dd}$, we have

$$\frac{\partial \mathcal{T}}{\partial x_j} = \frac{1}{3n}\sum_{d=1}^{3}\frac{\partial p_{dd}}{\partial x_j} - \frac{\mathcal{T}}{n}\frac{\partial n}{\partial x_j}, \quad j = 1,2,3. \tag{39}$$

Substituting (36), (38) and (39) into (33), we eliminate the time derivatives of $\boldsymbol{u}$ and $\mathcal{T}$ and the spatial derivatives of $\mathcal{T}$. Then the quasi-linear form of the moment system reads:

$$\begin{aligned}
&\frac{\partial f_\alpha}{\partial t} + \sum_{j=1}^{3}\left(\mathcal{T}\frac{\partial f_{\alpha-e_j}}{\partial x_j} + u_j\frac{\partial f_\alpha}{\partial x_j} + (\alpha_j+1)\frac{\partial f_{\alpha+e_j}}{\partial x_j}\right)\\
&\quad + \sum_{j=1}^{3}\sum_{d=1}^{3}\frac{\partial u_d}{\partial x_j}\left(\mathcal{T} f_{\alpha-e_d-e_j} + (\alpha_j+1) f_{\alpha-e_d+e_j} - \frac{p_{jd}}{3n}\sum_{k=1}^{3} f_{\alpha-2e_k}\right)\\
&\quad - \sum_{j=1}^{3}\sum_{d=1}^{3}\frac{f_{\alpha-e_d}}{n}\frac{\partial p_{jd}}{\partial x_j} - \frac{1}{3n}\left(\sum_{k=1}^{3} f_{\alpha-2e_k}\right)\sum_{j=1}^{3}\frac{\partial q_j}{\partial x_j}\\
&\quad + \sum_{j=1}^{3}\left(\left(-\frac{\mathcal{T}}{2n}\frac{\partial n}{\partial x_j} + \frac{1}{6n}\sum_{d=1}^{3}\frac{\partial p_{dd}}{\partial x_j}\right)\sum_{k=1}^{3}\left(\mathcal{T} f_{\alpha-2e_k-e_j} + (\alpha_j+1) f_{\alpha-2e_k+e_j}\right)\right)\\
&\quad = \sum_{d,k,m=1}^{3}\frac{q}{m}\varepsilon_{dkm}\left(\alpha_k+1\right)B_m f_{\alpha-e_d+e_k}, \qquad \forall\ |\alpha| \geq 2.
\end{aligned} \tag{40}$$

With (27), we can have the equations for p_{ij} by (40). Precisely, we have the equation for $p_{ii}/2$, $i = 1,2,3$, as

$$\begin{aligned}
&\frac{\partial p_{ii}/2}{\partial t} + \sum_{j=1}^{3} u_j\frac{\partial p_{ii}/2}{\partial x_j} + \sum_{j=1}^{3}\left(\frac{1}{2} + \delta_{ij}\right)\rho\mathcal{T}\frac{\partial u_j}{\partial x_j} + \sum_{j=1}^{3}\sum_{d=1}^{3}(2\delta_{ij}+1) f_{2e_i-e_d+e_j}\frac{\partial u_d}{\partial x_j}\\
&\qquad + \sum_{j=1}^{3}(2\delta_{ij}+1)\frac{\partial f_{2e_i+e_j}}{\partial x_j} = \frac{q}{m}\sum_{k,m=1}^{3}\varepsilon_{ikm}B_m f_{e_i+e_k}, \quad i = 1,2,3.
\end{aligned} \tag{41}$$

If $i \neq j$, we have $p_{ij} = f_{e_i+e_j}$, thus its equation is already in (40).

We collect the equations (34), (36), (41) and (40) together to obtain a moment system with an infinite number of equations. Noting that the relation between u_d and f_{e_d} given in (20) and the definition of q_i and of p_{ij} given in (25), one can see that the obtained system is quasi-linear for f_α.

4 Moment Closure with Global Hyperbolicity

The moment system derived from the Vlasov equation consists of (34), (36), (41) and (40). It is clear that this is a system with an infinite number of equations taken n, u_d, p_{ij} and f_α, $|\alpha| \geq 3$, as unknowns. To obtain a system with finite unknowns, we will truncate the expansion (14) and close the system following the method in [5].

With a truncation of (14), (40) will result in a finite moment system. Precisely, we let $M \geq 3$ be a positive integer and only the coefficients in the set $\mathcal{M} = \{f_\alpha\}_{|\alpha|\leq M}$ are considered. Let $F_M(\boldsymbol{u}, \mathcal{T})$ denotes the linear space spanned by all $\mathcal{H}_{\mathcal{T},\alpha}\left(\frac{\boldsymbol{p}-\boldsymbol{u}(t,\boldsymbol{x})}{\sqrt{\mathcal{T}(t,\boldsymbol{x})}}\right)$'s with $|\alpha| \leq M$, and the expansion (14) is truncated as

$$f(t, \boldsymbol{x}, \boldsymbol{p}) \approx \sum_{|\alpha|\leq M} f_\alpha(t, \boldsymbol{x}) \mathcal{H}_{\mathcal{T},\alpha}\left(\frac{\boldsymbol{p} - \boldsymbol{u}(t, \boldsymbol{x})}{\sqrt{\mathcal{T}(t, \boldsymbol{x})}}\right), \tag{42}$$

with $f(t, \boldsymbol{x}, \boldsymbol{p}) \in F_M(\boldsymbol{u}, \mathcal{T})$ and $f_\alpha \in \mathcal{M}$. The moment equations which contain $\partial f_\alpha/\partial t$ with $|\alpha| > M$ are disregarded in (40). Then, (34), (36), (41) and (40) with $2 \leq |\alpha| \leq M$ lead to a system with a finite number of equations.

Following [5], we let

$$\mathcal{S}_M = \{\alpha \in \mathbb{N}^3 \mid |\alpha| \leq M\}.$$

Then for any $\alpha \in \mathcal{S}_M$, let

$$\mathcal{N}(\alpha) = \sum_{i=1}^{3} \binom{\sum_{k=4-i}^{3} \alpha_k + i - 1}{i} + 1 \tag{43}$$

be the ordinal number of α in $\mathcal{S}_M$, and the cardinal number of set $\mathcal{S}_M$ be

$$N = \mathcal{N}(Me_3) = \binom{M+3}{3},$$

which is the total number of moments if a truncation with $|\alpha| \leq M$ is applied.

Let $\boldsymbol{w} = (w_1, \cdots, w_N)^T \in \mathbb{R}^N$ and for each $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$w_1 = n, \qquad w_{\mathcal{N}(e_i)} = u_i, \tag{44a}$$

$$w_{\mathcal{N}(2e_i)} = \frac{p_{ii}}{2}, \qquad w_{\mathcal{N}(e_i+e_j)} = p_{ij}, \tag{44b}$$

$$w_{\mathcal{N}(\alpha)} = f_\alpha, \quad 3 \leq |\alpha| \leq M. \tag{44c}$$

The moment system (34), (36), (41) and (40) is collected in quasi-linear format as

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{j=1}^{3} \mathbf{M}_j(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x_j} = \mathbf{G}\boldsymbol{w}, \tag{45}$$

by taking the derivatives of $f_{\alpha+e_j}(|\alpha| = M)$ to be zero, where $\mathbf{M}_j$ and $\mathbf{G}$ are $N \times N$ matrices. The entries of $\mathbf{M}_j$ are given as the coefficients of the terms in (34), (36), (41) and (40) with derivatives of $\boldsymbol{w}$. The entries of $\mathbf{G}$ arise from the electromagnetic force term. From (36), it is clear that

$$\mathbf{G}_{\mathcal{N}(e_i),1} = -\frac{q}{nm}\left(E_i + \sum_{k,m=1}^{3} \varepsilon_{ikm} u_k B_m\right), \quad i = 1, 2, 3. \tag{46}$$

The other nonzero entries of $\mathbf{G}$ are as

$$\mathbf{G}_{\mathcal{N}(\alpha-e_d+e_k),\mathcal{N}(\alpha)} = -\frac{q}{m}\sum_{m=1}^{3}\varepsilon_{dkm}\alpha_d B_m, \tag{47}$$

which is derived from

$$\mathbf{G}_{\mathcal{N}(\alpha),\mathcal{N}(\alpha-e_d+e_k)} = \frac{q}{m}\sum_{m=1}^{3}\varepsilon_{dkm}\left(\alpha_k+1\right)B_m. \tag{48}$$

All other entries of $\mathbf{G}$ vanish except for the ones specified above.

In (45), we following Grad [12] take $\partial f_{\alpha+e_j}/\partial x_j(|\alpha|=M)$ as zero to make the system to be closed. It has been pointed out in [5] that it is not appropriate to set $\partial f_{\alpha+e_j}/\partial x_j = 0$ $(|\alpha|=M)$ as the closure since the system is lack of hyperbolicity if the distribution function is far away from the equilibrium. To obtain a system with global hyperbolicity, we have to adopt the regularization given in [5]. For any α with $|\alpha|=M$, we define

$$\mathcal{R}_M^j(\alpha) = (\alpha_j+1)\left[\sum_{d=1}^{3} f_{\alpha-e_d+e_j}\frac{\partial u_d}{\partial x_j} + \frac{1}{2}\left(\sum_{d=1}^{3} f_{\alpha-2e_d+e_j}\right)\frac{\partial \mathcal{T}}{\partial x_j}\right]. \tag{49}$$

and

$$\hat{\mathbf{M}}_j\frac{\partial \boldsymbol{w}}{\partial x_j} = \mathbf{M}_j\frac{\partial \boldsymbol{w}}{\partial x_j} - \sum_{|\alpha|=M}\mathcal{R}_M^j(\alpha)I_{\mathcal{N}(\alpha)}, \qquad \text{for any admissible } \boldsymbol{w}, \tag{50}$$

where I_k is the k-th column of the $N\times N$ identity matrix. We regularize the system (45) as

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{j=1}^{3}\hat{\mathbf{M}}_j(\boldsymbol{w})\frac{\partial \boldsymbol{w}}{\partial x_j} = \mathbf{G}\boldsymbol{w}. \tag{51}$$

It has been proved in [5] that

Theorem 1. *The regularized moment system* (51) *is hyperbolic for any* $\boldsymbol{w}$ *with positive temperature. Precisely, for a given unit vector* $\boldsymbol{n}=(n_1,n_2,n_3)$*, the matrix*

$$\sum_{j=1}^{3} n_j\hat{\mathbf{M}}_j(\boldsymbol{w}) \tag{52}$$

is diagonalizable with real eigenvalues as

$$\boldsymbol{u}\cdot\boldsymbol{n} + \mathrm{C}_{k,m}\sqrt{\mathcal{T}}, \qquad 1\le k\le m\le M+1, \tag{53}$$

where $\mathrm{C}_{k,m}$ *is a root of* m*-order Hermite polynomial, and satisfies* $\mathrm{C}_{1,m} < \cdots < \mathrm{C}_{m,m}$. *The structure of the* N *eigenvectors can be fully clarified.*

Based on this theorem, the regularized moment system (51) is locally well-posed due to the hyperbolicity. We would like to mention here that the regularization here actually does not add any new terms to the system (45). On the contrary, it has erased the terms in (40) with a factor α_j+1 in its coefficient for the equations of f_α with $|\alpha|=M$ only.

Let us turn to the source term $\mathbf{G}\boldsymbol{w}$ coming from the phase space acceleration due to the electromagnetic force term. We first point out that the matrix $\mathbf{G}$ is block diagnal as

$$\mathbf{G} = \mathrm{diag}\{0,\mathbf{G}_1,\mathbf{G}_2,\cdots,\mathbf{G}_M\},$$

where

$$\mathbf{G}_m = \left[\mathbf{G}_{\mathcal{N}(\alpha),\mathcal{N}(\beta)}\right], \quad |\alpha| = |\beta| = m, \quad 1 \leq m \leq M,$$

and the nonzero entries of $\mathbf{G}_1$ is given by (46), and the nonzero entries of $\mathbf{G}_m$, $2 \leq m \leq M$ are given by (47). Omiting the convective term $\sum_{j=1}^{3} \mathbf{M}_j(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x_j}$ in (45) temporarily, let us consider the system with the source term only

$$\frac{\partial \boldsymbol{w}}{\partial t} = \mathbf{G}\boldsymbol{w}. \tag{54}$$

We define a diagonal matrix $\mathbf{D}$ with diagonal entries

$$\mathbf{D}_{\mathcal{N}(\alpha),\mathcal{N}(\alpha)} = \alpha! \triangleq \prod_{d=1}^{3} \alpha_d!. \tag{55}$$

It is clear that $\mathbf{D}$ can be divided into

$$\mathbf{D} = \mathrm{diag}\{0, \mathbf{D}_1, \mathbf{D}_2, \cdots, \mathbf{D}_M\},$$

where

$$\mathbf{D}_m = \mathrm{diag}\{\mathbf{D}_{\mathcal{N}(\alpha),\mathcal{N}(\alpha)}\}, \quad |\alpha| = m.$$

Correspondingly, the vector $\boldsymbol{w}$ is divided into

$$\boldsymbol{w} = [n, \boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_M]^T,$$

where

$$\boldsymbol{w}_1 = [u_1, u_2, u_3]^T, \text{ and } \boldsymbol{w}_m = [w_{\mathcal{N}(\alpha)}]^T, \quad |\alpha| = m, \ 2 \leq m \leq M.$$

We note that such a partition requires a re-permutation of the entries of $\boldsymbol{w}$. We have the following properties

Theorem 2. *The solution of system* (54) *satisfies*

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\sum_{i=1}^{3} u_i^2\right) = -\frac{q}{nm}\sum_{i=1}^{3} u_i E_i, \tag{56}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{w}_m^T \mathbf{D}_m \boldsymbol{w}_m\right) = 0, \quad 2 \leq m \leq M. \tag{57}$$

Proof. We notice that

$$\sum_{i,k,m=1}^{3} \varepsilon_{ikm} u_i u_k B_m = 0,$$

thus (56) is obtained. As for (57), we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{w}_m^T \mathbf{D}_m \boldsymbol{w}_m\right) = \boldsymbol{w}_m^T \mathbf{D}_m \frac{\partial \boldsymbol{w}_m}{\partial t} = \boldsymbol{w}_m^T \mathbf{D}_m \mathbf{G}_m \boldsymbol{w}_m.$$

Notice that the matrix $\mathbf{D}_m\mathbf{G}_m$ is a diagnol block of the matrix $\mathbf{DG}$, and each one of its nonzero entries satisfies

$$
\begin{aligned}
(\mathbf{DG})_{\mathcal{N}(\alpha),\mathcal{N}(\alpha-e_d+e_k)} &= \alpha!\mathbf{G}_{\mathcal{N}(\alpha),\mathcal{N}(\alpha-e_d+e_k)} && (58)\\
&= \frac{q}{m}\sum_{m=1}^{3}\varepsilon_{dkm}(\alpha_k+1)B_m(\alpha_d)!\alpha_k!\alpha_m! && (59)\\
&= \frac{q}{m}\sum_{m=1}^{3}\varepsilon_{dkm}\alpha_d B_m(\alpha_d-1)!(\alpha_k+1)!\alpha_m! && (60)\\
&= -(\alpha-e_d+e_k)!\mathbf{G}_{\mathcal{N}(\alpha-e_d+e_k),\mathcal{N}(\alpha)} && (61)\\
&= -(\mathbf{DG})_{\mathcal{N}(\alpha-e_d+e_k),\mathcal{N}(\alpha)}. && (62)
\end{aligned}
$$

It is turned out that $\mathbf{D}_m\mathbf{G}_m$ is a skew-symmetric matrix, thus

$$\boldsymbol{w}_m^T\mathbf{D}_m\mathbf{G}_m\boldsymbol{w}_m = 0.$$

This ends the proof. □

Since the Coulomb force provides an acceleration on the mean velocity only, which is the term $\frac{q}{m}E_i$ in $\mathbf{G}_1$, the Lorentz force in $\mathbf{G}_1$ exerting on the mean velocity is perpendicular to the mean velocity clearly. The result in Theorem 2 indicates that the Lorentz force will not change the magnitude of the high order moments for any order $m \geq 2$, too, taking the matrix $\mathbf{D}_m$ as the l_2 weight. One may observe that the Lorentz force in the Vlasov equation will rotate the distribution function in the velocity space only, and here we see such behavior is preserved in the moment system we derived.

5 An Exact Vlasov-Maxwell Equilibrium

Knowledge of the exact Vlasov-Maxwell equilibrium is often necessary when analyzing the stability of a plasma [9]. In this section, we present a simple example of exact Vlasov-Maxwell equilibrium. This makes it possible to examine the residual of the moment system if we substitute the exact solution into the moment system. The moment system we derived is then partially validated once it is observed that the residual of the system is very small.

For simplicity, we consider a situation in which all quantities vary only in the x_1 direction, and the magnetic field has components B_2 and B_3 in the x_2 and x_3 directions. The equilibrium is characterized by a zero electric field. We require exact charge neutrality,

$$n_i(x_1) = n_e(x_1), \tag{63}$$

where n_i and n_e denote the ion and electron number densities.

The magnetic field can be derived from a vector potential, $\mathbf{A}$, and

$$
\begin{aligned}
B_3 &= \frac{\mathrm{d}A_2}{\mathrm{d}x_1},\\
B_2 &= -\frac{\mathrm{d}A_3}{\mathrm{d}x_1}.
\end{aligned} \tag{64}
$$

Maxwell's equations (3) for the magnetic field become

$$\begin{aligned}\frac{\mathrm{d}^2 A_2}{\mathrm{d}x_1^2} &= -\mu_0 j_2,\\ \frac{\mathrm{d}^2 A_3}{\mathrm{d}x_1^2} &= -\mu_0 j_3,\end{aligned} \tag{65}$$

where $\boldsymbol{j}(x_1)$ is the current density. The constants of the motion for particles of species, $s(s = i$ or $e)$, are the Hamiltonian

$$H_s = \frac{m_s|\boldsymbol{p}|^2}{2}, \tag{66}$$

and the x_2 and x_3 components of momentum,

$$\begin{aligned}P_{2,s} &= m_s p_2 + q_s A_2,\\ P_{3,s} &= m_s p_3 + q_s A_3.\end{aligned} \tag{67}$$

To find a self-consistent equilibrium, we must solve the coupled Vlasov-Maxwell equations. The Vlasov equations (1) are easily satisfied by making the distribution functions depend only on constants of the motion. We assume it is of the form

$$f_s = \exp(-\beta_s H_s) g_s(P_{2,s}, P_{3,s}), \tag{68}$$

where the β_s are constants and the g_s are functions that satisfy (see details in [9])

$$\frac{1}{m_s^2}\left(\frac{2\pi}{m_s\beta_s}\right)^{1/2}\int \exp\left\{-\frac{\beta_s}{2m_s}\left[(P_2 - q_s A_2)^2 + (P_3 - q_s A_3)^2\right]\right\} g_s(P_2, P_3)\,\mathrm{d}P_2\,\mathrm{d}P_3$$
$$= \frac{\beta_e\beta_i U(A_2, A_3)}{\mu_0(\beta_e + \beta_i)}.$$

Consider a situation in which the magnetic field is unidirectional, we can take $A_3 = 0$. Let us assume that

$$U(A_2) = D\exp\left(-\gamma q_s^2 A_2^2\right), \tag{69}$$

where D and γ are constants, so that the potential now resembles a "hill". If we let

$$\delta_s = \frac{1}{4\gamma} - \frac{m_s}{2\beta_s}, \tag{70}$$

then the distribution functions are given by

$$f_s(H_s, P_{2,s}) = \frac{m_s^2\beta_s n_0}{4\pi^2}\left(\frac{\pi\beta_s}{2\gamma\delta_s m_s}\right)^{1/2}\exp\left(-\frac{P_{2,s}^2}{4\delta_s} - \beta_s H_s\right), \tag{71}$$

where n_0 is the density at $x_1 = +\infty$.

Using the expression of $f_s(x_1, \boldsymbol{p})$, we obtain density $n_s(x_1)$ by (4), mean velocity $\boldsymbol{u}_s(x_1)$ by (5) and temperature $\mathcal{T}_s(x_1)$ by (13)

$$n_s = \int_{\mathbb{R}^3} f_s(t, x_1, \boldsymbol{p})\,\mathrm{d}\boldsymbol{p} = n_0\exp\left(-\gamma q_s^2 A_2^2\right), \tag{72}$$

$$\boldsymbol{u}_s = \frac{1}{n_s}\int_{\mathbb{R}^3} \boldsymbol{p} f_s(t, x_1, \boldsymbol{p})\,\mathrm{d}\boldsymbol{p} = \left(0, \frac{-2\gamma q_s}{\beta_s}A_2, 0\right), \tag{73}$$

$$\mathcal{T}_s = \frac{1}{3n_s}\int_{\mathbb{R}^3} (\boldsymbol{p} - \boldsymbol{u}_s)^2 f_s(t, x_1, \boldsymbol{p})\,\mathrm{d}\boldsymbol{p} = \left(\frac{1}{m_s\beta_s} - \frac{2\gamma}{3\beta_s^2}\right). \tag{74}$$

And the pressure tensor $P_s(x_1)$ by (23)

$$
\begin{aligned}
P_{ij,s} &= \int_{\mathbb{R}^3} (p_i - u_{i,s})(p_j - u_{j,s}) f_s \, \mathrm{d}\boldsymbol{p} = 0, \ \text{if } i \neq j, && (75)\\
P_{11,s} &= \frac{n_s}{m_s \beta_s} \triangleq n_s \mathcal{T}_{1,s}, && (76)\\
P_{22,s} &= \frac{4\delta_s \gamma}{m_s \beta_s} n_s = \left(\frac{1}{m_s \beta_s} - \frac{2\gamma}{\beta_s^2} \right) n_s \triangleq n_s \mathcal{T}_{2,s}, && (77)\\
P_{33,s} &= \frac{n_s}{m_s \beta_s} \triangleq n_s \mathcal{T}_{3,s}, && (78)\\
\mathcal{T}_s &= \frac{1}{3n_s} \left(P_{11,s} + P_{22,s} + P_{33,s} \right) = \frac{1}{3} \left(\mathcal{T}_{1,s} + \mathcal{T}_{2,s} + \mathcal{T}_{3,s} \right). && (79)
\end{aligned}
$$

Then we have

$$
f_s(x_1, \boldsymbol{p}) = \frac{n_s}{\sqrt{(2\pi)^3 \mathcal{T}_{1,s} \mathcal{T}_{2,s} \mathcal{T}_{3,s}}} \exp\left(-\frac{(p_1 - u_{1,s})^2}{2\mathcal{T}_{1,s}} - \frac{(p_2 - u_{2,s})^2}{2\mathcal{T}_{2,s}} - \frac{(p_3 - u_{3,s})^2}{2\mathcal{T}_{3,s}} \right). \quad (80)
$$

Let us calculate the moments and plug them into our moment equations and find the residual is going to zero as the order goes to infinity. We expand $f_s(\boldsymbol{x}, \boldsymbol{p})$ into the Hermite series

$$
f_s(\boldsymbol{x}, \boldsymbol{p}) = \sum_{\alpha} f_{\alpha,s} \mathcal{H}_{\mathcal{T}_s,\alpha} \left(\frac{\boldsymbol{p} - \boldsymbol{u}_s}{\sqrt{\mathcal{T}_s(\boldsymbol{x})}} \right). \quad (81)
$$

The expression of $f_{\alpha,s}$ can be calculated as

$$
f_{\alpha,s} = \begin{cases} \displaystyle\prod_{d=1}^{3} \frac{(\mathcal{T}_{d,s} - \mathcal{T}_s)^{k_d}}{(2k_d)!!} n_s, & \text{if } \alpha = (2k_1, 2k_2, 2k_3), k_i = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases} \quad (82)
$$

It is clear that

$$
\lim_{|\alpha| \to \infty} f_{\alpha,s} = 0, \quad (83)
$$

Actually all the moment equations which are not modified by closure are satisfied by the moments of $f_s(\boldsymbol{x}, \boldsymbol{p})$. We need only examine the moment equations of order $|\alpha| = M$ in the regularized moment system of order M which has been modified due to the truncation and closure. Substituting the exact moments into the regularized moment system and calculating the residual yields

$$
\mathrm{Res}(\alpha) = \sum_{j=1}^{3} \mathcal{R}_M^j(\alpha) + \sum_{j=1}^{3} (\alpha_j + 1) \frac{\partial f_{\alpha+e_j,s}}{\partial x_j} \quad (84)
$$

where the closure term $\mathcal{R}_M^j(\alpha)$ is defined in (49) and the truncation term $\sum_{j=1}^{3}(\alpha_j + 1)\dfrac{\partial f_{\alpha+e_j,s}}{\partial x_j}$ is easy to be identified by observing the original moment equation (40). The residue (84) is reduced into

$$
\begin{aligned}
\mathrm{Res}(\alpha) &= (\alpha_1+1)\left(\frac{\mathrm{d}u_{2,s}}{\mathrm{d}x_1} f_{\alpha-e_2+e_1,s} + \frac{\partial f_{\alpha+e_1,s}}{\partial x_1}\right) \\
&= \begin{cases}
\dfrac{(\mathcal{T}_{1,s}-\mathcal{T}_s)^{\frac{\alpha_1+1}{2}} (\mathcal{T}_{2,s}-\mathcal{T}_s)^{\frac{\alpha_2-1}{2}} (\mathcal{T}_{3,s}-\mathcal{T}_s)^{\frac{\alpha_3}{2}}}{(\alpha_1-1)!!(\alpha_2-1)!!\alpha_3!!} n_s \dfrac{\mathrm{d}u_{2,s}}{\mathrm{d}x_1}, & \text{if } \alpha = (2k_1-1, 2k_2+1, 2k_3), k_i > 0 \\
\dfrac{(\mathcal{T}_{1,s}-\mathcal{T}_s)^{\frac{\alpha_1+1}{2}} (\mathcal{T}_{2,s}-\mathcal{T}_s)^{\frac{\alpha_2}{2}} (\mathcal{T}_{3,s}-\mathcal{T}_s)^{\frac{\alpha_3}{2}}}{(\alpha_1-1)!!\alpha_2!!\alpha_3!!} \dfrac{\mathrm{d}n_s}{\mathrm{d}x_1}, & \text{if } \alpha = (2k_1-1, 2k_2, 2k_3), \quad k_i > 0 \\
0, & \text{others.}
\end{cases}
\end{aligned}
$$

The residue goes to zero as the truncation order M going to infinity, i.e.,

$$\lim_{|\alpha|\to\infty} \mathrm{Res}(\alpha) = 0. \tag{85}$$

6 Conclusion

We extend the moment closure method [5] for the Boltzmann equation to the Vlasov-Maxwell system and a hyperbolic moment system is derived. The systems for arbitrary number of moments are obtained at once by a systematic approach. We are developing the numerical method for the moment system obtained.

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