A REMARK ON GEODESIC GEOMETRY OF TEICHMÜLLER SPACES

LI ZHONG

Abstract. Let $\mathcal{T}(S)$ be the Teichmüller space of a hyperbolic Riemann surface S. In this paper, it is shown that, if μ is an extremal Beltrami differential on S of landslide-type, then there exist infinitely many geodesic rays, all of which tangent to each other at the basepoint of $\mathcal{T}(S)$ but intersect at $[\mu]$.

§1. Introduction

Let $\mathcal{T}(S)$ be the Teichmüller space of a hyperbolic Riemann surface S and let Belt(S) be the Banach space of bounded measurable Beltrami differentials $\mu = \mu\overline{dz}/dz$ on S with L_∞-norms. Suppose $\mathcal{M}(S)$ is the open unit ball of Belt(S).

For any μ in $\mathcal{M}(S)$, f^μ stands for a quasiconformal mapping of S onto $f^\mu(S)$, whose complex dilatation is μ. Two elements μ and ν in $\mathcal{M}(S)$ are said to be Teichmüller equivalent, denoted by $\mu \sim \nu$, if and only if, there exits a conformal mapping ψ of $f^\mu(S)$ onto $f^\nu(S)$ such that $(f^\nu)^{-1} \circ \psi \circ f^\mu$ is homotopic to the identity mapping (Mod ∂S). The Teichmüller space $\mathcal{T}(S)$ is defined as $\mathcal{M}(S)/\sim$, i.e.,

$$\mathcal{T}(S) := \{[\mu] : \mu \in \mathcal{M}(S)\},$$

where $[\mu]$ is the Teichmüller equivalence class of μ.

A Beltrami differential $\mu \in \mathcal{M}(S)$ is said to be extremal, if and only if,

$$\|\mu\|_\infty \le \|\mu'\|_\infty, \quad \forall\, \mu' \in [\mu].$$

As is well-known, $\mathcal{T}(S)$ is a complex manifold. When S is compact with genus $g > 1$, or more generally speaking, when S is of (g, n)-type with $3g - 3 + n > 0$, $\mathcal{T}(S)$ is finite-dimensional. Otherwise, it is infinite-dimensional.

$\mathcal{T}(S)$ has a natural metric $d_T(\cdot, \cdot)$, which is coincides with the Kobayashi metric and can be induced from a Finsler form.

The geometry of the Teichmüller metric has been studied by many authors, for example, [7],[1],[16],[2],[3],[9],[10] and many others, including some preprints [19],[14] and [17].

This paper is a further study on angels between two geodesic rays in Teichmüller spaces, which is firstly defined in [19].

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To state our question and results, we need some notation and terminologies. Suppose $\mu \in \mathcal{M}(S) \setminus \{0\}$ is extremal. Then the mapping [1]

$$\gamma_\mu : [0,1) \to \mathcal{T}(S); \quad r \mapsto [r\mu/\|\mu\|_\infty]$$

is an isometry embedding with respect to the Poincaré matric and the Teichmüller metric, respectively. We called γ_μ a *geodesic ray*. The restriction $\gamma_\mu|_{[0,\|\mu\|_\infty]}$ of γ_μ is called a *geodesic segment* between $[0]$ and $[\mu]$.

Suppose both μ and ν are extremal with

$$\|\mu\|_\infty = \|\nu\|_\infty = k \neq 0. \tag{1.1}$$

Following [19] (or see [12]), if the limit

$$\lim_{r\to 0+} \frac{d_T([r\mu],[r\nu])}{r} \tag{1.2}$$

exists, the angel $<\gamma_\mu, \gamma_\nu>$ at $[0]$ between two geodesic rays γ_μ and γ_ν is defined as follows :

$$<\gamma_\mu, \gamma_\nu> := 2\arcsin\left(\frac{1}{2}\lim_{r\to 0+}\frac{d_T([r\mu],[r\nu])}{rk}\right). \tag{1.3}$$

For the reason why we definite the angle $<\gamma_\mu, \gamma_\nu>$ like this, refer to [12].

If $<\gamma_\mu, \gamma_\nu>= 0$, we say γ_μ is tangent to γ_ν at $[0]$, or say γ_μ and γ_ν are tangent to each other at $[0]$. Naturally, in this case, we also say geodesic segment $\gamma_\mu|_{[0,k]}$ is tangent to the geodesic segment $\gamma_\nu|_{[0,k]}$ at $[0]$, or say they are tangent to each other at $[0]$.

It is natural to ask the following question that was firstly proposed in [14]:

Are there two distinct geodesic rays γ_μ and γ_ν that tangent at $[0]$ but intersect at another point?

If $\mathcal{T}(S)$ is finite-dimensional, the answer to this question is no. However, for the infinite-dimensional case, it is yes. The following theorem provides an affirmative answer to this question: There exist infinitely many geodesic rays that start at $[0]$ and intersect at another point $[\mu]$ in $\mathcal{T}(S)$.

Theorem 1.1 *Suppose $\mu \in \mathcal{M}(S) \setminus \{0\}$ is extremal with the following property:*

$$\mu(z) \equiv 0, \quad \forall z \in U \tag{1.4}$$

where U is an open subset of S. Then there exits a family $\mathcal{F} = \{\nu_\alpha : \ 0 < \alpha < \delta\}$ of extremal Beltrami differentials ν_α in $[\mu]$ such that each geodesic rays γ_{ν_α} is tangent to γ_μ, namely

$$<\gamma_\mu, \gamma_{\nu_\alpha}>= 0, \quad \forall\ \alpha \in (0,\delta).$$

Moreover, if $\alpha \neq \alpha'$ and both of them are in (δ_0, δ), then geodesic ray γ_α is distinct from $\gamma_{\alpha'}$.

Remark 1.1: It is known that condition (1.4) implies the existence of infinitely many geodesic segments connecting $[0]$ and $[\mu]$ (see [18] or [11]). Now Theorem 1.1 tells us that condition (1.4) implies much more: such segments can be required to be tangent to each other.

Remark 1.2: For any infinite-dimensional Teichmüller space, one can easily show the existence of such a μ in Theorem 1.1. So such a geometric phenomenon appears in any infinite-dimensional Teichmüller space.

[1] Here we regard the interval $[0,1)$ as a non-Euclidean ray in the Poincaré disk $\mathbb{D}$.

Remark 1.3: By a result [20] of Z. Zhou, if there is an open set V of S such that

$$\sup\{|\mu|(z) : z \in V\} < \|\mu\|_\infty \tag{1.5}$$

then (1.4) holds for some open subset U of S. So condition (1.4) can be replaced by (1.5).

An extremal Beltrami differential μ with condition (1.5) is said to be of landslide type. So Theorem 1.1 holds for any extremal Beltrami differential of landslide type.

Remark 1.4 : Recently we got a preprint of paper [17] by Y-L. Shen and Y. Hu. In their interesting paper, it is shown that the limit (1.2) always exists for any Beltrami differentials μ and ν and

$$\lim_{r\to 0+} \frac{d_T([r\mu],[r\nu])}{r} = \sup_{\phi\in\mathcal{Q}_1(S)} \left| \int_S [\mu-\nu]\phi \right|,$$

where $\mathcal{Q}_1(S) := \{\phi \in \mathcal{Q}(S) : \|\phi\| = 1\}$. Moreover, they answered a question on triangles posted in [12], by showing the following fact: In an infinite-dimensional Teichmüller space, there is an equiangular triangle whose inner angle θ may take any given values in $[0, \pi]$. Now our paper investigates two-sided polygons whose sides are geodesic segments.

§2. Preliminary and notation

Without loss of generality, throughout this paper we assume that S is a Riemann surface whose universal covering surface is the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The corresponding covering map of S is denoted by $\pi : \mathbb{D} \to S$ and the covering transformation group of π is denoted by Γ.

For convenient sake, we identify S with $\mathbb{D}/\Gamma$ and all discussions on S are transformed to $\mathbb{D}$ with the action of Γ. For example, the Banach space Belt(S) is regarded as the Banach space of functions $\mu(z) \in L_\infty(\mathbb{D})$ that satisfy the following condition:

$$\mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z), \text{ for a.e. } z \in \mathbb{D} \text{ and } \forall\gamma \in \Gamma. \tag{2.1}$$

The ideal boundary ∂S of S is regarded as $\partial\mathbb{D}/\Gamma$.

With this agreement, for each element $\mu \in \mathcal{M}(S)$, there is a uniquely determined quasiconformal mapping of $\mathbb{D}$ onto itself that keeps 1, i and -1 fixed, whose complex dilatation is μ. Such a quasiconformal mapping is denoted by f^μ. Actually, f^μ represents a quasiconformal mapping of S onto $S^\mu := \mathbb{D}/\Gamma^\mu$ with the complex dilatation μ, where

$$\Gamma^\mu := \{f^\mu \circ \gamma \circ (f^\mu)^{-1} : \forall\gamma \in \Gamma\}.$$

Sometimes we need to deal with some other Riemann surface expect for S. In this case, a quasiconformal mapping of this Riemann surface onto another one is also expressed as a quasiconformal mapping of $\mathbb{D}$ onto itself that is compatible with a group.

It is known that a quasiconformal mapping $f^\mu : S \to S^\mu$ can be extended to $\overline{S} = S \cup \partial S$ as a homeomorphism of $\overline{S}$ onto $\overline{S^\mu}$. Such an extension of f^μ is denoted by $\hat{f}^\mu$.

With this notation, the Teichmüller equivalence can be simply expressed as follows: $\mu \sim \nu$, if and only if,

$$\hat{f}^{\mu}|_{\partial\mathbb{D}} = \hat{f}^{\nu}|_{\partial\mathbb{D}}\,.$$

Let μ be any element of $\mathcal{M}(S)$. As usually, $K(f^{\mu})$ denotes the maximal dilatation of f^{μ}, namely

$$K(f^{\mu}) := \frac{1+\|\mu\|_{\infty}}{1-\|\mu\|_{\infty}}.$$

In this paper, we define

$$K([\mu]) := \inf\{K(f^{\mu'}) : \mu' \sim \mu\},$$

and call it the extremal maximal dilatation of $[\mu]$. We also need the notation of the boundary dilatation. Let $h([\mu])$ be defined as

$$h([\mu]) := \inf\{h(\mu') : \mu' \sim \mu\},$$

where $h(\mu')$ is the boundary norm of μ', that is

$$h(\nu) := \inf_{E}\{\|\nu|_{S\setminus E}\|_{\infty}\},$$

where E ranges over all compact subsets of S.

By $\mathcal{Q}(S)$ we denote the Banach space of integrable holomorphic quadratic differentials $\phi = \phi(z)dz^2$ on S with L_1-norms $\|\phi\|$. According to our agreement that S is identified with $\mathbb{D}/\Gamma$, a holomorphic quadratic differential ϕ on S is regarded as a holomorphic function $\phi(z)$ on $\mathbb{D}$ that satisfies the following condition:

$$\phi(\gamma(z))[\gamma'(z)]^2 = \phi(z), \quad \forall\, \gamma \in \Gamma.$$

The norm $\|\phi\|$ will be written in both ways:

$$\|\phi\| = \int_S |\phi|, \quad \text{or} \quad \|\phi\| = \iint_{\Omega} |\phi(z)|dxdy \ (z = x+iy),$$

where Ω is a fundamental domain of Γ.

The Banach dual space of $\mathcal{Q}(S)$ is the tangent space to $\mathcal{T}(S)$ at $[0]$, which is usually called *the infinitesimal Teichmüller space* of S and denoted by $\mathfrak{B}(S)$. More precisely, two elements μ and ν of $\mathrm{Belt}(S)$ are called *infinitesimal Teichmüller equivalent*, denoted by $\mu \approx \nu$, if and only if,

$$\int_S (\mu-\nu)\phi = 0, \quad \forall \phi \in \mathcal{Q}(S).$$

The infinitesimal Teichmüller equivalence class of μ is denoted by $[\mu]_{\mathfrak{B}}$. Then $\mathfrak{B}(S)$ is defined as the quotient space $\mathrm{Belt}(S)/\approx$, namely $\mathfrak{B}(S) = \{[\mu]_{\mathfrak{B}} : \mu \in \mathrm{Belt}(S)\}$. $[\mu]_{\mathfrak{B}}$ has a standard sup norm:

$$\|[\mu]_{\mathfrak{B}}\| := \sup_{\phi\in\mathcal{Q}_1(S)} \left|\int_S \mu\phi\right|,$$

where $\mathcal{Q}_1(S) = \{\phi \in \mathcal{Q}(S) : \|\phi\| = 1\}$.

§3. Proof of Theorem 1.1

To show Theorem 1.1, we need some lemmas. The first lemma is a generalization of the Polygon Inequality of Reich-Strebel. For its proof, refer to [9].

Lemma 3.1 *Let $\tilde S$ be a hyperbolic Riemann surface and $\mathcal{T}(\tilde S)$ the Teichmüller space of $\tilde S$. Suppose $\tilde\sigma$ is a Beltrami differential on $\tilde S$ with $\|\tilde\sigma\| < 1$ and $[\tilde\sigma]$ is the Teichmüller equivalence class of $\tilde\sigma$ (Mod $\partial\tilde S$). Then we have*

$$K([\tilde\sigma]) \leq \sup_{\tilde\phi\in\mathcal{Q}_1(\tilde S)} \int_{\tilde S} \frac{\left|1+\tilde\sigma\frac{\tilde\phi}{|\tilde\phi|}\right|^2}{1-|\tilde\sigma|^2}|\tilde\phi|,$$

where $\mathcal{Q}_1(\tilde S) := \{\tilde\phi \in \mathcal{Q}(\tilde S) : \|\tilde\phi\| = 1\}$.

In what follows, we will use the notation "O" in the following sense: Suppose $f(r)$ and $g(r)$ are two complex valued functions of $r \in (0,1)$. We say $f(r) = O(|g(r)|)$ (as $r \to 0+$), if there are two constants $C(>0)$ and r_0 with $0 < r_0 < 1$ such that

$$|f(r)| \leq C|g(r)|, \quad \text{provided } 0 < r < r_0.$$

The constants C and r_0 are called *the constants contained in the "O"*.

The second lemma is a special case of the "*good approximations*" . For its proof, refer to [15].

Lemma 3.2 *Suppose $\{\sigma_r : r \in (0,1)\}$ is a family of elements in $\mathcal{M}(X)$ with the following condition*

$$\|\sigma_r\|_\infty \leq 3r \quad (0 < r < 1). \tag{3.1}$$

Then we have

$$|f^{\sigma_r}(z) - z| = O(r)\ (\text{as } r \to 0+), \quad \forall z \in \overline{\mathbb{D}} \tag{3.2}$$

[2]

and

$$\|\partial_z f^{\sigma_r} - 1\|_{L_2(\mathbb{D})} = O(r)\ (\text{as } r \to 0+). \tag{3.3}$$

The constants contained in the "O"s in (3.2) and (3.3) are universal.

Remark 3.1 : In the general case, (3.1) should be $|\sigma_r| \leq Mr$, where $M > 0$ is a constant. However, for our discussion below, $M = 3$ is good enough and in this case, the constants contained in "O" are universal.

As a consequence of (3.3) we have the following:

Corollary 3.1 *For any sequence $\{r_n\}$ in (0,1) with $r_n \to 0$ as $n \to \infty$, there is a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that*

$$\partial_z f^{r_{n_k}}(z) \to 1, \quad \text{for a. e. } z \in \Omega. \tag{3.4}$$

The third lemma is new version of the main inequality of Reich-Strebel (see [13]):

Lemma 3.3 *Let μ and ν be arbitrarily given two elements of $\mathcal{M}(X)$. Suppose κ is a Beltrami differential on $f^\mu(X)$, such that $f^\kappa \sim f^\nu \circ (f^\mu)^{-1}(\mathrm{Mod}\,\partial f^\mu(X))$.*

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[2] Here f^{σ_r} actually is a quasiconformal mapping of $\mathbb{D}$ onto $\mathbb{D}$ that is compactible with the group Γ (see §2).

Let τ be the Beltrami coefficient of $f^\kappa \circ f^\mu$. Then for any $\phi \in \mathcal{Q}(X)$ with $\|\phi\| = 1$, we have

$$1 \leq \int_X \frac{\left|1 - \mu \frac{\phi}{|\phi|}\right|^2}{1 - |\mu|^2} \frac{\left|1 - \kappa \circ f^\mu \Omega_\mu(\phi) \frac{\phi}{|\phi|}\right|^2}{1 - |\kappa \circ f^\mu|^2} \frac{\left|1 - \nu_1 \circ f^\tau \Omega_\tau(\phi) \frac{\phi}{|\phi|}\right|^2}{1 - |\nu_1 \circ f^\tau|^2} |\phi|,$$

where ν_1 is the Beltrami coefficient of $(f^\nu)^{-1}$,

$$\Omega_\mu(\phi) := \frac{\overline{\partial_z f^\mu}}{\partial_z f^\mu} \frac{1 - \overline{\mu\phi}/|\phi|}{1 - \mu\phi/|\phi|} \quad \text{and} \quad \Omega_\tau(\phi) := \frac{\overline{\partial_z f^\tau}}{\partial_z f^\tau} \frac{1 - \overline{\tau\phi}/|\phi|}{1 - \tau\phi/|\phi|}.$$

Proof of Theorem 1.1 . We divide our proof into four parts.

Part $\mathcal{A}$: Contraction of the family $\mathcal{F} := \{\nu_\alpha : \ 0 < \alpha < \delta\}$.

Suppose μ is the given Beltrami differential on S in Theorem 1.1; namely, $\mu \in \mathcal{M}(S) \setminus \{0\}$ is extremal and

$$\mu(z) \equiv 0, \quad \forall z \in U,$$

where U is an open subset of S.

Let $\pi : \mathbb{D} \to S$ be the covering mapping of S and Γ the covering transformation group of π. Suppose Ω is a fundamental domain of Γ. Without loss of generality, one may assume that $\pi^{-1}(U) \cap \Omega$ contains a disk

$$D := \{z \in \mathbb{D} : |z - z_0| < \rho\}$$

with $\overline{D} \subset \Omega$. So we have

$$\mu(z) \equiv 0, \quad \forall z \in D. \tag{3.5}$$

We look at the following function:

$$\eta_\alpha(z) := z + \alpha(z - z_0)(|z - z_0|^2 - \rho^2)|z - z_0|^2, \quad z \in D,$$

where α is a real parameter. A simple computation shows

$$\partial_{\overline{z}} \eta_\alpha = \alpha(z - z_0)^2(2|z - z_0|^2 - \rho^2)$$

and

$$\partial_z \eta_\alpha = 1 + \alpha(3|z - z_0|^2 - 2\rho^2)|z - z_0|^2.$$

Let $\tau_\alpha := \partial_{\overline{z}} \eta_\alpha / \partial_z \eta_\alpha$. Then we have

$$\tau_\alpha(z) = (z - z_0)^2 h_\alpha(|z - z_0|), \quad \forall z \in D_\alpha, \tag{3.6}$$

where h_α is a function of $r \in [0, \rho)$:

$$h_\alpha(r) := \frac{\alpha(2r^2 - \rho^2)}{1 + \alpha(3r^2 - 2\rho^2)}.$$

Now we assume that $0 < \alpha < \delta$ and δ is sufficiently small, such that

$$|\partial_z \eta_\alpha|(z) > \frac{1}{2} \quad \text{and} \quad |\partial_{\overline{z}} \eta_\alpha|(z) < \frac{\|\mu\|_\infty}{2}, \quad \forall z \in D.$$

This leads to

$$|\tau(z)| < \|\mu\|_\infty < 1 \quad \forall z \in D. \tag{3.7}$$

Clearly, η_α satisfies the following Beltrami equation

$$\partial_{\overline{z}} \eta_\alpha(z) = \tau_\alpha(z) \partial_z \eta_\alpha(z), \quad \forall z \in D.$$

On the other hand, by the definition of η_α, the restriction of η_α to ∂D is an identity mapping of ∂D. Therefore η_α is a quasiconformal mapping of D onto itself.

Now for each fixed α, we define a quasiconformal mapping g_α of Ω onto itself:

$$z \mapsto g_\alpha(z) := \begin{cases} \eta_\alpha(z), & \text{as } z \in D; \\ z, & \text{as } z \in \Omega \setminus D. \end{cases}$$

Let $\Gamma(D) := \cup_{\gamma\in\Gamma}\, \gamma(D)$ and $\mathfrak{D} := \Gamma(D)/\Gamma$. It is clear that g_α induces a quasiconformal mapping $\tilde{g}_\alpha$ of S onto itself, which is an identity mapping of $S \setminus \mathfrak{D}$. The complex dilatation of $\tilde{g}_\alpha$ is denoted by $\tilde{\tau}_\alpha$. Obviously, we have

$$\tilde{\tau}_\alpha(z) \equiv 0, \quad \forall z \in S \setminus \mathfrak{D}. \tag{3.8}$$

Now ν_α is defined to be the complex dilatation of $f^\mu \circ \tilde{g}_\alpha$. In the other words,

$$f^{\nu_\alpha} = f^\mu \circ \tilde{g}_\alpha. \tag{3.9}$$

From (3.7) we see that $\|\nu_\alpha|_{\mathfrak{D}}\|_\infty \le \|\mu\|_\infty$. On the other hand, $\nu_\alpha(z) = \mu(z)$ as $z \in S \setminus \mathfrak{D}$. So ν_α is extremal.

Then $F = \{\nu_\alpha \ : 0 < \alpha < \delta\}$ is the family of extremal Beltrami differentials. In the following parts of the proof, we will show each element ν_α in $\mathfrak{F}$ satisfies the requirements of Theorem 1.1.

Part $\mathcal{B}$: Proof of $\|[\mu - \nu_\alpha]_{\mathfrak{B}}\| = 0$.

In this part, we want to show

$$\|[\mu - \nu_\alpha]_{\mathfrak{B}}\| \equiv \sup_{\phi\in\mathcal{Q}_1(S)} \left| \int_S (\mu - \nu_\alpha)\phi \right| = 0, \tag{3.10}$$

where $\mathcal{Q}_1(S) := \{\phi \in \mathcal{Q}(S) : \|\phi\| = 1\}$.

By the chain rule of complex dilatations, from (3.9) we have

$$\nu_\alpha = \frac{\tilde{\tau}_\alpha + \mu \circ \tilde{g}_\alpha \omega_{\tilde{g}_\alpha}}{1 + \overline{\tilde{\tau}_\alpha} \mu \circ \tilde{g}_\alpha \omega_{\tilde{g}_\alpha}}, \tag{3.11}$$

where $\omega_{\tilde{g}_\alpha} = \overline{\partial_z \tilde{g}_\alpha} / \partial_z \tilde{g}_\alpha$.

Recalling the fact that $\tilde{g}_\alpha|_{S\setminus\mathfrak{D}}$ is an identity mapping, we see $\omega_{\tilde{g}_\alpha}|_{S\setminus\mathfrak{D}} = 1$. On the other hand, $\mu|_{\mathfrak{D}} \equiv 0$. So it follows from (3.11) that

$$\nu_\alpha(z) = \begin{cases} \tilde{\tau}_\alpha(z), & \text{as } z \in \mathfrak{D}; \\ \mu(z), & \text{as } z \in S \setminus \mathfrak{D}. \end{cases} \tag{3.12}$$

Because $\mu|_{\mathfrak{D}} \equiv 0$ and $\tilde{\tau}_\alpha(z) = 0$ when $z \in S \setminus \mathfrak{D}$, (3.12) leads to

$$\nu_\alpha(z) - \mu(z) = \tilde{\tau}_\alpha(z), \quad \forall z \in S. \tag{3.13}$$

In particular, we have

$$\nu_\alpha(z) - \mu(z) = \tau_\alpha(z), \quad \forall z \in \Omega. \tag{3.14}$$

Then it follows from (3.13) and (3.14) that

$$\begin{aligned}
&\sup_{\phi\in\mathcal{Q}_1(S)}\left|\int_S(\tilde{\nu}_\alpha-\mu)\phi\right| \\
&=\sup_{\phi\in\mathcal{Q}_1(\Omega)}\left|\iint_\Omega(\nu_\alpha(z)-\mu(z))\phi(z)\right|dxdy, \\
&=\sup_{\phi\in\mathcal{Q}_1(\Omega)}\left|\iint_{D_\alpha}\tau_\alpha(z)\phi(z)\right|dxdy,
\end{aligned} \tag{3.15}$$

where $\mathcal{Q}_1(\Omega)$ is the set of all local expressions in terms of parameters in Ω of elements in $\mathcal{Q}_1(S)$.

Let ϕ be any elementary in $\mathcal{Q}_1(\Omega)$ and let its restriction to D be

$$\phi|_D(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

We have

$$\begin{aligned}
&\iint_D\tau_\alpha(z))\phi|_D(z)dxdy \\
&=\iint_D h_\alpha(|z-z_0|)\sum_{n=0}^{\infty}a_n(z-z_0)^{n+2}dxdy \\
&=\int_0^\rho h_\alpha(r)rdr\int_0^{2\pi}\sum_{n=0}^{\infty}a_nr^ne^{i(n+2)\theta}d\theta=0.
\end{aligned} \tag{3.16}$$

Then (3.10) follows from (3.15) and (3.16).

Part $\mathcal{C}$: Proof of $<\gamma_\mu,\gamma_{\nu_\alpha}>=0$.

Now we are going to show $<\gamma_\mu,\gamma_{\nu_\alpha}>=0$ by using (3.10).

Remak 3.2 : If one uses the result of [17], the conclusion $<\gamma_\mu,\gamma_{\nu_\alpha}>=0$ can be gotten directly from (3.10). However, so far [17] has not published yet. For the completeness of this paper, here we give a proof that is different from [17].

Suppose S_r is the Riemann surface $f^{r\mu}(S)$ and $\sigma_{\alpha,r}$ is the complex dilatation of $f^{r\nu_\alpha}\circ(f^{r\mu})^{-1}$. Then $\sigma_{\alpha,r}$ is a Beltrami differential on S_r and

$$f^{\sigma_{\alpha,r}}=f^{r\nu_\alpha}\circ(f^{r\mu})^{-1}. \tag{3.17}$$

Let $K([\sigma_{\alpha,r}])$ be the extremal maximal dilatation of $[\sigma_{\alpha,r}]$. Then the Teichmüller distance between $[r\mu]$ and $[r\nu_\alpha]$ is

$$d_T([r\mu],[r\nu_\alpha])=\frac{1}{2}\log K([\sigma_{\alpha,r}]).$$

Now we apply Lemma 3.1 with the following notation changes: $\tilde{S}$ and $\tilde{\sigma}$ in Lemma 3.1 are replaced by S_r and $\sigma_{\alpha,r}$, respectively. Then we get

$$K([\sigma_{\alpha,r}])\leq\sup_{\phi_r\in\mathcal{Q}_1(S_r)}\int_{S_r}\frac{\left|1+\sigma_{\alpha,r}\frac{\phi_r}{|\phi_r|}\right|^2}{1-|\sigma_{\alpha,r}|^2}|\phi_r|, \tag{3.18}$$

where $\mathcal{Q}_1(S_r):=\{\phi_r\in\mathcal{Q}(S_r):\|\phi_r\|=1\}$.

By the chain rule of complex dilatations and (3.17), we have

$$\sigma_{\alpha,r} \circ f^{r\mu} = \frac{r\nu_\alpha - r\mu}{1 - r^2\overline{\mu}\nu_\alpha}\overline{\omega_{f^{r\mu}}},$$

where $\omega_{f^{r\mu}} = \overline{\partial_z f^{r\mu}}/\partial_z f^{r\mu}$. Then we get

$$\|\sigma_{\alpha,r}\|_\infty \leq \frac{2r}{1-r^2} \quad (0 < r < 1),$$

and hence, from (3.18),

$$\begin{aligned} K([\sigma_{\alpha,r}]) &\leq \sup_{\phi \in \mathcal{Q}_1(S_r)} \int_{S_r} \left|1 + \sigma_{\alpha,r}\frac{\phi_r}{|\phi_r|}\right|^2 |\phi_r| + O(r^2) \\ &= 1 + 2 \sup_{\phi \in \mathcal{Q}_1(S_r)} \mathrm{Re} \int_{S_r} \sigma_{\alpha,r}\phi_r + O(r^2) \text{ (as } r \to 0+), \end{aligned}$$

where the constants contained in the "O"s here are universal. Then a simple computation shows

$$\begin{aligned} 0 \leq d_T([r\mu], [r\nu_\alpha]) &\leq \frac{1}{2}\log[1 + (K([\sigma_{\alpha,r}]) - 1)] \\ &\leq \frac{1}{2}(K([\sigma_{\alpha,r}]) - 1) \\ &\leq \sup_{\phi_r \in \mathcal{Q}_1(S_r)} \mathrm{Re} \int_{S_r} \sigma_{\alpha,r}\phi_r + O(r^2)(\text{as } r \to 0+). \end{aligned} \tag{3.19}$$

Let Ω be the fundamental domain of Γ which is the same as in Part $\mathcal{A}$. Let $\Omega_r := f^{r\mu}(\Omega)$. It is a fundamental domain of the group

$$\Gamma_r := \{f^{r\mu} \circ \gamma \circ (f^{r\mu})^{-1} : \ \forall \gamma \in \Gamma\}.$$

Then (3.19) can be rewritten as

$$\begin{aligned} &0 \leq d_T([r\mu], [r\nu_\alpha]) \\ &\leq \sup_{\phi_r \in \mathcal{Q}_1(\Omega_r)} \mathrm{Re} \iint_{\Omega_r} \sigma_{\alpha,r}(\zeta)\phi_r(\zeta)d\xi d\eta + O(r^2)(\text{as } r \to 0+), \end{aligned} \tag{3.20}$$

where $\mathcal{Q}_1(\Omega_r)$ is the set of local expressions in Ω_r of all $\phi_r \in \mathcal{Q}_1(S_r)$. The constants contained in the "O" here are universal.

It is easy to see the $f^{r\mu}$ is a good approximation of the identity mapping. By Lemma 3.2, we see

$$|f^{r\mu}(z) - z| = O(r) \quad (\text{as } r \to 0+), \forall z \in \mathbb{D},$$

where the constants contained in the "O" are universal. Then we have

$$\begin{aligned} \sigma_{\alpha,r} \circ f^{r\mu}(z) &= \frac{r\nu_\alpha(z) - r\mu(z)}{1 - r^2\overline{\mu}(z)\nu_\alpha(z)}\overline{\omega_{f^{\mu_r}}(z)} \\ &= r[\nu_\alpha(z) - \mu(z)]\overline{\omega_{f^{r\mu}}}(z) + O(r^2) \text{ (as } r \to 0+), \end{aligned}$$

where $\omega_{f^{r\mu}} = \overline{\partial_z f^{r\mu}}/\partial_z f^{r\mu}$ and the constants contained in the "O" are universal.

Noting the facts that $\nu_\alpha(z)-\mu(z)=0$ when $z\in\Omega\setminus D$ and $\mu(z)=0$ when $z\in D$, we have

$$\begin{aligned}
&\iint_{\Omega_r}\sigma_r(z)\phi_r(z)dxdy=\iint_{\Omega}\sigma_r\circ f^{r\mu}(z)\phi_r\circ f^{r\mu}(z)J_r(z)dxdy\\
&=\iint_D r\nu_\alpha(z)\overline{\omega_{f^{r\mu}}}(z)\phi_r\circ f^{r\mu}(z)J_r(z)dxdy\\
&\quad+O(r^2)\quad(\text{as } r\to 0+),
\end{aligned}\tag{3.21}$$

where $J_r=|\partial_z f^{r\mu}|^2-|\partial_{\overline{z}}f^{r\mu}|^2$. Then it follows from (3.20) and (3.21) that

$$\begin{aligned}
&0\le\frac{d_T([r\mu],[r\nu_\alpha])}{r}\\
&\le\sup_{\phi_r\in\mathcal{Q}_1(\Omega_r)}\operatorname{Re}\iint_D\nu_\alpha(z)\overline{\omega_{f^{\mu_r}}}(z)\phi_r\circ f^{\mu_r}(z)J_r(z)dxdy\\
&\quad+O(r)\quad(\text{as } r\to 0+).
\end{aligned}\tag{3.22}$$

The constants contained in the "O" here are universal.

Now we choose a sequence $\{r_n\}$ in $(0,1)$ with $r_n\to 0+$ (as $n\to\infty$) such that

$$\limsup_{r\to 0+}\frac{d_T([r\mu],[r\nu_\alpha])}{r}=\lim_{n\to\infty}\frac{d_T([r_n\mu],[r_n\nu_\alpha])}{r_n}.$$

Then from (3.22) we get

$$\frac{d_T([r_n\mu],[r_n\nu_\alpha])}{r_n}\le I_{r_n}+O(r_n\to 0+)\quad(\text{as } r_n\to 0+),\tag{3.23}$$

where

$$I_{r_n}:=\sup_{\phi_{r_n}\in\mathcal{Q}_1(\Omega_{r_n})}\operatorname{Re}\iint_D\nu_\alpha(z)\overline{\omega_{f^{r_n\mu}}}(z)\phi_{r_n}\circ f^{r_n\mu}(z)J_{r_n}(z)dxdy.$$

Now for each fixed r_n, we choose a $\psi_{r_n}\in\mathcal{Q}_1(\Omega_{r_n})$ such that

$$\operatorname{Re}\iint_D\nu_\alpha(z)\overline{\omega_{f^{r_n\mu}}}(z)\psi_{r_n}\circ f^{r_n\mu}(z)J_{r_n}dxdy>I_{r_n}-\frac{1}{n},\tag{3.24}$$

Noting the fact that the constants contained in "O" in (3.23) are universal, from (3.24) we get

$$\begin{aligned}
&\limsup_{r\to 0+}\frac{d_T([r\mu],[r\nu_\alpha])}{r}\\
&\le\limsup_{n\to\infty}\operatorname{Re}\iint_D\nu_\alpha(z)\overline{\omega_{f^{r_n\mu}}}(z)\psi_{r_n}\circ f^{r_n\mu}(z)J_{r_n}(z)dxdy.
\end{aligned}\tag{3.25}$$

Now we look at the family $\Phi:=\{\psi_{r_n}(z):\ z\in\Omega\}$. For any open subset V of Ω with $\overline{V}\subset\Omega$, when n is sufficiently, $f^{r_n\mu}(V)\subset\Omega$ and

$$\iint_V|\psi_{r_n}(z)|dxdy\le\int_{S_{r_n}}|\psi_{r_n}|=1.\tag{3.26}$$

This means that Φ is a normal family. We can choose a subsequence of $\{\psi_{r_n}\}$, which is uniformly convergent on any compact subset of Ω. Without loss of generality, we may assume that such a subsequence is $\{\psi_{r_n}\}$ itself.

We suppose the limit function of $\{\psi_{r_n}\}$ is ϕ_0. Then $\phi_0(z)$ is holomorphic on Ω. By the Fatou lemma and (3.26), we see

$$\iint_V |\phi_0(z)|dxdy \le 1.$$

for any open set V with $\overline{V} \subset \Omega$. This implies

$$\iint_\Omega |\phi_0(z)|dxdy \le 1. \tag{3.27}$$

Now we claim that

$$\psi_0(z) = \psi_0(\gamma(z))[\gamma'(z)]^2, \quad \forall\gamma \in \Gamma \ \&\ \forall z \in \Omega. \tag{3.28}$$

In fact, for any fixed $\gamma \in \Gamma$, let $\gamma_{r_n} = f^{r_n\mu} \circ \gamma \circ (f^{r_n\mu})^{-1}$. We have

$$\psi_{r_n}(z) = \psi_{r_n}(\gamma_{r_n}(z))[\gamma'_{r_n}(z)]^2 \quad (\forall z \in \Omega), \tag{3.29}$$

It is easy to check by using Lemma 3.2 that

$$\gamma_{r_n} \to \gamma \quad \text{and} \quad \gamma'_{r_n} \to \gamma' \text{ (as } n \to \infty).$$

Then we can get (3.28) by taking the limits of both sides in (3.29).

From (3.28) we see that $\phi_0(z)dz^2$ represents a quadratic differential on S.

Because ψ_{r_n} is locally uniformly convergent to ϕ_0 in Ω and $f^{r_n\mu}$ is a good proximation of the identity mapping, $\psi_{r_n\mu} \circ f^{r_n\mu}$ uniformly converges to ϕ_0 on D On the other hand, by Lemma 3.2, one may choose a subsequence of $\omega_{f^{r_n\mu}}$ and assume such a subsequence is $\omega_{f^{r_n\mu}}$ itself, such that

$$\omega_{f^{r_n\mu}}(z) \to 1 \quad (\text{as } r \to 0+) \quad \text{for a.e. } z \in D.$$

Using Lemma 3.2 again, it is easy to see that

$$\begin{aligned}&\iint_{\mathbb{D}} |J_{r_n}(z) - 1|dxdy \\ =& \iint_{\mathbb{D}} |\partial_z f^{r_n}(z)|^2 - 1 + |r_n\mu(z)|^2\| dxdy \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Similarly as above, choosing a subsequence of J_{r_n} and assuming such subsequence is $J_{f^{r_n}}$ itself, we may assume

$$J_{r_n}(z) \to 1 \quad (\text{as } r \to 0+) \quad \text{for a.e. } z \in D.$$

By using the Lebesgue Theorem, we get

$$\lim_{n\to\infty} \iint_D |\nu_\alpha(z)[\overline{\omega_{f^{r_n\mu}}}(z)\psi_{r_n} \circ f^{r_n\mu}(z)J_{r_n}(z) - \phi_0(z)]|dxdy = 0,$$

which implies

$$\begin{aligned}&\lim_{n\to\infty} \operatorname{Re} \iint_D \nu_\alpha(z)\overline{\omega_{f^{r_n\mu}}}(z)\psi_{r_n} \circ f^{r_n\mu}(z)J_{r_n}(z)dxdy \\ =& \lim_{n\to\infty} \operatorname{Re} \iint_D \nu_\alpha(z)\phi_0(z)dxdy.\end{aligned} \tag{3.30}$$

Then it follows from (3.25) and (3.30) that

$$\limsup_{r\to 0+} \frac{d_T([r\mu],[r\nu_\alpha])}{r} \le \operatorname{Re} \iint_D \nu_\alpha(z)\phi_0(z)dxdy. \tag{3.31}$$

In Party $\mathcal{B}$, we have shown

$$\iint_D \nu_\alpha(z)\phi_0(z)dxdy = 0.$$

So it follows from (3.31) that

$$\limsup_{r\to 0+} \frac{d_T([r\mu],[r\nu_\alpha])}{r} = 0,$$

which clearly implies

$$\lim_{r\to 0+} \frac{d_T([r\mu],[r\nu_\alpha])}{r} = 0.$$

By the definition of angles, we have $<\gamma_\mu, \gamma_{\nu_\alpha}>= 0$. This means each γ_{ν_α} is tangent to γ_μ at $[0]$.

Part $\mathcal{D}$: Proof of the conclusion that $\gamma_{\nu_\alpha} \neq \gamma_{\nu_{\alpha'}}$ $(\alpha \neq \alpha')$.

To complete the proof of Theorem 1.1, we need to show that, if $\alpha \neq \alpha'$, the geodesic ray γ_α is distinct from $\gamma_{\alpha'}$. The proof is based on Lemma 3.3, i.e., the generalized main inequality of Reich-Strebel [13].

Now we apply Lemma 3.3 with the following notation changes: μ and ν are replaced by $r\nu_\alpha$ and $r\nu_{\alpha'}(0<r<1)$, respectively. Let ϕ be any given element in $\mathcal{Q}(S)$ with $\|\phi\|=1$. Then we have

$$\begin{aligned}1 \le \iint_\Omega &\frac{\left|1-r\nu_\alpha\frac{\phi}{|\phi|}\right|^2}{1-r^2|\nu_\alpha|^2}\frac{\left|1-\kappa_r\circ f^{r\nu_\alpha}\Omega_{r\nu_\alpha}(\phi)\frac{\phi}{|\phi|}\right|^2}{1-|\kappa_r\circ f^{r\nu_\alpha}|^2}\\ &\times \frac{\left|1-\nu_{r,1}\circ f^{\tau_r}\Omega_{\tau_r}(\phi)\frac{\phi}{|\phi|}\right|^2}{1-|\nu_{1,r}\circ f^{\tau_r}|^2}|\phi|dxdy,\end{aligned} \tag{3.32}$$

where κ_r,τ_r, $\Omega_{r\nu_\alpha}(\phi)$ and $\Omega_{\tau_r}(\phi)$ are the corresponding terms of κ, τ, $\Omega_\mu(\phi)$ and $\Omega_\nu(\phi)$ in Lemma 3.3, respectively, and $\nu_{1,r}$ is the complex dilatation of $(f^{r\nu_{\alpha'}})^{-1}$.

Noting the fact that $|\Omega_{\kappa_r}(\phi)|=1$, we have

$$\frac{\left|1-\kappa_r\circ f^{r\nu_\alpha}\Omega_{r\nu_\alpha}(\phi)\frac{\phi}{|\phi|}\right|^2}{1-|\kappa_r\circ f^{r\nu_\alpha}|^2} \le K(f^{\kappa_r}).$$

Then from (3.32) we get

$$\frac{1}{K(f^{\kappa_r})} \le \iint_\Omega \frac{\left|1-r\nu_\alpha\frac{\phi}{|\phi|}\right|^2}{1-r^2|\nu_\alpha|^2}\frac{\left|1-\nu_{r,1}\circ f^{\tau_r}\Omega_{\tau_r}(\phi)\frac{\phi}{|\phi|}\right|^2}{1-|\nu_{1,r}\circ f^{\tau_r}|^2}|\phi|dxdy,$$

which implies

$$\frac{1}{K([\kappa_r])} \le \iint_\Omega \frac{\left|1-r\nu_\alpha\frac{\phi}{|\phi|}\right|^2}{1-r^2|\nu_\alpha|^2}\frac{\left|1-\nu_{r,1}\circ f^{\tau_r}\Omega_{\tau_r}(\phi)\frac{\phi}{|\phi|}\right|^2}{1-|\nu_{1,r}\circ f^{\tau_r}|^2}|\phi|dxdy,$$

where $K([\kappa_r])$ is the extremal maximal dilatation of $[\kappa_r]$.

Let

$$L_r := \iint_\Omega \frac{\left|1-r\nu_\alpha\frac{\phi}{|\phi|}\right|^2}{1-r^2|\nu_\alpha|^2}\frac{\left|1-\nu_{r,1}\circ f^{\tau_r}\Omega_{\tau_r}(\phi)\frac{\phi}{|\phi|}\right|^2}{1-|\nu_{1,r}\circ f^{\tau_r}|^2}|\phi|dxdy.$$

Then we have

$$\frac{1}{K([\kappa_r])} \le L_r. \tag{3.33}$$

Now we take κ_r to be the complex dilatation of $f^{r\nu_{\alpha'}} \circ (f^{r\nu_\alpha})^{-1}$, namely

$$f^{\kappa_r} = f^{r\nu_{\alpha'}} \circ (f^{r\nu_\alpha})^{-1}. \tag{3.34}$$

According to the assumption in Lemma 3.3, τ_r should be the complex dilatation of $f^{\kappa_r} \circ f^{r\nu_\alpha}$. So from (3.34) we get $f^{\tau_r} = f^{\kappa_r} \circ f^{r\nu_\alpha} = f^{r\nu_{\alpha'}}$, namely

$$\tau_r = r\nu_{\alpha'}. \tag{3.35}$$

Because $\nu_{r,1}$ is the dilatation of $(f^{r\nu_{\alpha'}})^{-1}$, so we have

$$\nu_{r,1} \circ f^{r\nu'_\alpha} = -r\nu_{\alpha'}\overline{\omega_{f^{r\nu_{\alpha'}}}}. \tag{3.36}$$

Then we get

$$\nu_{r,1} \circ f^{r\nu_{\alpha'}} \Omega_{r\nu_{\alpha'}}(\phi)\frac{\phi}{|\phi|} = -r\nu_{\alpha'}\frac{1 - r\overline{\nu_{\alpha'}}\overline{\phi}/|\phi|}{1 - r\nu_{\alpha'}\phi/|\phi|}\frac{\phi}{|\phi|}.$$

A simple computation shows

$$\frac{1 - r\overline{\nu_{\alpha'}}\overline{\phi}/|\phi|}{1 - r\nu_{\alpha'}\phi/|\phi|} = 1 - r\overline{\nu_{\alpha'}} + r\nu_{\alpha'} + O(r^2) \quad (\text{as } r \to 0+),$$

So we have

$$\begin{aligned} &\nu_{r,1} \circ f^{r\nu_{\alpha'}} \Omega_{r\nu_{\alpha'}}(\phi)\frac{\phi}{|\phi|} \\ &= [-r\nu_{\alpha'} + r^2|\nu_{\alpha'}|^2 - r^2(\nu_{\alpha'})^2]\frac{\phi}{|\phi|} + O(r^3),\ (\text{as } r \to 0+). \end{aligned} \tag{3.37}$$

Then it follows from (3.35) to (3.37) that

$$\begin{aligned} L_r = &\iint_\Omega \frac{\left|1 - r\nu_\alpha\frac{\phi}{|\phi|}\right|^2}{1 - r^2|\nu_\alpha|^2}\frac{\left|1 + [r\nu_{\alpha'} + r^2|\nu_{\alpha'}|^2 - r^2(\nu_{\alpha'})^2]\frac{\phi}{|\phi|}\right|^2}{1 - r^2|\nu_{\alpha'}|^2}|\phi|dxdy \\ &+ O(r^3) \qquad (\text{as } r \to 0+). \end{aligned}$$

A further computation shows

$$\begin{aligned} L_r = &\iint_\Omega \frac{1 + r^2|\nu_\alpha|^2 + r^2|\nu'_\alpha|^2}{(1 - r^2|\nu_\alpha|^2)(1 - r^2|\nu_{\alpha'}|^2)}|\phi|dxdy \\ + &\iint_\Omega \frac{2\mathrm{Re}\left(r^2|\nu_{\alpha'}|^2\frac{\phi}{|\phi|}\right) - 2\mathrm{Re}\left\{\left[r(\nu_\alpha - \nu_{\alpha'}) - r^2(\nu_{\alpha'})^2\right]\frac{\phi}{|\phi|}\right\}}{(1 - r^2|\nu_\alpha|^2)(1 - r^2|\nu_{\alpha'}|^2)}|\phi|dxdy \\ + &O(r^3) \qquad (\text{as } r \to 0+). \end{aligned} \tag{3.38}$$

By the construction of $\mathfrak{F} = \{\nu_\alpha : 0 < \alpha < \delta\}$, we see that both $\nu_\alpha(z)$ and $\nu_{\alpha'}(z)$ are zero when z is in $\Omega \setminus D$. So we have

$$\begin{aligned} &\iint_\Omega \frac{r(\nu_\alpha - \nu_{\alpha'}) - r^2(\nu_{\alpha'})^2}{(1 - r^2|\nu_\alpha|^2)(1 - r^2|\nu_{\alpha'}|^2)}\,\phi\,dxdy \\ = &\iint_D \frac{r(\nu_\alpha - \nu_{\alpha'}) - r^2(\nu_{\alpha'})^2}{(1 - r^2|\nu_\alpha|^2)(1 - r^2|\nu_{\alpha'}|^2)}\,\phi\,dxdy. \end{aligned} \tag{3.39}$$

Noting the fact that

$$(1 - r^2|\nu_\alpha|^2(z))(1 - r^2|\nu_{\alpha'}|^2(z)) \quad (z \in D)$$

is a function of $|z-z_0|$ and and the facts that

$$\nu_\alpha = h_\alpha(|z-z_0|)(z-z_0)^2 \quad \text{and} \quad \nu_{\alpha'} = h_{\alpha'}(|z-z_0|)(z-z_0)^2,$$

similarly as done in Part $\mathcal{B}$, we have

$$\iint_D \frac{r(\nu_\alpha - \nu_{\alpha'})}{(1-r^2|\nu_\alpha|^2)(1-r^2|\nu_{\alpha'}|^2)}\,\phi\,dxdy = 0, \quad \forall r \in (0,1). \tag{3.40}$$

Noting the fact that $(\nu_{\alpha'})^2 = [h_{\alpha'}(|z-z_0|)]^2(z-z_0)^4$, the same discussion leads to

$$\iint_D \frac{r^2(\nu_{\alpha'})^2}{(1-r^2|\nu_\alpha|^2)(1-r^2|\nu_{\alpha'}|^2)}\,\phi\,dxdy = 0, \quad \forall r \in (0,1). \tag{3.41}$$

Then it follows from (3.38),(3.40) and (3.41) that

$$\begin{aligned} L_r = & \iint_\Omega \frac{1+r^2|\nu_\alpha|^2 + r^2|\nu'_\alpha|^2}{(1-r^2|\nu_\alpha|^2)(1-r^2|\nu_{\alpha'}|^2)}|\phi|dxdy \\ & + \iint_\Omega \frac{2r^2|\nu_{\alpha'}|^2\mathrm{Re}\left[\frac{\phi}{|\phi|}\right]}{(1-r^2|\nu_\alpha|^2)(1-r^2|\nu_{\alpha'}|^2)}|\phi|dxdy + O(r^3)\ (\text{as } r \to 0+). \end{aligned}$$

Then we have

$$\begin{aligned} 1 - L_r = \iint_\Omega & \frac{2r^2|\nu_\alpha|^2 + 2r^2|\nu_{\alpha'}|^2\left\{1 - \mathrm{Re}\left[\frac{\phi}{|\phi|}\right]\right\}}{(1-r^2|\nu_\alpha|^2)(1-r^2|\nu_{\alpha'}|^2)}|\phi| + O(r^3) \\ & (\text{as } r \to 0+). \end{aligned} \tag{3.42}$$

Because $1 - \mathrm{Re}(\phi/|\phi|) \geq 0$, it is easy from (3.42) to see

$$\lim_{r\to 0+} \frac{1-L_r}{r} = 0 \quad \text{and} \quad \lim_{r \to 0+} \frac{1-L_r}{r^2} > 0.$$

This implies

$$1 - L_r > 0, \quad \text{as } r(>0) \text{ is sufficiently small.} \tag{3.43}$$

However, from (3.33) we have

$$1 - \frac{1}{K([\kappa_r])} \geq 1 - L_r \quad \forall r \in (0,1). \tag{3.44}$$

Then it follows from (3.43) and (3.44) that

$$1 - 1/K([\kappa_r]) > 0, \quad \text{as } r(>0) \text{ is sufficiently small.}$$

In the other words, if $r(>0)$ is sufficiently small, $K([\kappa_r]) > 1$.

However $K([\kappa_r]) = e^{2d_T([r\nu_\alpha],[r\nu_{\alpha'}])}$. So what we have shown is

$$d_T([r\nu_\alpha],[r\nu_{\alpha'}]) > 0, \quad \text{as } r(>0) \text{ is sufficiently small.}$$

Therefore, γ_{ν_α} and $\gamma_{\nu_{\alpha'}}$ are distinct.

The proof of Theorem 1.1 is completed. □

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Z. Li: School of Mathematical Sciences, Peking University, Beijing 100871, China.
E-mail address: `lizhong@math.pku.edu.cn`