Modern Control Theory

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Chapter 4:

Controllability and Observability

4.1 Concept of Controllability and Observability

4.1.1 Definition of Controllability

For linear system $\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)u$, given the initial valve X(0) at instant t_0 , if there exist $t_a > t_0, t_a \in J$ (J is definition domain) ,and a admissible control u(t), such that make $x(t_a)=0$, then the system is controllable at $[t_0, t_\alpha]$

explanation:

(1) If a state is affected by input , it is controllable. A system is uncontrollable if any state variable is unaffected by input.

(2) X(0) is non-zero finite dot and $X(t_a)$ is origin of state space.

(3) u(t) must satisfy condition of solution's uniqueness.

(4) definition domain is a finite interval $[t_0, t_\alpha]$.

(5) Controllability is concept reflecting the ability of a system reaching any given state.

4.1.2 Definition of Observability

For linear system $\begin{cases} \dot{X} = A(t)X + B(t)u \\ y = C(t)X \end{cases}$,given $t_\alpha > t_0 \in J$.

if the initial valve X_0 can be uniquely determined according the measured valve y(t) of $[t_0, t_\alpha]$. Then the system is observable.

If a state affect the output , system is observable. A system is unobservable if any state variable does not appear in output equation. Obersvability studies the relation of state and output, That is , the identification of initial state.

Example:

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & -6 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = 4x_1 + u \\ \dot{x}_2 = -5x_2 + 2u \\ y = -6x_2 \end{cases}$$

State variables all associated with u, so the system is complete controllable. y reflect only x_2 , not x_1. so x_2 is observable , x_1 is unobservable .

return

4.2 Criteria of Controllability

4.2.1 First form of state controllability Criteria

Theorem1. System $\sum = (A, B)$ *or*

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

the necessary and sufficient condition of system being comp lete controllable is that the controllablity matrix

$$Q_k = [B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B]$$ has full-rank

or $$rankQ_k = rank[B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B] = n$$

Deduction of Theorem1：

The case of single input The complete controllable necessary and sufficient condition is that the controllability matrix

$$Q_k = [B \vdots AB \vdots A^2 B \vdots \cdots \vdots A^{n-1} B]$$

is regular matrix (nonsingular matrix), or the inverse matrix of Q_k exist ($|Q_k| \neq 0$)

The case of multi-input Q_k is not square matrix

$$rankQ_k = rankQ_k \cdot Q_k^T$$

$\left|Q_k Q_k^T\right| \neq 0$ is controllability criteria

Example 1: study the controllability of following system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-1 & -2 & -2\\ 0 & -1 & 1\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}u$$

Analysis:

$$\dot{x}_1 = -x_1 - 2x_2 - 2x_3 + 2u$$
$$\dot{x}_2 = -x_2 + x_3$$
$$\dot{x}_3 = x_1 - x_3 + u$$

From appearances , $x_1$and x_3 involve with control action u, x_2 do not associate with u visually , system looks like not complete controllable. But x3 associate with u , so system is complete controllable.

Sol:

$$B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, AB = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

$$A^2 B = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

$$rankQ_r = rank \begin{bmatrix} 2 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix} = 3$$

So that system is complete controllable.

Example 2: Given system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Judge the controllability.

Sol :

$$rank[B \quad AB \quad AB^2] = rank \begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ -1 & -1 & -2 & -2 & -4 & -4 \end{bmatrix}$$

$$= rank \begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 2 < 3$$

So system has no controllability, or not complete controllable.

If the number of row is less than the number of column, the following calculations are more convenient:

$$rank[B \vdots AB \vdots \cdots \vdots A^{n-1}B] = rank[(B \vdots AB \vdots \cdots \vdots A^{n-1}B)(B \vdots AB \vdots \cdots \vdots A^{n-1}B)^T]$$

$$= rank\left(\begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ -1 & -1 & -2 & -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ -1 & -1 & -2 & -2 & -4 & -4 \end{bmatrix}^T\right)$$

$$= rank\begin{bmatrix} 59 & 49 & 49 \\ 49 & 42 & 42 \\ -49 & -42 & -42 \end{bmatrix} = rank\begin{bmatrix} 59 & 49 & 49 \\ 49 & 42 & 42 \\ 0 & 0 & 0 \end{bmatrix} = 2 < 3$$

Example 3: Given system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

Judge the controllability.

Sol :

$$rank[B \quad AB \quad A^2B]=rank\begin{bmatrix}1 & 0 & 1 & 2 & 0 & 4\\ 0 & 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 & 4 & 2\end{bmatrix}=3$$

So system has controllability, or complete controllable.

Specially point out: when controllability matrix has full-rank , complete controllability matrix calculation is not needed.

$$rank[B \quad AB \quad] = rank\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = 3$$

4.2.2 Second form of state controllability Criteria

Theorem 2:

Suppose system has distinct eigenvalues $\lambda_1\ \lambda_2 \cdots\cdots \lambda_n$, the necessary and sufficient condition of system being complete controllable is :

$\hat{B}$ do not contain row with all 0 element in diagonal canonical form of state equation obtained by nonsingular transform

$$\dot{\hat{X}} = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix} \hat{X} + \hat{B}u$$

Example: Study systems controllability

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-7 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}-2\\ 5\\ 7\end{bmatrix}u$$

complete controllable

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-7 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}2\\ 0\\ 9\end{bmatrix}u$$

Not complete controllable,
x_2 is not controllable

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-7 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0 & 1\\ 4 & 0\\ 7 & 5\end{bmatrix}u$$

complete controllable

Theorem 3:

Suppose that system has repeated eigenvalue $\lambda_1(m_1-repeated)$,

$$\lambda_2(m_2-repeated)\cdots\cdots\ \lambda_k(m_k-repeated),\sum_{i=1}^{k}m_i=n,\lambda_i\neq\lambda_j(i\neq j)$$

the Jordan canonical form of state equation obtained by nonsi ngular transform is

$$\dot{\hat{X}}=\begin{bmatrix}J_1 & & & \\ & J_2 & & 0 \\ & & \ddots & \\ 0 & & & J_k\end{bmatrix}\hat{X}+\hat{B}u$$

The necessary and sufficient condition of system being complet e controllable is the row elements of $\hat{B}$ which correspond to th e last row of each Jordan block are not all 0

Example: Study systems controllability

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-4 & 1 & 0\\ 0 & -4 & 0\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 4\\ 3\end{bmatrix}u$$

complete controllable

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}-4 & 1 & 0 & 0\\ 0 & -4 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 2\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}u$$

complete controllable

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-4 & 1 & 0\\ 0 & -4 & 0\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}4 & 2\\ 0 & 0\\ 3 & 0\end{bmatrix}u$$

Not complete controllable, x_2 is not controllable

4.2.3 Third form of state controllability Criteria

4.2.3.1 Determining transfer function by

state space descripation

For SISO system, system equation is $\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu \\ y = C\mathbf{x} + Du \end{cases}$

Doing Laplace transform, suppose initial condition is 0

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$W(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$= C\frac{adj(sI - A)}{|sI - A|}B + D$$

$$D = 0, \quad W(s) = C\frac{adj(sI - A)}{\det(sI - A)}B$$

$$= \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Define state-input transfer function

$$(sI - A)^{-1}B$$

Define state-output transfer function

$$C(sI - A)^{-1}$$

Remark:

(1)The denominator polynomial of transfer function equal to the characteristic polynomial of matrix A,
(2)The poles of transfer function are eigenvalues of matrix A,
(3)The necessary and sufficient condition of system stability is that eigenvalues of matrix A have negative real part.

4.2.3.2 Criteria 3 of controllability

For SISO system , the necessary and sufficient condition of system being complete controllable is that state-input transfer function

$$(sI - A)^{-1} B$$

do not exist cancellation factor, or do not exist zero-pole cancellation phenomenon.

return

4.3 Criteria of Observability

4.3.1 First form of state observability Criteria

Theorem 1. System $\sum = (A, C)$ *or*

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

the necessary and sufficient condition of system being comp lete observables is that the observability matrix

$$Q_g = \left[C^T \vdots A^T C^T \vdots \cdots \vdots (A^T)^{n-1} C^T \right]$$ has full-rank

or
$$Q_g^T = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ has full-rank

The rank of observability matrix means the number of obse rvable state .

Observability is identification of initial state in essence .

The deduction of theorem 1:

The case of single output The necessary and sufficient condition of system being complete observable is that the observability matrix

$$Q_g = \left[C^T \vdots A^T C^T \vdots \cdots \vdots (A^T)^{n-1} C^T \right]$$

is regular matrix (nonsingular matrix), or the inverse matrix of Q_k exist ($|Q_g| \neq o$)

The case of multi-output Q_g is not square matrix.

$rankQ_g = rankQ_g Q_g^T$ $|Q_g Q_g^T| \neq 0$ is obversability criteria

Example 1: Given system equation

$$\begin{cases} \dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix} X \end{cases}$$

Judge the observability

Sol:

$$C = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} -6 & -7 & -1 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} -6 & -7 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 5 & -1 \end{bmatrix}$$

$$Q_k^T = \begin{bmatrix} 4 & 5 & 1 \\ -6 & -7 & -1 \\ 6 & 5 & -1 \end{bmatrix}$$

$$rankQ_g^T = rank\begin{bmatrix} 4 & 5 & 1 \\ -6 & -7 & -1 \\ 6 & 5 & -1 \end{bmatrix} = 2 < 3$$

(Rank is determined by column vector)

∴ system is not complete observable

Example 2: Given system equation

$$\dot{X} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-3} \end{bmatrix} X + \begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix} u$$

$$y = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} \end{bmatrix} X$$

Judge the observability..

Sol:

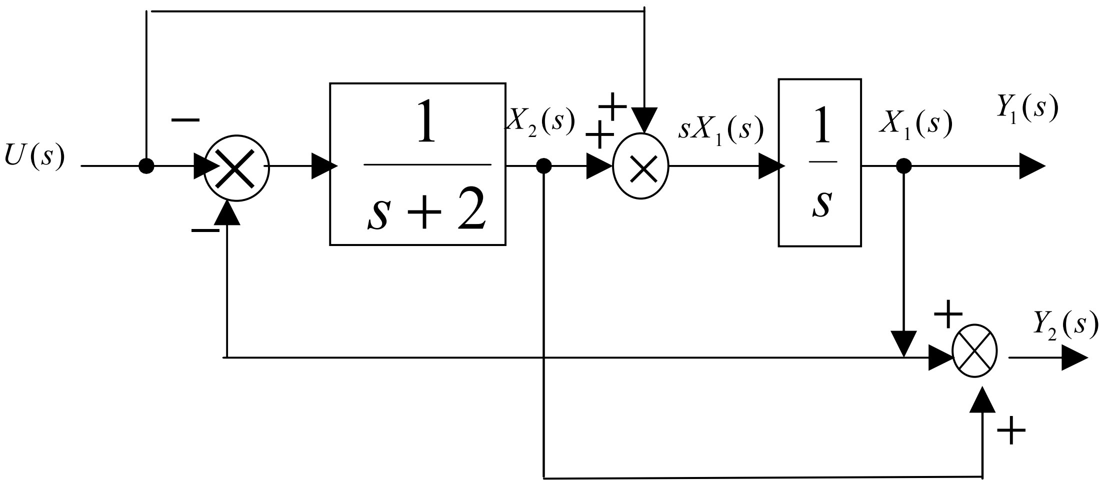
$$CA = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

$$Q_g^T = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 2 & -1 \\ -2 & 1 \end{bmatrix} \quad , \quad rank Q_g^T = 2$$

(Rank is determined by column vector)

∴ system is complete observable.

Example 3: Given system block diagram as follow:



Judge the controllability and observability

Sol:

$$sX_1(s) = X_2(s) + u$$

$$[-X_1(s) - U(s)]\frac{1}{s+2} = X_2(s)$$

$$Y_1(s) = X_1(s)$$

$$Y_2(s) = X_1(s) + X_2(s)$$

$$\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = -x_1 - 2x_2 - u \end{cases}$$

$$\begin{cases} y_1 = x_1 \\ y_2 = x_1 + x_2 \end{cases}$$

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x \end{cases}$$

$$Q_k = [B \vdots AB] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad rankQ_k = 1 < 2$$

$\therefore$ system is not complete controllable

$$Q_g^T = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$rankQ_g^T = 2$$

$\therefore$ system is complete observable.

4.3.2 Second form of state observability Criteria

Theorem 2:

Suppose system has distinct eigenvalues $\lambda_1 \lambda_2 \cdots\cdots \lambda_n$, the necessary and sufficient condition of system being complete observable is :

$\hat{C}$ do not contain column with all 0 element in diagonal canonical form of state equation obtained by nonsingular transform

$$\dot{\hat{X}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \hat{X} + \hat{B}u$$

$$y = \hat{C}X$$

Example: Study the observability of systems:

$$\begin{cases} \dot{\hat{X}} = \begin{bmatrix} -7 & & 0 \\ & -5 & \\ 0 & & -1 \end{bmatrix} \hat{X} \\ y = \begin{bmatrix} 0 & 4 & 5 \end{bmatrix} \hat{X} \end{cases}$$

system states are not complete observable

$$\begin{cases} \dot{\hat{X}} = \begin{bmatrix} -7 & & 0 \\ & -5 & \\ 0 & & -1 \end{bmatrix} \hat{X} \\ y = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \hat{X} \end{cases}$$

system states are complete observable

Theorem 3:

Suppose system has repeated eigenvalues $\lambda_1(m_1 - repeated)$

$$\lambda_2(m_2 - repeated)\cdots\cdots \lambda_k(m_k - repeated), \sum_{i=1}^{k} m_i = n, \lambda_i \neq \lambda_j (i \neq j)$$

the Jordan canonical form of state equation obtained by non-singular transition is

$$\dot{\hat{X}} = \begin{bmatrix} J_1 & & & \\ & J_2 & & 0 \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix} \hat{X} + \hat{B}u$$

$$y = \hat{C}\hat{X}$$

The necessary and sufficient condition of system being complete observable is the column elements of $\widehat{C}$ which correspond to the first row of each Jordan block is not all 0.

Example: Study the observability of systems:

$$1)\begin{cases}\dot{\hat{X}}=\begin{bmatrix}3 & 1 & 0 & & \\ 0 & 3 & 1 & & 0 \\ 0 & 0 & 3 & & \\ & 0 & & -2 & 1 \\ & & & 0 & -2\end{bmatrix}\hat{X} \\ y=\begin{bmatrix}1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0\end{bmatrix}\hat{X}\end{cases}$$

system states are complete observable

$$2)\begin{cases}\dot{\hat{X}}=\begin{bmatrix}2 & 1 & & 0 \\ 0 & 2 & & \\ & & 3 & 1 \\ 0 & & 0 & 3\end{bmatrix}\hat{X} \\ y=\begin{bmatrix}0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1\end{bmatrix}\hat{X}\end{cases}$$

system states are not complete observable

$\hat{x}_1$ is not observable

4.3.3Third form of state observability Criteria

Theorem 3: For single-input single-output system, the neces sary and sufficient condition of system being complete observa ble is that the state-output transfer function

$$C(sI - A)^{-1}$$

do not exist cancellation factor, or do not exist zero-pole cancell ation phenomenon.

Theorem4: For single-input single-output system, the necess ary and sufficient condition of system being complete controlla ble and observable is that the input-output transfer function

$$C(sI - A)^{-1}B$$

do not exist cancellation factor, or do not exist zero-pole cancell ation phenomenon

Comprehension of zero-pole cancellation

- **Modern** unallowable If exist→not controllable and obser vable , no optimum control exist
- **Classical** allowable If exist, zero-pole located left s-plane –s table , no optimum control system structure simple

return

4.4 Criteria of Controllability and Observability for linear discrete system

4.4.1 Controllability Criteria of linear discrete system

4.4.1.1 Concept of Discrete System Controllability

For linear discrete system
$$\begin{cases} X(k+1) = GX(k) + Hu(k) \\ y(k) = CX(k) + Du(k) \end{cases}$$

given the initial valve X(0) at instant t_0 , if there exist a admissible u(k) , such that make x(k)=0 after finite sampling periods, then the system is controllable .

4.4.1.2 Criteria of Discrete System Controllability

the necessary and sufficient condition of system $\sum = (G, H)$ being complete controllable is that the controllable matrix

$$Q_k = \left[H \vdots GH \vdots \cdots \vdots G^{n-1} H \right] \quad \text{has full-rank}$$

Or $$rank\left[H \vdots GH \vdots \cdots \vdots G^{n-1} H \right] = n$$

Example 1: The state equation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Judge the state controllability of system

Sol:

$$H = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad GH = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$G^2H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

$$rankQ_k = rank \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 1 & -1 & -3 \end{bmatrix} = 3$$

System states are complete controllable

Example 2: The state equation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(k)$$

Judge the state controllability of system

Sol:

$$H=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix},\qquad GH=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & -2\\ -1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 2\\ -1 & 1\end{bmatrix}$$

$$G^2H=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & -2\\ -1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 2\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\ 2 & 2\\ -1 & 1\end{bmatrix}$$

$$rankQ_k=rank\begin{bmatrix}1 & 0 & 1 & 1 & 1 & 3\\ 0 & 1 & 0 & 2 & 2 & 2\\ 0 & 0 & -1 & 1 & -1 & 1\end{bmatrix}=3$$

System states are complete controllable

Example 3: The state equation of continuous system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Judge the state controllability of this system and its discrete system.

Sol: (1)

$$A=\begin{bmatrix}0 & 1\\ -\omega^2 & 0\end{bmatrix} \qquad B=\begin{bmatrix}0\\ 1\end{bmatrix} \qquad AB=\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$Q_k=[A \quad AB]=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$\mathrm{rank}Q_k=2$

System states are complete controllable

(2) Discretization， suppose that sampling period is T

$$G=e^{AT}=L^{-1}[(sI-A)^{-1}]\ \Big|_{t=T}$$

$$=L^{-1}\begin{bmatrix}\frac{s}{s^2+\omega^2} & \frac{1}{s^2+\omega^2}\\ \frac{-\omega^2}{s^2+\omega^2} & \frac{s}{s^2+\omega^2}\end{bmatrix}\Bigg|_{t=T}=\begin{bmatrix}\cos\omega T & \frac{\sin\omega T}{\omega}\\ -\omega\sin\omega T & \cos\omega T\end{bmatrix}$$

$$H=\int_0^T e^{At}Bdt=\int_0^T\begin{bmatrix}\cos\omega t & \dfrac{\sin\omega t}{\omega}\\ -\omega\sin\omega t & \cos\omega t\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}dt=\begin{bmatrix}\dfrac{1-\cos\omega T}{\omega^2}\\ \dfrac{\sin\omega T}{\omega}\end{bmatrix}$$

$$GH=\begin{bmatrix}\cos\omega T & \dfrac{\sin\omega T}{\omega}\\ -\cos\omega T & \cos\omega T\end{bmatrix}\begin{bmatrix}\dfrac{1-\cos\omega T}{\omega^2}\\ \dfrac{\sin\omega T}{\omega}\end{bmatrix}=\begin{bmatrix}\dfrac{\cos\omega T-\cos^2\omega T+\sin^2\omega T}{\omega^2}\\ \dfrac{2\sin\omega T\cos\omega T-\sin\omega T}{\omega}\end{bmatrix}$$

$$Q_k=\begin{bmatrix}H & GH\end{bmatrix}=\begin{bmatrix}\dfrac{1-\cos\omega T}{\omega^2} & \dfrac{\cos\omega T-\cos^2\omega T+\sin^2\omega T}{\omega^2}\\ \dfrac{\sin\omega T}{\omega} & \dfrac{2\sin\omega T\cos\omega T-\sin\omega T}{\omega}\end{bmatrix}$$

$$\left|Q_k\right| = \frac{2}{\omega^2}\sin\omega T(\cos\omega T - 1)$$

In order to make $\left|Q_k\right| \neq 0$

$$T \neq \frac{k\pi}{\omega}(k = 0,1,2,\cdots)$$

If T selected improperly, the complete controllable contin uous system may be not complete controllable after discre tization.

4.4.2 Observability Criteria of linear discrete system

4.4.2.1 Concept of Discrete System Observability

If the arbitrary initial state value X_0 can be determined uniquely according to y(k) measured in finite sampling periods, the discrete system is complete observable.

4.4.2.2 Criteria of Discrete System Observability

For linear time invariant discrete system $\sum = (G, C)$, the necessary and sufficient condition of system being complete observable is that the observability matrix

$$Q_g = \left[C^T \vdots G^T C^T \vdots \cdots \vdots (G^T)^{n-1} C^T \right]$$ has full-rank

or $$rank Q_g^T = rank \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix} = n$$

Example: linear time invariant discrete system

$$\begin{cases} X(k+1) = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} X(k) \\ y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} X(k) \end{cases}$$

Judge the system observability

$$CG = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix}$$

$$CG^2 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$$rank \begin{bmatrix} C \\ CG \\ CG^2 \end{bmatrix} = rank \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{0} \\ \mathbf{4} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{4} & \mathbf{0} \end{bmatrix} = \mathbf{2}$$

System is not complete observable.

return

4.5 Controllable Canonical Form and Observability Canonical Form

4.5.1.The introduction of problem

For linear time invariant system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

If system $\sum = (A, B)$ is complete controllable

Then $rank\left[B \vdots AB \vdots \cdots \vdots A^{n-1}B\right] =$

$$rank\left[b_1 \cdots b_r \vdots Ab_1 \cdots Ab_r \vdots \cdots \vdots A^{n-1}b_1 \cdots A^{n-1}b_r\right] = n$$

That is, controllable matrix has and only has n column v ectors which are linear irrespective.

If selecting any linear combination of n column vectors , we can obtain another linear irrespective n column vectors. So there exist a basis vector, through nonsingular transform, the basis vector is changed to canonical form, this canonical form is called controllable canonical form.

If system $\sum = (A, C)$ is complete observable

Then $rank\left[C^T \vdots A^T C^T \vdots \cdots \vdots (A^T)^{n-1} C^T\right] =$

$$rank\left[c_1^T \cdots c_m^T \vdots A^T c_1^T \cdots A^T c_m^T \vdots \cdots \vdots (A^T)^{n-1} c_1^T \cdots (A^T)^{n-1} C_m^T\right] = n$$

That is, observable matrix has and only has n column vec tors which are linear irrespective.

If selecting any linear combination of n column vectors , we can obtain another linear irrespective n column vectors. So there exist a basis vector, through nonsingular transform, the basis vector is changed to canonical form, this canonical form is called observable canonical form.

For SISO system controllable matrix or observable matrix has only sole linear irrespective vector, so the canonical expression is sole.

But for MIMO system, the basis vector has different selecti on, so the canonical expression is not sole.

4.5.2 The controllable canonical of SISO system

Given the state space description

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Where X— $n \times 1$

A— $n \times n$

B— $n \times 1$

C— $1 \times n$

If the system is complete controllable, that is , the controllable matrix $Q_k = \left[B \vdots AB \vdots \cdots \vdots A^{n-1}B \right]$ is nonsingular matrix

Then there exist nonsingular transform

$$\hat{X} = PX \quad \text{or} \quad X = P^{-1}\hat{X} \qquad (1)$$

the nonsingular transform change the state equation to controllable canonical form

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \hat{A}\hat{\mathbf{x}} + \hat{B}u \\ y = \hat{C}\hat{\mathbf{x}} \end{cases} \qquad (2)$$

Where $\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ (3)

The transform matrix $P = \begin{bmatrix} P_1 \\ P_1 A \\ \\ P_1 A^{n-1} \end{bmatrix}$ (4)

$\hat{A} = PAP^{-1}$

$\hat{B} = PB$

$\hat{C} = CP^{-1}$

Where $P_1 = [\mathbf{0} \quad \cdots \quad \mathbf{0} \quad \mathbf{1}] \left[B \vdots AB \vdots \cdots \vdots A^{n-1}B\right]^{-1}$

review

Proof: Let

$$\hat{X} = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \vdots \\ \hat{x}_n(t) \end{bmatrix} \qquad P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}$$

1. The proof of P

From $\hat{X} = PX$, We have

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \vdots \\ \hat{x}_n(t) \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} X(t) \qquad \begin{matrix} \hat{\mathbf{x}}_1(t) = P_1\mathbf{x}(t) \\ \vdots \\ \hat{\mathbf{x}}_n(t) = P_n\mathbf{x}(t) \end{matrix} \qquad (5)$$

Derivate two sides of (5), and consider (2), (3)

$$\dot{\hat{x}}_1(t) = \hat{x}_2(t) = P_1\dot{\mathbf{x}}(t) = P_1A\mathbf{x}(t) + P_1Bu(t) \qquad (6)$$

Compare (1) and (6) $\quad \therefore P_1B = 0$

(6) turn to $\quad \dot{\hat{x}}_1(t) = \hat{x}_2(t) = P_1Ax(t) \qquad (7)$

Derivate two sides of (7), and consider (2), (3)

$$\dot{\hat{x}}_2(t) = \hat{x}_3(t) = P_1A^2x(t) \qquad P_1A\ B = 0$$

$$\vdots$$

$$\dot{\hat{x}}_{n-1}(t) = \hat{x}_n(t) = P_1A^{n-1}x(t) \qquad P_1A^{n-2}B = 0$$

Or $\quad \hat{X}(t) = PX(t) = \begin{bmatrix} P_1 \\ P_1A \\ \\ P_1A^{n-1} \end{bmatrix} X(t)$

$$P = \begin{bmatrix} P_1 \\ P_1 A \\ \\ P_1 A^{n-1} \end{bmatrix} \quad (8)$$

$$P_1 B = P_1 AB = \cdots = PA^{n-2} B = 0 \quad (9)$$

2. the proof of P_1

Derivate $\hat{X}(t) = PX(t)$ two sides

$$\dot{\hat{x}}(t) = P\dot{x}(t) = PAx(t) + PBu(t)$$

$$= PAP^{-1}\hat{x} + PBu(t)$$

$$y = Cx = CP^{-1}\hat{x}$$

So $\hat{A} = PAP^{-1}$

$$\hat{B} = PB$$

$$\hat{C} = CP^{-1}$$

Consider (3) and (9) $PB = \begin{bmatrix} P_1 B \\ P_1 AB \\ \vdots \\ P_1 A^{n-1} B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

or $P_1\left[B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B\right] = [0 \quad 0 \cdots \quad 0 \quad 1]$

$$P_1 = [0 \quad 0 \cdots \quad 0 \quad 1]\left[B \vdots AB \vdots \cdots \vdots A^{n-1}B\right]^{-1} = [0 \quad 0 \cdots \quad 0 \quad 1]Q_k^{-1}$$

Example: Given the state space description of linear time invariant system

$$\dot{X}(t) = AX(t) + Bu(t)$$

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Turn it to controllable canonical form

Sol: (1) controllability discrimination

$$Q_k = [B \vdots AB] = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$rankQ_k = rank \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = 2$$

System is complete controllable

(2) Find P_1

$$P_1 = [0 \quad 1]\ [B \vdots AB]^{-1} = [0 \quad 1] \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= [0 \quad 1] \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = [1 \quad -1]$$

(3) Find P

$$P = \begin{bmatrix} P_1 \\ P_1 A \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

(4) Find $\hat{A}, \ \hat{B}$

$$\hat{A} = PAP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B = PB = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(5) Write out canonical form

$$\dot{\hat{X}} = \hat{A}\hat{X} + \hat{B}u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{X} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

4.5.3 The observable canonical of SISO system

Given the state space description

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Where X—vector $n \times 1$

A— $n \times n$

B— $n \times 1$

C — $1 \times n$

If the system is complete observable, that is , the observable matrix

$$Q_g^T = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ is nonsingular matrix

Then there exist nonsingular transform $X = T\hat{X}$ or $\hat{X} = T^{-1}X$ (1)

the nonsingular transform change the state equation to controllable canonical form

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \hat{A}\hat{\mathbf{x}} + \hat{B}u \\ y = \hat{C}\hat{\mathbf{x}} \end{cases} \quad (2)$$

Where

$$\hat{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -a_n \\ \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} & -a_{n-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & -a_2 \\ \mathbf{0} & \mathbf{0} & & \mathbf{1} & -a_1 \end{bmatrix}, \quad \hat{C} = [\mathbf{0} \cdots \quad \mathbf{0} \quad \mathbf{1}] \quad (3)$$

The transform matrix $T = \begin{bmatrix} T_1 & AT_1 & \cdots & A^{n-1}T_1 \end{bmatrix}$ (4)

$$\hat{A} = T^{-1}AT$$

$$\hat{B} = T^{-1}B$$

$$\hat{C} = CT$$

Where $$T_1 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = [Q_g^T]^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Example: Given the state space description

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} \end{bmatrix} \mathbf{x}(t) \\ y(t) = \begin{bmatrix} \mathbf{-1} & -\dfrac{\mathbf{1}}{\mathbf{2}} \end{bmatrix} \mathbf{x}(t) \end{cases}$$

Turn it to observable canonical form

Sol:

$$Q_g^T = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ -1 & 0 \end{bmatrix} \qquad Rank Q_g^T = 2$$ system is complete observable

$$T_1 = \begin{bmatrix} -1 & -\frac{1}{2} \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$T = \begin{bmatrix} T_1 & AT_1 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix}$$

$$\hat{A} = T^{-1}AT = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\hat{C} = CT = \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\therefore \begin{cases} \dot{\mathrm{x}}(\mathrm{t}) = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \mathrm{x}(\mathrm{t}) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathrm{x}(\mathrm{t}) \end{cases}$$

4.5.4 Determine controllable canonical form and observable canonical form from state variables diagram

From state state variables diagram we has obtained

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

Theorem 1: Suppose the SISO system is complete controllable, the transfer function is

$$W(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

The controllable canonical form is

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_1 \end{bmatrix}$$

Theorem 2: Suppose the SISO system is complete observable, the transfer function is

$$W(s)=\frac{Y(s)}{U(s)}=\frac{b_1 s^{n-1}+\cdots+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n}$$

The controllable canonical form is

$$\hat{A}=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}, \hat{B}=\begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix}$$

$$\hat{C}=[0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

4.6 Duality principle of controllability and obersvability

Σ_1 $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

x——n-dimension state vector
u——r -dimension control vector
y——m-dimension output vector

Σ_2 $\begin{cases} \dot{z} = A^T z + C^T v \\ w = B^T z \end{cases}$

z——n -dimension state vector
v——m -dimension control vector
w——r -dimension output vector

1 The necessary and sufficient condition of systems being complete controllable

Σ_1 $$rankQ_k = rank[B \vdots AB \vdots \cdots \vdots A^{n-1}B] = n$$

Σ_2 $$rankQ_k = rank[C^T \vdots A^T C^T \vdots \cdots \vdots (A^T)^{n-1} C^T] = n$$

2 The necessary and sufficient condition of systems being complete observable

Σ_1 $$rankQ_g = rank[C^T \vdots A^T C^T \vdots \cdots \vdots (A^T)^{n-1} C^T] = n$$

Σ_2 $$rankQ_g = rank[B \vdots AB \vdots \cdots \vdots A^{n-1} B] = n$$

So Σ_1 Complete controllable condition= Σ_2 Complete observable condition

Σ_1 Complete observable condition= Σ_2 Complete controllable condition

Controllability and observability have duality

return

Exercise

4.4 System $\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{a} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{b} \\ -\mathbf{1} \end{bmatrix} u$ has complete controllability.

Try to determine the relation of a and b.

4.9 Turn the state equation to controllable canonical form

$$\dot{\mathbf{x}} = \begin{bmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{1} & -\mathbf{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix} u$$

4.11 $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, y = C\mathbf{x}$ $A = \begin{bmatrix} -2 & 2 & -1 \\ 0 & -2 & 0 \\ 1 & -4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$

(1) Discriminate controllability and observability,

(2) Find transfer function.

4.14 $\sum_1$ and $\sum_2$ are complete controllable and complete observable

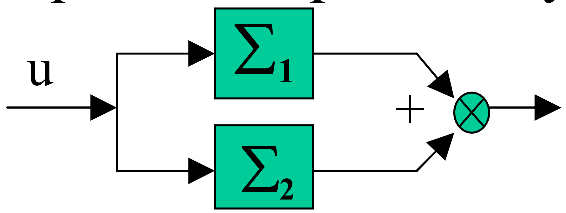
$$\sum_1 : \quad \dot{\mathbf{x}}_1 = A_1\mathbf{x}_1 + B_1 u, \quad y_1 = C\mathbf{x}_1$$

$$\sum_2 : \quad \dot{\mathbf{x}}_2 = A_2\mathbf{x}_2 + B_2 u, \quad y_2 = C\mathbf{x}_2$$

where $A_1 = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}$

$A_2 = -1, \quad B_2 = 1, \quad C_2 = 1$

(1) deduce the state equation of parallel system



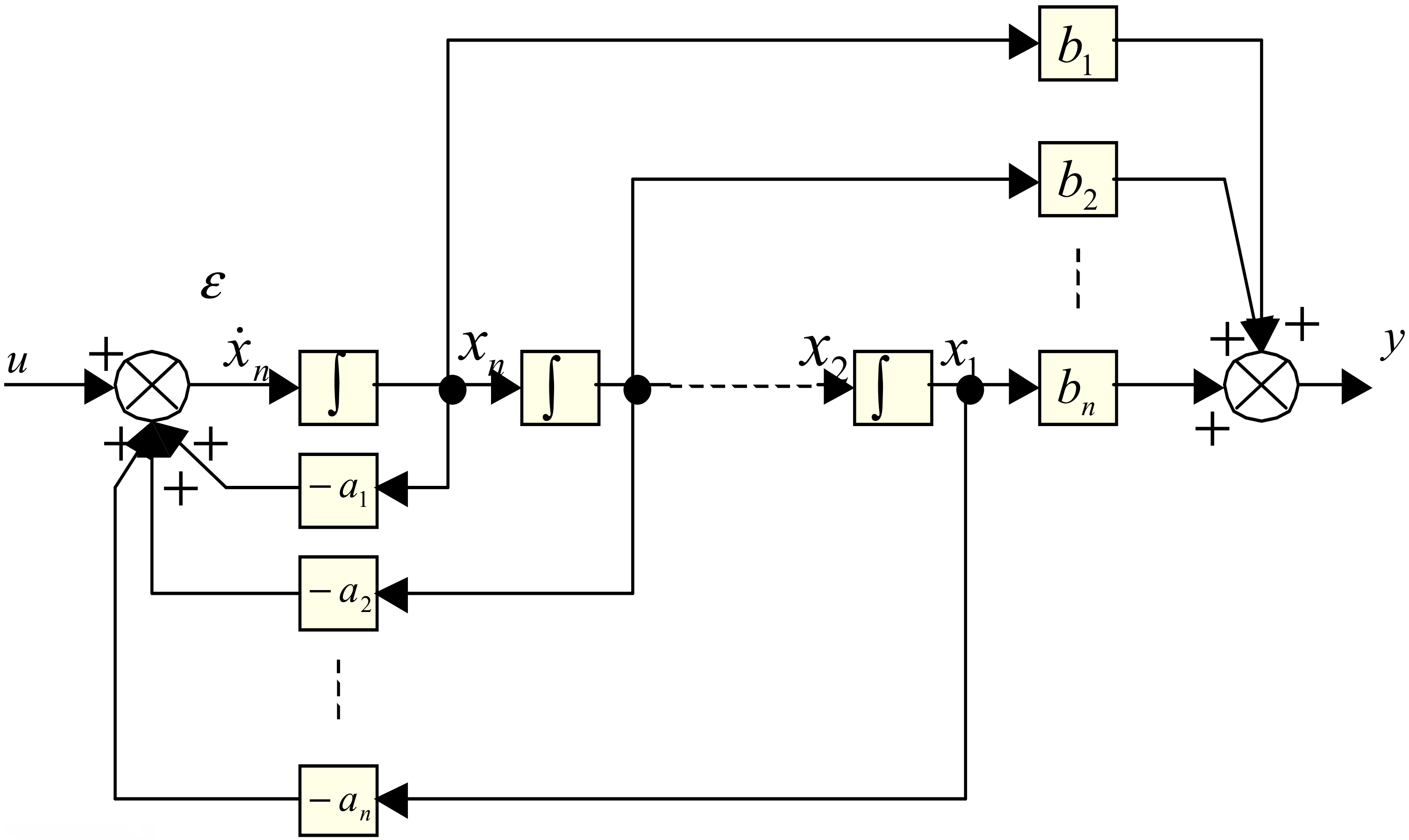
(2) Discriminate controllability and observability

(3) Find transfer function

4.17 Given the transfer function of SISO system

$$\frac{U(s)}{Y(s)} = \frac{K}{(s+a)^2(s+c)(s+d)}$$

Where a, b and c are different。Find the state equation , and discuss controllability.



ε
u
+
$\dot{x}_n$
x_n
x_2
x_1
y
b_1
b_2
b_n
$-a_1$
$-a_2$
$-a_n$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

return