An Overpartition Analogue of Bressoud's Theorem of Rogers-Ramanujan-Gordon Type

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Abstract. For $k \geq 2$ and $k \geq i \geq 1$, let $B_{k,i}(n)$ denote the number of partitions of n such that part 1 appears at most $i-1$ times, two consecutive integers l and $l+1$ appear at most $k-1$ times and if l and $l+1$ appear exactly $k-1$ times then the sum of the parts l and $l+1$ is congruent to $i-1$ modulo 2. Let $A_{k,i}(n)$ denote the number of partitions with parts not congruent to i, $2k-i$ and $2k$ modulo $2k$. Bressoud's theorem states that $A_{k,i}(n) = B_{k,i}(n)$. Corteel, Lovejoy, and Mallet found an overpartition analogue of Bressoud's theorem for $i = 1$, that is, for partitions not containing non-overlined part 1. We obtain an overpartition analogue of Bressoud's theorem in the general case. For $k \geq 2$ and $k \geq i \geq 1$, let $D_{k,i}(n)$ denote the number of overpartitions of n such that the non-overlined part 1 appears at most $i-1$ times, for any integer l, l and non-overlined $l+1$ appear at most $k-1$ times and if the parts l and the non-overlined part $l+1$ together appear exactly $k-1$ times then the sum of the parts l and non-overlined parts $l+1$ has the same parity as the number of overlined parts that are less than $l+1$ plus $i-1$. Let $C_{k,i}(n)$ denote the number of overpartitions of n with the non-overlined parts not congruent to $\pm i$ and $2k-1$ modulo $2k-1$. We show that $C_{k,i}(n) = D_{k,i}(n)$. Note that this relation can also be considered as a Rogers-Ramanujan-Gordon type theorem for overpartitions.

Keywords: the Rogers-Ramanujan-Gordon theorem, overpartition, Bressoud's theorem

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1 Introduction

The Rogers-Ramanujan-Gordon theorem is a combinatorial generalization of the Rogers-Ramanujan identities [15, 16], see Gordon [10]. It establishes the equality between the number of partitions of n with parts satisfying certain residue conditions and the number of partitions of n with certain difference conditions. Gordon found an involution for an equivalent form of the generating function identity for this relation. An algebraic proof was given by Andrews [1] by using a recursive approach. It should be noted that the Rogers-Ramanujan-Gordon theorem is concerned only with odd moduli. Bressoud [4] succeeded in finding a theorem of Rogers-Ramanujan-Gordon type for even moduli by using an algebraic approach in the spirit of Andrews [1].

The objective of this paper is to give an overpartition analogue of Bressoud's theorem. We derive the equality between the number of overpartitions of n such that the non-overlined parts belong to certain residue classes modulo an odd positive integer and the number of overpartitions of n with parts satisfying certain difference conditions. A special case of this relation has been discovered by Corteel, Lovejoy, and Mallet [8].

An overpartition analogue of the Rogers-Ramanujan-Gordon theorem was obtained by Chen, Sang and Shi [6], which states that the number of overpartitions of n with non-overlined parts belonging to certain residue classes modulo an even positive integer equals the number of overpartitions of n with parts satisfying certain difference conditions. However, as will be seen, the proof of the overpartition analogue of the Rogers-Ramanujan-Gordon theorem does not seem to be directly applicable to the case for the overpartition analogue of Bressoud's theorem.

Let us give an overview of some definitions. A partition λ of a positive integer n is a non-increasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_s > 0$ such that $n = \lambda_1 + \cdots + \lambda_s$. The partition of zero is defined to be the partition with no parts. An overpartition λ of a positive integer n is also a non-increasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_s > 0$ such that $n = \lambda_1 + \cdots + \lambda_s$ and the first occurrence of each integer may be overlined. For example, $(\overline{7}, 7, 6, \overline{5}, 2, \overline{1})$ is an overpartition of 28. Many q-series identities have combinatorial interpretations in terms of overpartitions, see, for example, Corteel and Lovejoy [7]. Furthermore, overpartitions possess many analogous properties to ordinary partitions, see Lovejoy [11, 13]. For example, various overpartition theorems of the Rogers-Ramanujan-Gordon type have been obtained by Corteel and Lovejoy [9], Corteel, Lovejoy and Mallet [8] and Lovejoy [11, 12, 14]. For a partition or an overpartition λ and for any integer l, let $f_l(\lambda)(f_{\overline{l}}(\lambda))$ denote the number of occurrences of a non-overlined part l (an overlined part $\overline{l}$) in λ. Let $V_\lambda(l)$ denote the number of overlined parts in λ that are less than or equal to l.

We shall adopt the common notation as used in Andrews [3]. Let

$$(a)_\infty = (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

and

$$(a)_n = (a;q)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

We also write

$$(a_1,\ldots,a_k;q)_\infty = (a_1;q)_\infty \cdots (a_k;q)_\infty.$$

The Rogers-Ramanujan-Gordon theorem reads as follows.

Theorem 1.1 *(Rogers-Ramanujan-Gordon) For $k \geq 2$ and $k \geq i \geq 1$, let $F_{k,i}(n)$ denote the number of partitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$, where $\lambda_j \geq \lambda_{j+1}$, $\lambda_j - \lambda_{j+k-1} \geq 2$ and part 1 appears at most $i-1$ times. Let $E_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Then for any $n \geq 0$, we have*

$$E_{k,i}(n) = F_{k,i}(n). \tag{1.1}$$

In the algebraic proof of the above relation, Andrews [1, 3] introduced a hypergeometric function $J_{k,i}(a; x; q)$ as given by

$$J_{k,i}(a; x; q) = H_{k,i}(a; xq; q) - axqH_{k,i-1}(a; xq; q), \tag{1.2}$$

where

$$H_{k,i}(a; x; q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n-in} a^n (1 - x^i q^{2ni})(axq^{n+1})_\infty (1/a)_n}{(q)_n (xq^n)_\infty}. \tag{1.3}$$

To prove (1.1), Andrews considered a refinement of $F_{k,i}(n)$, that is, the number of partitions enumerated by $F_{k,i}(n)$ with exactly m parts, denoted by $F_{k,i}(m, n)$, and he showed that $J_{k,i}(0; x; q)$ and the generating function of $F_{k,i}(m, n)$ satisfy the same recurrence relation with the same initial values. Setting $x = 1$ and using Jacobi's triple product identity, Andrews deduced that $J_{k,i}(0; 1; q)$ equals the generating function for $E_{k,i}(n)$. This yields $E_{k,i}(n) = F_{k,i}(n)$.

The following Rogers-Ramanujan-Gordon type theorem for even moduli is due to Bressoud [4].

Theorem 1.2 *For $k \geq 2$ and $k \geq i \geq 1$, let $B_{k,i}(n)$ denote the number of partitions of n of the form $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that*

(i) $f_1(\lambda) \leq i - 1$,

(ii) $f_l(\lambda) + f_{l+1}(\lambda) \leq k - 1$,

(iii) if $f_l(\lambda) + f_{l+1}(\lambda) = k - 1$, *then* $lf_l(\lambda) + (l+1)f_{l+1}(\lambda) \equiv i - 1 \pmod 2$.

Let $A_{k,i}(n)$ denote the number of partitions of n with parts not congruent to $0, \pm i$ modulo $2k$. Then we have

$$A_{k,i}(n) = B_{k,i}(n). \tag{1.4}$$

In the proof of Bressoud, he also used the hypergeometric function $J_{k,i}(a; x; q)$ and used a recurrence relation for $(-xq)_\infty J_{(k-1)/2,i/2}(a; x^2; q^2)$.

Lovejoy [11] found the following overpartition analogues of Rogers-Ramanujan-Gordon theorem for the cases $i = 1$ and $i = k$.

Theorem 1.3 *For $k \geq 2$, let $\overline{B}_k(n)$ denote the number of overpartitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that $\lambda_j - \lambda_{j+k-1} \geq 1$ if λ_j is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of n into parts not divisible by k. Then we have*

$$\overline{A}_k(n) = \overline{B}_k(n). \tag{1.5}$$

Theorem 1.4 *For $k \geq 2$, let $\overline{D}_k(n)$ denote the number of overpartitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that 1 cannot occur as a non-overlined part, and $\lambda_j - \lambda_{j+k-1} \geq 1$ if λ_j is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. Let $\overline{C}_k(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k$. Then we have*

$$\overline{C}_k(n) = \overline{D}_k(n). \tag{1.6}$$

Chen, Sang and Shi [6] obtained an overpartition analogue of the Rogers-Ramanujan-Gordon theorem in the general case.

Theorem 1.5 *For $k \geq 2$ and $k \geq i \geq 1$, let $P_{k,i}(n)$ denote the number of overpartitions of n of the form $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that part 1 occurs as a non-overlined part at most $i-1$ times, and $\lambda_j - \lambda_{j+k-1} \geq 1$ if λ_j is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. For $k \geq 2$ and $k > i \geq 1$, let $Q_{k,i}(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm i$ modulo $2k$ and let $Q_{k,k}(n)$ denote the number of overpartitions of n with parts not divisible by k. Then we have*

$$P_{k,i}(n) = Q_{k,i}(n). \tag{1.7}$$

As an overpartition analogue of Bressoud's theorem for the case $i = 1$, Corteel, Lovejoy, and Mallet [8] obtained the following overpartition theorem.

Theorem 1.6 *For $k \geq 2$, let $\overline{A}_k^3(n)$ denote the number of overpartitions whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k-1$. Let $\overline{B}_k^3(n)$ denote the number of overpartitions λ of n such that*

(i) $f_1(\lambda) = 0$,

(ii) $f_l(\lambda) + f_{\overline{l}}(\lambda) + f_{l+1}(\lambda) \leq k-1$,

(iii) *if* $f_l(\lambda) + f_{\overline{l}}(\lambda) + f_{l+1}(\lambda) = k-1$, *then* $lf_l(\lambda) + lf_{\overline{l}}(\lambda) + (l+1)f_{l+1}(\lambda) \equiv V_\lambda(l) \pmod 2$.

Then we have

$$\overline{A}_k^3(n) = \overline{B}_k^3(n). \tag{1.8}$$

In this paper, we give an overpartition analogue of the Bressoud's theorem in the general case.

2 The Main Result

The main result of this paper can be stated as follows.

Theorem 2.1 *For $k \geq 2$ and $k \geq i \geq 1$, let $D_{k,i}(n)$ denote the number of overpartitions of n of the form $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that* *(i)* $f_1(\lambda) \le i-1$;

(ii) $f_l(\lambda) + f_{\overline{l}}(\lambda) + f_{l+1}(\lambda) \le k-1$;

(iii) *if* $f_l(\lambda)+f_{\overline{l}}(\lambda)+f_{l+1}(\lambda) = k-1$, *then* $lf_l(\lambda)+lf_{\overline{l}}(\lambda)+(l+1)f_{l+1}(\lambda) \equiv V_\lambda(l)+i-1 \pmod 2$.

Let $C_{k,i}(n)$ *denote the number of overpartitions of* n *whose non-overlined parts are not congruent to* $0, \pm i$ *modulo* $2k-1$. *Then we have*

$$C_{k,i}(n) = D_{k,i}(n). \tag{2.9}$$

Instead of using the function $\widetilde{J}_{k,i}(a;x;q)$ as in the proof of Theorem 1.6 given by Corteel, Lovejoy, and Mallet [8], we find that the function $\widetilde{H}_{k,i}(a;x;q)$, also introduced by Corteel, Lovejoy, and Mallet [8], is related to the generating functions of the numbers $C_{k,i}(n)$ and $D_{k,i}(n)$. Recall that

$$\widetilde{J}_{k,i}(a;x;q) = \widetilde{H}_{k,i}(a;xq;q) + axq\widetilde{H}_{k,i-1}(a;xq;q), \tag{2.10}$$

where

$$\widetilde{H}_{k,i}(a;x;q) = \sum_{n\ge 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n-in} x^{(k-1)n}(1-x^iq^{2ni})(-x,-1/a)_n(-axq^{n+1})_\infty}{(q^2;q^2)_n(xq^n)_\infty}. \tag{2.11}$$

It should be noticed that the function $\widetilde{J}_{k,i}(a;x;q)$ can be expressed as $F_{1,k,i}(-q,\infty;-1/a;q)$ in the notation of Bressoud [5], and the function $(-q)_\infty\widetilde{H}_{k,i}(a;x;q)$ can be written as $H_{k,i}(-1/a,-x;x;q)_2$ in the notation of Andrews [2].

Let $\widetilde{B}_k^3(m,n)$ denote the number of overpartitions enumerated by $\widetilde{B}_k^3(n)$ with exactly m parts. Corteel, Lovejoy and Mallet [8] have shown that the coefficients of x^mq^n in $\widetilde{J}_{k,1}(1/q;x;q)$ and $\widetilde{B}_k^3(m,n)$ satisfy the same recurrence relation with the same initial values. Moreover, they proved that the generating function of $\widetilde{B}_k^3(m,n)$ also equals $\widetilde{J}_{k,1}(1/q;x;q)$, that is,

$$\sum_{m,n\ge 0} \overline{B}_k^3(m,n)x^mq^n = \widetilde{J}_{k,1}(-1/q;x;q). \tag{2.12}$$

Setting $a=1/q$, $x=1$ and using Jacobi's triple product identity, the function $\widetilde{J}_{k,i}(a;x;q)$ can be expressed as an infinite product, namely,

$$\widetilde{J}_{k,1}(1/q;1;q) = \frac{(q,q^{2k-2},q^{2k-1};q^{2k-1})_\infty(-q)_\infty}{(q)_\infty}.$$

Clearly, this is the generating function for $\overline{A}_k^3(n)$. It follows that $\overline{A}_k^3(n) = \overline{B}_k^3(n)$.

However, the proof of Corteel, Lovejoy and Mallet does not seem to apply to the general case, since $\widetilde{J}_{k,i}(1/q;x;q)$ does not seem to have an infinite product expression for $i\ge 2$. Our strategy goes as follows. For $C_{k,i}(n)$, we show that the generating function for $C_{k,i}(n)$ can be expressed in terms of $\widetilde{H}_{k,i}(a;x;q)$ with $a=1/q$ and $x=q$. For $D_{k,i}(n)$, let $D_{k,i}(m,n)$ denote the number of overpartitions enumerated by $D_{k,i}(n)$ with exactly m parts. We find a combinatorial interpretation of $D_{k,i}(m,n)-D_{k,i-1}(m,n)$ from which we can derive a recurrence relation for $D_{k,i}(m,n)$. Furthermore, we see that the recurrence relation and initial values of $D_{k,i}(m,n)$ coincide with the recurrence relation and the initial values of the coefficients of $x^m q^n$ in $\widetilde{H}_{k,i}(1/q;xq;q)$. Thus we reach the conclusion that the generating function of $D_{k,i}(m,n)$ equals $\widetilde{H}_{k,i}(-1/q;xq;q)$. Setting $x=1$, we deduce that the generating function of $D_{k,i}(n)$ equals the generating function of $C_{k,i}(n)$.

For convenience, we write $W_{k,i}(x;q)$ for $\widetilde{H}_{k,i}(1/q;xq;q)$, that is,

$$W_{k,i}(x;q)=\sum_{n\geq 0}\frac{(-1)^n q^{(2k-1)\binom{n+1}{2}-in}x^{(k-1)n}(1-x^i q^{(2n+1)i})(-xq)_\infty}{(q)_n(xq^{n+1})_\infty}. \tag{2.13}$$

Recall that Andrews found the following recurrence relation for $H_{k,i}(a;x;q)$:

$$H_{k,i}(a;x;q)-H_{k,i-1}(a;x;q)=x^{i-1}H_{k,k-i+1}(a;xq;q)-ax^i qH_{k,k-i}(a;xq;q). \tag{2.14}$$

A recurrence relation for $W_{k,i}(x;q)$ is given below.

Theorem 2.2 *For $k\geq 2$ and $k\geq i\geq 1$, we have*

$$W_{k,i}(x;q)-W_{k,i-1}(x;q)=(1+xq)(xq)^{i-1}W_{k,k-i}(xq;q). \tag{2.15}$$

Proof. Since

$$q^{-in}-x^i q^{(n+1)i}-q^{(-i+1)n}+x^{i-1}q^{(n+1)(i-1)}=q^{-in}(1-q^n)+x^{i-1}q^{(n+1)(i-1)}(1-xq^{n+1}),$$

it can be checked that $W_{k,i}(x;q)-W_{k,i-1}(x;q)$ can be written as

$$\sum_{n=1}^{\infty}q^{-in}\frac{(-1)^n x^{(k-1)n}q^{(2k-1)\binom{n+1}{2}}(-xq)_\infty}{(q)_{n-1}(xq^{n+1})_\infty}+\sum_{n=0}^{\infty}(xq^{n+1})^{i-1}\frac{(-1)^n x^{(k-1)n}q^{(2k-1)\binom{n+1}{2}}(-xq)_\infty}{(q)_n(xq^{n+2})_\infty}. \tag{2.16}$$

Now, replacing n with $n+1$, the first sum in (2.16) can be expressed as

$$\sum_{n=0}^{\infty}q^{-i(n+1)}\frac{(-1)^{(n+1)}x^{(k-1)(n+1)}q^{(2k-1)\binom{n+2}{2}}(-xq)_\infty}{(q)_n(xq^{n+2})_\infty}. \tag{2.17}$$

Hence $W_{k,i}(x;q)-W_{k,i-1}(x;q)$ equals

$$\begin{aligned}
&-(xq)^{i-1}\sum_{n=0}^{\infty}\frac{(-1)^n(xq)^{(k-1)n}q^{(2k-1)\binom{n+1}{2}}x^{k-i}q^{(2k-1)(n+1)-in-2i+1-(k-1)n}(-xq)_\infty}{(q)_n(xq^{n+2})_\infty}\\
&\quad+(xq)^{i-1}\sum_{n=0}^{\infty}\frac{(-1)^n(xq)^{(k-1)n}q^{(2k-1)\binom{n+1}{2}+(i-1)n-(k-1)n}(-xq)_\infty}{(q)_n(xq^{n+2})_\infty}\\
&\quad=(1+xq)(xq)^{i-1}\sum_{n\geq 0}\frac{(-1)^n(xq)^{(k-1)n}q^{(2k-1)\binom{n+1}{2}-(k-i)n}(1-x^{k-i}q^{(2n+2)(k-i)})(-xq^2)_\infty}{(q)_n(xq^{n+2})_\infty}\\
&\quad=(1+xq)(xq)^{i-1}W_{k,k-i}(xq),
\end{aligned}$$

as desired. ∎

The following relation can be considered as a combinatorial interpretation of $D_{k,i}(m,n) - D_{k,i-1}(m,n)$.

Theorem 2.3 *For $k \geq 2$, $k \geq i \geq 1$ and for $m,n \geq 0$, let $S_{k,i}(m,n)$ denote the set of overpartitions enumerated by $D_{k,i}(m,n)$ with exactly one overlined part 1 and exactly $i-1$ non-overlined part 1. Let $T_{k,i}(m,n)$ denote the set of overpartitions enumerated by $D_{k,i}(m,n)$ with exactly one overlined part 1 and exactly $i-2$ non-overlined part 1. Then we have*

$$D_{k,i}(m,n) - D_{k,i-1}(m,n) = |S_{k,i}(m,n)| + |T_{k,i}(m,n)|. \tag{2.18}$$

Proof. Let $U_{k,i}(m,n)$ denote the set of overpartitions enumerated by $D_{k,i}(n)$ with exactly m parts. By the definitions of $D_{k,i}(m,n)$ and $D_{k,i-1}(m,n)$, it can be easily seen that $U_{k,i-1}(m,n)$ is not contained in $U_{k,i}(m,n)$. To compute $D_{k,i}(m,n) - D_{k,i-1}(m,n)$, we wish to construct an injection φ from overpartitions in $U_{k,i-1}(m,n)$ to overpartitions in $U_{k,i}(m,n)$. We proceed to give a characterization of the images of this map, which leads to relation (2.18).

Let λ be an overpartition in $U_{k,i-1}(m,n)$. If there exists an overlined part of λ with the smallest underlying part, then we switch this overlined part to a non-overlined part, otherwise we choose a smallest non-overlined part and switch it to an overlined part. Let λ' denote the resulting overpartition. It can be checked that this map is an injection. It is not difficult to verify that $\lambda' \in U_{k,i}(m,n)$. Hence the number $D_{k,i}(m,n) - D_{k,i-1}(m,n)$ can be interpreted as the number of overpartitions in $U_{k,i}(m,n)$ which cannot be obtained by using the above map.

By the construction of the map φ, we may generate all the overpartitions in $U_{k,i}(m,n)$ with no overlined part equal to 1 and all the overpartitions in $U_{k,i}(m,n)$ with an overlined 1 and with at most $i-3$ non-overlined part 1. Therefore, $D_{k,i}(m,n) - D_{k,i-1}(m,n)$ is exactly the number of overpartitions in $U_{k,i}(m,n)$ with exactly one overlined part 1 such that the non-overlined part 1 appears either $i-1$ times or $i-2$ times. This completes the proof. ∎

Theorem 2.4 *For $k \geq 2$, $k \geq i \geq 1$, and $m,n \geq 0$, we have*

$$|S_{k,i}(m,n)| = D_{k,k-i}(m-i, n-m). \tag{2.19}$$

Proof. We define a bijection ϕ from $S_{k,i}(m,n)$ to $U_{k,k-i}(m-i,n-m)$ which implies (2.19). Let λ be an overpartition in $S_{k,i}(m,n)$, the map ϕ is defined as follows.

Step 1. Remove all the i parts with underlying part 1.

Step 2. Subtract 1 from each part.

Clearly, the resulting overpartition λ' is an overpartition of $n-m$ with $m-i$ parts. Moreover, we claim that $\lambda' \in U_{k,k-i}(m-i,n-m)$.

We first show that $f_1(\lambda') \leq k-i-1$. By the construction of ϕ, it is easy to see that $f_1(\lambda') = f_2(\lambda)$ and $f_1(\lambda) = i-1$. From the condition (ii) in Theorem 2.1, that is, $f_1(\lambda) + f_{\overline{1}}(\lambda) + f_2(\lambda) \leq k-1$, we find that $f_2(\lambda) \leq k-1-i$.

We still need to verify that if there is an integer l such that

$$f_l(\lambda') + f_{\overline{l}}(\lambda') + f_{l+1}(\lambda') = k-1, \tag{2.20}$$

then we have

$$l f_l(\lambda') + l f_{\overline{l}}(\lambda') + (l+1) f_{l+1}(\lambda') \equiv V_{\lambda'}(l) + k - i - 1 \pmod 2. \tag{2.21}$$

By the construction of ϕ, it is easily checked that (2.20) implies

$$f_{l+1}(\lambda) + f_{\overline{l+1}}(\lambda) + f_{l+2}(\lambda) = k - 1.$$

Since $\lambda \in S_{k,i}(m,n)$, we have

$$(l+1) f_{l+1}(\lambda) + (l+1) f_{\overline{l+1}}(\lambda) + (l+2) f_{l+2}(\lambda) \equiv V_\lambda(l+1) + i - 1 \pmod 2,$$

Clearly, $f_t(\lambda') = f_{t+1}(\lambda)$ and $f_{\overline{t}}(\lambda') = f_{\overline{t+1}}(\lambda)$ for any $t \geq 1$. Thus we deduce that

$$l f_l(\lambda') + l f_{\overline{l}}(\lambda') + (l+1) f_{l-1}(\lambda') \equiv V_\lambda(l+1) + i - 1 - (k-1) \pmod 2.$$

Again, by the construction of ϕ, we find $V_\lambda(l+1) = V_{\lambda'}(l) + 1$. So we arrive at relation (2.21), which implies $\lambda' \in U_{k,k-i}(m-i, n-m)$.

It is not difficult to verify that the above construction is reversible, that is, from any overpartition in $U_{k,k-i}(m-i, n-m)$, we can recover an overpartition in $S_{k,i}(m,n)$. This completes the proof. ∎

For example, let $k = 5$ and $i = 3$, and let $\lambda = (\overline{9}, 9, 9, 9, 8, 8, 7, 7, 7, 6, 6, \overline{5}, 5, 5, 4, 4, 4, \overline{3}, 3, 2, 2, \overline{1}, 1, 1)$ be an overpartition in $S_{5,3}(24, 125)$. Then we get $\phi(\lambda) = (\overline{8}, 8, 8, 8, 7, 7, 6, 6, 6, 5, 5, \overline{4}, 4, 4, 3, 3, 3, \overline{2}, 2, 1, 1)$, which is an overpartition in $U_{5,2}(21, 101)$.

Theorem 2.5 *For $k \geq 2$, $k \geq i \geq 1$, and $m, n \geq 0$, we have*

$$|T_{k,i}(m,n)| = D_{k,k-i}(m-i+1, n-m). \tag{2.22}$$

Proof. We proceed to construct a bijection χ from $T_{k,i}(m,n)$ to $U_{k,k-i}(m-i+1, n-m)$. Let λ be an overpartition in $T_{k,i}(m,n)$, the map χ is defined as follows.

Step 1. Remove all $i-1$ parts equal to 1.

Step 2. Subtract 1 from each part.

Clearly, the resulting overpartition λ' is an overpartition of $n-m$ with $m-i+1$ parts. We shall show that $\lambda' \in U_{k,k-i}(m-i+1, n-m)$.

We first verify that $f_1(\lambda') \leq k-i-1$. It is obvious that $f_1(\lambda') = f_2(\lambda)$. So it suffices to prove that $f_2(\lambda) \leq k-i-1$. Since $\lambda \in T_{k,i}(m,n)$, we have $f_1(\lambda) = i-2$, $f_{\overline{1}}(\lambda) = 1$ and

$$f_1(\lambda) + f_{\overline{1}}(\lambda) + f_2(\lambda) \leq k-1. \tag{2.23}$$

It follows that $f_2(\lambda) \leq k-i$.

It remains to show that the non-overlined part 2 cannot occur $k-i$ times. Assume to the contrary that $f_2(\lambda) = k-i$. Then the equality in (2.23) holds, that is,

$$f_1(\lambda) + f_{\overline{1}}(\lambda) + f_2(\lambda) = k-1.$$

We proceed to derive a contradiction to the condition (iii) in Theorem 2.1. By the facts $f_1(\lambda) = i-2$ and $f_{\overline{1}}(\lambda) = 1$, we find

$$1f_1(\lambda) + 1f_{\overline{1}}(\lambda) + 2f_2(\lambda) = 2k - i - 1. \tag{2.24}$$

Since $V_\lambda(1) = 1$, from (2.24) it follows that

$$1f_1(\lambda) + 1f_{\overline{1}}(\lambda) + 2f_2(\lambda) \not\equiv V_\lambda(1) + i - 1 \pmod 2,$$

which is a contradiction to assumption that the non-overlined 2 occurs $k-i$ times. Thus we reach the conclusion that the non-overlined part 2 occurs at most $k-i-1$ times in λ, or equivalently, the non-overlined part 1 occurs at most $k-i-1$ times in λ'.

Next, we check condition (ii) in Theorem 2.1 for λ'. For any $l \geq 1$, we see that

$$f_{l+1}(\lambda) = f_l(\lambda') \quad \text{and} \quad f_{\overline{l+1}}(\lambda) = f_{\overline{l}}(\lambda'). \tag{2.25}$$

From condition (ii) for λ, we get

$$f_l(\lambda') + f_{\overline{l}}(\lambda') + f_{l+1}(\lambda') \leq k - 1.$$

Finally, we verify the condition that if there is an integer l such that

$$f_l(\lambda') + f_{\overline{l}}(\lambda') + f_{l+1}(\lambda') = k - 1, \tag{2.26}$$

then we have

$$lf_l(\lambda') + lf_{\overline{l}}(\lambda') + (l+1)f_{l+1}(\lambda') \equiv V_{\lambda'}(l) + k - i - 1 \pmod 2. \tag{2.27}$$

Notice that (2.26) implies

$$f_{l+1}(\lambda) + f_{\overline{l+1}}(\lambda) + f_{l+2}(\lambda) = k - 1. \tag{2.28}$$

Since $\lambda \in T_{k,i}(m,n)$, by condition (iii) for λ, we have

$$(l+1)f_{l+1}(\lambda) + (l+1)f_{\overline{l+1}}(\lambda) + (l+2)f_{l+2}(\lambda) \equiv V_\lambda(l+1) + i - 1 \pmod 2. \tag{2.29}$$

Substituting (2.25) into (2.29), we obtain

$$lf_l(\lambda') + lf_{\overline{l}}(\lambda') + (l+1)f_{l+1}(\lambda') \equiv V_\lambda(l+1) + i - 1 - (k-1) \pmod 2. \tag{2.30}$$

Observing that $V_\lambda(l+1) = V_{\lambda'}(l) + 1$, (2.30) can be rewritten as (2.27). This leads to the conclusion that $\lambda' \in U_{k,k-i}(m-i+1, n-m)$.

It is routine to verify that the above procedure is reversible, that is, from any overpartition in $U_{k,k-i}(m-i+1, n-m)$, one can recover an overpartition in $T_{k,i}(m,n)$. This completes the proof. ∎

By relations (2.18), (2.19) and (2.22), we obtain a recurrence relation of $D_{k,i}(m,n)$.

Theorem 2.6 *For $k \geq 2$, $k \geq i \geq 1$ and for $m, n \geq 0$, we have*

$$D_{k,i}(m,n) - D_{k,i-1}(m,n) = D_{k,k-i}(m-i, n-m) + D_{k,k-i}(m-i+1, n-m). \tag{2.31}$$

By Theorem 2.2 and Theorem 2.6, we obtain a combinatorial interpretation of $W_{k,i}(x;q)$ in terms of overpartitions.

Theorem 2.7 *For $k \geq 2$, $k \geq i \geq 1$, we have*

$$W_{k,i}(x;q) = \sum_{m,n\geq 0} D_{k,i}(m,n)x^m q^n. \tag{2.32}$$

Proof. For $m,n \geq 0$, and for $k \geq 2$ and $k \geq i \geq 1$, let $w_{k,i}(m,n)$ denote the coefficient of $x^m q^n$ in $W_{k,i}(x;q)$, that is,

$$W_{k,i}(x;q) = \sum_{m,n\geq 0} w_{k,i}(m,n)x^m q^n. \tag{2.33}$$

We proceed to show that $D_{k,i}(m,n)$ and $w_{k,i}(m,n)$ satisfy the same recurrence relation with the same initial values.

Clearly, we have $w_{k,i}(0,0) = 1$ for $k \geq 2$ and $k \geq i \geq 1$, and $w_{k,0}(m,n) = 0$ for $k \geq 2, m,n \geq 0$. Moreover, we have $w_{k,i}(m,n) = 0$ if m or n is zero but not both. By Theorem 2.2, we find that

$$w_{k,i}(m,n) - w_{k,i-1}(m,n) = w_{k,k-i}(m-i,n-m) + w_{k,k-i}(m-i+1,n-m), \tag{2.34}$$

which is the same recurrence relation as $D_{k,i}(m,n)$ as given in Theorem 2.6.

It is clear that $D_{k,i}(0,0) = 1$ for $k \geq 2$ and $k \geq i \geq 1$, and $D_{k,0}(m,n) = 0$ for $k \geq 2$ and $m,n \geq 0$. Moreover, $D_{k,i}(m,n) = 0$ if m or n is zero but not both. Now, we see that $D_{k,i}(m,n)$ and $w_{k,i}(m,n)$ have the same recurrence relation and the same initial values. This completes the proof. ∎

We are now ready to finish the proof of Theorem 2.1.

Proof of Theorem 2.1. Setting $x = 1$ in (2.32), we find that the generating function for $D_{k,i}(n)$ equals $W_{k,i}(1;q)$. In other words,

$$\sum_{n\geq 0} D_{k,i}(n)q^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(2k-1)\binom{n+1}{2}-in}(1-q^{(2n+1)i})(-q)_\infty}{(q)_n(q^{n+1})_\infty}. \tag{2.35}$$

The right hand side of (2.35) can be expressed as

$$\frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{(2k-1)\binom{n+1}{2}-in} + \frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{(2k-1)\binom{n+1}{2}+i(n+1)}. \tag{2.36}$$

By substituting n with $-(n+1)$ in the second sum of (2.36), we get

$$\frac{(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k-1)\binom{n+1}{2}-in}.$$

In view of Jacobi's triple product identity, we obtain

$$\sum_{n\geq 0} D_{k,i}(n)q^n = \frac{(q^i,q^{2k-1-i},q^{2k-1};q^{2k-1})_\infty(-q)_\infty}{(q)_\infty}. \tag{2.37}$$

By the definition of $C_{k,i}(n)$, it is easily seen that

$$\sum_{n=0}^{\infty} C_{k,i}(n)q^n = \frac{(q^i, q^{2k-1-i}, q^{2k-1}; q^{2k-1})_\infty(-q)_\infty}{(q)_\infty}. \tag{2.38}$$

Comparing (2.37) and (2.38) we deduce that $C_{k,i}(n) = D_{k,i}(n)$ for $k \geq 2$, $k \geq i \geq 1$ and $n \geq 0$. This completes the proof. ∎

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