On a fully non-linear elliptic PDE in conformal geometry

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In Memory of José Escobar

Abstract

We give an expository survey on the subject of the Yamabe-type problem and applications. With a recent technique in hand, we also present a simplified proof of the result by Chang-Gursky-Yang on 4-manifolds.

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1 Introduction

One of the fundamental contribution of José Escobar in mathematics is his work on the solution of the Yamabe problem on manifolds with boundary. In this paper, we will describe some recent development on a class of fully nonlinear elliptic equations of second order in conformal geometry, which in the special case when the equation is semi-linear is the Yamabe equation. We will also discuss the state of the art of this type of fully nonlinear equations on compact manifolds with boundary with natural matching boundary conditions. The problem of finding solutions of these equations corresponds to the problem of prescribing some (Ricci) curvatures under a conformal change of metrics on a Riemannian manifold with some prescribed boundary curvature. Thus the problem can be viewed as a generalization of the Yamabe problem.

Recall on a Riemannian manifold (M^n, g), the full Riemannian tensor Rm decomposes as

$$Rm = W \ \oplus \ A \owedge g,$$

where W denotes the Weyl tensor,

$$A \ = \ \frac{1}{n-2}(Ric - \frac{R}{2(n-1)}g)$$

denotes the Schouten tensor, and $\owedge$ is the Kulkarni-Nomizu wedge product (see [3], p 110).

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Under a conformal change of metrics $g_u = e^{-2u}g$, the Weyl curvature changes pointwisely as $W_{g_u} = e^{-2u}W_g$. Thus all the information of the Riemannian tensor under a conformal change of metrics is reflected by the change of the Schouten tensor:

$$A_{g_u} = A_g + \{\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g\}. \tag{1}$$

It is thus natural to study the equation (1) and to consider the eigenvalues of the Schouten tensor. Denote $\sigma_k(A_g)$ the k-th elementary symmetric function of the eigenvalues of the Schouten tensor. By this we mean

$$\begin{aligned}
\sigma_1(A_g) &= \sum_i \lambda_i = \frac{1}{2(n-1)} R_g, \\
\sigma_2(A_g) &= \sum_{i<j} \lambda_i \lambda_j \\
&= \frac{1}{2}(|Tr\ A_g|^2 - |A_g|^2) \\
&= \frac{n}{8(n-1)(n-2)^2} R_g^2 - \frac{1}{2(n-2)^2} |Ric_g|^2, \\
\sigma_n(A_g) &= det(A_g).
\end{aligned}$$

The equation

$$\sigma_k(A_{g_u}) = f \tag{2}$$

for a given function f, is a fully nonlinear, Monge-Ampere-type equation which in the case, when $k = 1$ and when f is the constant function, is the Yamabe equation. We remark that in a Riemannian setting, the correct version of equation (2) should be written as

$$\sigma_k(g_u^{-1} A_{g_u}) = f, \tag{3}$$

where $g^{-1}A_g$ denotes the $(1,1)$ tensor $g^{ik}(A_g)_{kj}$.

For a symmetric $n \times n$ matrix M, we say $M \in \Gamma_k^+$ in the sense of Garding ([20]) if $\sigma_k(M) > 0$ and M is in the connected component of $\{\sigma_k > 0\}$ containing the identity matrix. There is a rich literature concerning the Dirichlet problem of solving the equation

$$\sigma_k(\nabla^2 u) = f, \tag{4}$$

for a given positive function f (see [7], [33], [19], [6]). Some of the techniques in these work can be modified to study equation (3). However, there are features of the equation (3) that are distinct from those of the equation (4); some of which we will explain in Section 2 of this article.

When the manifold (M, g) is locally conformally flat, and when $k \neq \frac{n}{2}$, Viaclovsky [45] showed that the equation (3) when f is the constant function is the Euler-Lagrange equation of the functional $\int \sigma_k(A_{g_u}) dv_{g_u}$. In the exceptional case when $k = n/2$, the integral $\int \sigma_k(A_g)dv_g$ is a conformal invariant. We say $g \in \Gamma_k^+$ (or g in the positive k-cone) if the corresponding Weyl-Schouten tensor $A_g(x) \in \Gamma_k^+$ for every point $x \in M$. We remark that when $k = 1$ the Yamabe equation for prescribing scalar curvature is a semilinear equation; hence the condition for $g \in \Gamma_1^+$ is the same as requiring the conformal Laplacian operator $L = -\frac{4(n-1)}{n-2}\Delta + R_g$ to be positive. It turns out that the existence of a metric with $g \in \Gamma_k^+$ for $k \geq \frac{n}{2}$ implies the positivity of the Ricci curvature of g ([29], [12], [24]). Hence the condition gives a constraint on the topology of the manifold.

To study the above fully nonlinear version of the Yamabe problem on manifolds with boundary, we first recall that for most compact manifolds with boundary Escobar [18] proved that we can conformally deform the metric to constant scalar curvature with vanishing mean curvature on the boundary. This tells us that the mean curvature equation is a matching boundary condition for the Yamabe equation. A fundamental question which arises at this stage is to define a notion of suitable boundary value problems for the given fully nonlinear equation (3). In other words, we need to find "natural" matching boundary curvatures for the $\sigma_k(A_g)$ curvature defined on the manifold. Some of recent progress in this direction and existence results of the corresponding boundary value problem of fully nonlinear equations will be discussed in Section 4 below; see also [16].

In this paper, we will discuss existence and uniqueness results concerning solutions of the equation (3), and some generalization to compact manifolds with boundary. In Section 2 of this paper, we will give a brief survey of some known results in the past few years for equation (3) on compact manifolds without boundary. In Section 3, we will present a streamlined version of the proof of one of the main results in [12],[11], [30], which corresponds to the existence of a solution for the equation (3) when f is the constant function for k=2 and n=4. The proof depends on some a priori local estimates of the equation, which was first established in ([26]) and later simplified and generalized to more general equations by the second author ([14]). This latter version of the proof has the advantage that it establishes local C^2 estimates directly from local C^0 estimates; thus obtaining C^1 estimates as a consequence.

In Section 4, we will first recall some work of Escobar on the Yamabe problem and then we summarize the work by the second author of the equation (3) on manifolds with boundary. We remark that the matching boundary curvature proposed here is again a nonlinear version of the mean curvature equation, but in the special case–i.e., when the boundary is umbilic– the boundary curvature condition is reduced to the condition on the mean curvature. In general, boundary value problems for this type of generalized Yamabe problem remain largely open.

2 Existence of solutions on closed manifolds

In this section, we will briefly survey some of the recent development in the study of the equation

$$\sigma_k(g_u^{-1}A_{g_u}) = 1 \tag{5}$$

under a conformal change of metrics $g_u = e^{-2u}g \in [g]$ on closed manifolds (M^n, g). We will break the existence results into two different categories: the existence result starting from the sign of some integral conformal invariants, and the existence results starting from the assumption that g is already in Γ_k^+.

(A) Existence result from the sign of some integral invariants.

We recall that for the Yamabe problem, the Yamabe constant is defined as

$$Y(M, [g]) = \inf_{g_u \in [g], Vol_{g_u}=1} \int R_{g_u} dv_{g_u}.$$

One can solve the equation $R_{g_u} =$ constant, with the sign of the constant depending on the sign of the Yamabe constant $Y(M, [g])$ ([47], [43], [2], [40]).

Thus, a natural question is whether there are some natural geometric conditions on the sign of some conformal invariants like that of the Yamabe constant under which equation (5) is solvable. It turns out that this question has a partial satisfactory answer in the special case when $k = 2,\ n = 4$. In this case, we first observe that the integral $\int \sigma_2(A_g) dv_g$ is itself a conformal invariant quantity. To see this fact, we recall the Chern-Gauss-Bonnet formula in dimension four

$$8\pi^2\chi(M) = \int_M (4\sigma_2(A_g) + \frac{1}{4}|W_g|^2) dv_g. \tag{6}$$

The term $|W_g|^2 dv_g$ is pointwisely conformally invariant and the term $\chi(M)$ is a topological, and hence conformal invariant. In this case, it is possible to find a criterion:

Theorem 1. *(Chang-Gursky-Yang [11]) For a closed 4-manifold (M, g) satisfying the following conformally invariant conditions:*
(i) $Y(M, g) > 0$, and
(ii) $\int \sigma_2(A_g) dv_g > 0$;
there exists a conformal metric $g_u \in \Gamma_2^+$, and moreover the equation (5) is solvable.

Remark: The original proof [11] of the existence result above depends on the solution of a family of fourth order equations involving the Paneitz operator ([38]), and some associated 4th order Q curvature named after Branson. There is a vast literature on the study of Q curvature; the readers are referred to ([1] and [8]) for some of the recent progress of this subject. The equation (2) becomes elliptic on 4-manifolds which admit a metric $g \in \Gamma_2^+$. In the article ([12]), when the manifold (M, g) is not conformally equivalent to $(\mathbb{S}^4, g_c)$, we provide a priori estimates for solutions to the equation (3) where f is any given positive smooth function; the estimates were established through a contradiction argument. To do so, we assume that there is a sequence of solutions with no a priori bounds. Then we use a blow up analysis to deduce that the limiting metric is an entire solution of the equation (3) on the Euclidean space $(\mathbb{R}^4, dx^2)$. The next step is to establish a Liouville Theorem to identify all the solutions of the equation (5) on $(\mathbb{R}^4, dx^2)$–and all of them with the conformal invariant $\int \sigma_2(A_g)dv_g$ the same as that of the standard metric on $(\mathbb{R}^4, dx^2)$ or $(\mathbb{S}^4, g_c)$. This in term implies that the manifold is conformally equivalent to $(\mathbb{S}^4, g_c)$, a contradiction to our assumption. Finally, we apply the degree theory for fully nonlinear elliptic equations by Y. Li ([34]) to the following path of equations

$$\sigma_2(A_{g_t}) = tf + (1-t) \tag{7}$$

to deform the original metric to the one with constant $\sigma_2(A_g)$.

In [30], based on some "local estimates" results developed by Guan-Wang ([26]), Gursky-Viaclovsky ([30]) gave a different proof of the first part of Theorem 1 above. The local estimates techniques were further simplified and generalized to more general equations later by S. Chen ([14]). In this paper, we will combine all the techniques above and present a simple, complete proof of Theorem 1 in the next section.

It turns out that in dimension 4, by some simple algebraic computation, one can see that any metric in Γ_2^+ has positive *Ricci* curvatures. Thus, as an immediate consequence of Theorem 1, we have the following result, which was established earlier by a different argument by Gursky ([27]).

Corollary 1. *A closed manifold (M^4, g) satisfying conditions (i) and (ii) in Theorem 1 has vanishing first Betti number.*

In terms of geometric applications, this circle of ideas may be applied to characterize a number of interesting conformal classes in terms of the ratio of the conformal invariant $\int \sigma_2(A_g)dV_g$ and the Euler number.

Theorem 2. *(Chang-Gursky-Yang [13]) Suppose (M, g) is a closed 4-manifold with $Y(M, g) > 0$.*
(I) If $\int_M \sigma_2(A_g)dv_g > \frac{1}{4}\int_M |W_g|^2\, dv_g$, then M is diffeomorphic to $(\mathbb{S}^4, g_c)$ or $(\mathbb{R}P^4, g_c)$.
(II) If M is not diffeomorphic to $(\mathbb{S}^4, g_c)$ or $(\mathbb{R}P^4, g_c)$ and $\int_M \sigma_2(A_g)dv_g = \frac{1}{4}\int_M |W_g|^2\, dv_g$, then either
(a) (M, g) is conformally equivalent to $(\mathbb{C}P^2, g_{FS})$, or
(b) (M, g) is conformal equivalent to $((\mathbb{S}^3 \times \mathbb{S}^1)/\Gamma, g_{prod})$.

The theorem above is an L^2 version of an earlier result of Margerin ([37]). The first part of the theorem should be compared to a result of Hamilton ([32]); where he pioneered the method of Ricci flow and established the diffeomorphism of M^4 to the 4-sphere under the assumption that the curvature operator is positive.

One might ask for a suitable generalization of Theorem 1 to manifolds of even dimension higher than four. One of the difficulty in doing so is that when $n \neq 4$, the functional $\int \sigma_{n/2}(A_g)dv_g$ is conformally invariant only when the manifolds are locally conformally flat ([45]). Similarly, when $k \neq \frac{n}{2}$ and $k \neq 2$, the equation (5) is the Euler equation of the functional $\int \sigma_k(A_g)dv_g$ only when manifolds are locally conformally flat ([4]). Thus, all the existence results of equation (5) when $k > 2$ (including those mentioned in part (B) below) in this section are restricted to locally conformally flat manifolds. It is in this context, Theorem 1 above has been generalized to $k \leq \frac{n}{2}$ ([23]).

In a recent preprint ([21]), Ge-Lin-Wang established some existence result for equation (5) under some sign condition of some conformal invariants which they have defined, and the condition that the scalar curvature be pointwisely positive. Their result also includes some existence result of conformal metrics with $\frac{\sigma_2(A_g)}{\sigma_1(A_g)}$ = constant, with the sign of the constant not necessarily positive.

(B) Existence results assuming that g is in Γ_k^+.

There has been a lot of progress in this direction, via different approaches. In the case of $1 \leq k \leq n$, assuming (M, g) is locally conformally flat, the existence of a metric $g_u \in [g]$ satisfying (5) has been proved by Y.Li and A. Li ([35]). They also have established the Liouville theorem ([36]). Similar existence results have also been established by Guan-Wang ([25]) using the parabolic flow method; see also Shen-Trudinger-Wang ([42]) for another evolutionary approach for the cases $k < \frac{n}{2}$. In the case when $k > \frac{n}{2}$, the existence result was established by Gursky-Viaclovsky [31] for all manifolds; their proof is a beautiful interplay between the analytic and geometric aspects of the equation, and depends heavily on the geometric property that in these cases, the Ricci curvature of the metric g is positive. Trudinger-Wang [44] also established some Harnack inequality for metrics in positive k-cone in this case.

There are many other work related to the study of the $\sigma_k(A_g)$ equation, for example the study of the singular set of the σ_k equations on punctured domains ([22]), and the classification of the singular solutions on annular domains ([10]).

We would like to add that with the possible exception of [21], the existence result for solutions of this geometric type of σ_k equation is largely restricted to metrics in *positive* k-cone. When a metric is in the negative k-cone (for example, when $k = 2$, $R_g < 0$ and $\sigma_2(A_g) > 0$), it is known that on any compact manifold, there are a priori C^0 and C^1 estimates for solutions of the equation (2) depending on the data of the function f, but there is no local C^2 estimates (c.f., the example by Heinz-Levy in [41], also a modification in [42]). The reader is referred to the recent survey articles by Gursky ([28]) and by Viaclovsky ([46]) for a more complete survey for the current status of the problem.

3 Skipping gradient estimates

In this section, we present local estimates by Chen [14] for some fully nonlinear equations. The main technique we introduce is to derive local C^2 estimates directly from local C^0 estimates, and obtain C^1 estimates as a consequence. Applying the method of local estimates, we also give a simplified proof of the main result in Chang-Gursky-Yang [11] [12].

We begin with prescribing a class of fully nonlinear equations which have similar structure to the Monge-Ampere equations. Let Γ be an open convex cone in $\mathbb{R}^n$ with vertex at the origin satisfying $\{\lambda : \lambda_i > 0, \forall i\} \subset \Gamma \subset \{\lambda : \sum_i \lambda_i > 0\}$. Suppose that $F(\lambda)$ is a homogeneous symmetric function of degree one normalized with $F(1, \cdots, 1) = 1$.

Consider the equation

$$F(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A) = fe^{-2u} + C_0 \quad (\text{resp.}\, fe^{2u} + C_0), \tag{8}$$

where F satisfies the following conditions in Γ :
(S0) F is positive;
(S1) F is concave (i.e., $\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}$ is negative semi-definite);
(S2) F is monotone (i.e., $\frac{\partial F}{\partial \lambda_i}$ is positive).
The model case is when $F = \sigma_k^{\frac{1}{k}}$ with $\Gamma = \Gamma_k^+ = \{\lambda : \sigma_i > 0, 1 \le i \le k\}$.

Theorem 3. *(Chen [14]) Let $u \in C^4$ be a solution to (8). Then*

$$\sup_{x\in B_{\frac{r}{2}}} (|\nabla u|^2 + |\nabla^2 u|) \le C(1 + \sup_{x\in B_r} e^{-2u}) \quad (resp.\, C(1 + \sup_{x\in B_r} e^{2u})),$$

where $C = C(r, n, C_0, \|f\|_{C^2(B_r)}, \inf_{B_r} f)$.

The idea of proof is to derive the Hessian bounds directly from C^0 bounds. There is a magic cancellation phenomenon coming from the structure of this kind of equations. The idea of skipping gradient estimates has appeared before in the literature in the study of complex Monge-Ampere equations by Yau [48]. The equation (8) has a more complicated structure. In below we will show that the same idea of skipping gradient estimates also apply to this type of real valued fully nonlinear equations.

Proof. For simplicity, we present the proof when g is flat. For non-flat metrics, the computations are the same up to negligible lower order terms.

Let $\hat{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g$. The condition $\Gamma_1^+ \subset \Gamma$ gives

$$0 < tr_g\, \hat{A} = \Delta u - \frac{n-2}{2}|\nabla u|^2.$$

Thus, Δu is positive and

$$|\nabla u|^2 < C\Delta u. \tag{9}$$

We will show that Δu is bounded. Let $H = \eta(\Delta u + |\nabla u|^2) = \eta K$. Denote $r^2 :\equiv \sum_i x_i^2$. Let $\eta(r)$ be a cutoff function such that $0 \le \eta \le 1$, $\eta = 1$ in $\overline{B}^+_{\frac{r}{2}}$ and $\eta = 0$ outside $\overline{B}^+_r$, and also $|\nabla \eta| < C\frac{\eta^{\frac{1}{2}}}{r}$ and $|\nabla^2\eta| < \frac{C}{r^2}$. Without loss of generality, we may assume $r = 1$ and $K = \Delta u + |\nabla u|^2 \gg 1$.

At the maximal point x_0, we have

$$H_i = \eta_i K + \eta K_i = 0, \tag{10}$$

and

$$H_{ij} = \eta_{ij}K + \eta_i K_j + \eta_j K_i + \eta K_{ij} = (\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)K + \eta K_{ij}$$

is negative semi-definite. Using the positivity of F^{ij}, we get

$$0 \geq F^{ij} H_{ij} = F^{ij}((\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)K + \eta K_{ij}) \geq \eta F^{ij} K_{ij} - C \sum_i F^{ii} K, \tag{11}$$

where we use conditions on η.

Now we compute $F^{ij}K_{ij}$:

$$F^{ij} K_{ij} = F^{ij}(u_{ijll} + 2u_{li}u_{lj} + 2u_l u_{ijl}).$$

We denote $I = F^{ij}u_{ijll}$ and $II = F^{ij}(2u_{li}u_{lj} + 2u_l u_{ijl})$. For term I, we notice that

$$\hat{A}_{ij,ll} = u_{ijll} + 2u_{il}u_{jl} + u_i u_{jll} + u_j u_{ill} - (u_k u_{kll} + u_{kl}^2) g_{ij}.$$

Then

$$I = F^{ij}(\hat{A}_{ij,ll} - 2u_{li}u_{lj} - 2u_{ill}u_j + (u_{lk}^2 + u_k u_{kll}) g_{ij}),$$

where $F^{ij}(u_i u_{jll}) = F^{ij}(u_j u_{ill})$ because F^{ij} is symmetric. Now using (10) to replace u_{lli} and u_{kll} yields

$$I = F^{ij}\hat{A}_{ij,ll} + F^{ij}(-2u_{li}u_{lj} - 2u_j(-2u_l u_{li} - \frac{\eta_i}{\eta}K) + (|\nabla^2 u|^2 + u_k(-2u_l u_{lk} - \frac{\eta_k}{\eta}K)) g_{ij}).$$

By (9) and the conditions on η, we have

$$I \geq F^{ij}\hat{A}_{ij,ll} + F^{ij}(-2u_{li}u_{lj} + 4u_j u_l u_{li} + (|\nabla^2 u|^2 - 2u_k u_l u_{lk}) g_{ij}) - C\sum_i F^{ii}\eta^{-\frac{1}{2}}(1 + |\nabla^2 u|^{\frac{3}{2}}).$$

For term II, we use the formula

$$\hat{A}_{ij,l} = u_{ijl} + u_i u_{jl} + u_j u_{il} - u_k u_{kl} g_{ij}$$

to obtain

$$II = F^{ij}(2u_{li}u_{lj} + 2u_l(\hat{A}_{ij,l} - 2u_i u_{jl} + u_k u_{kl} g_{ij})).$$

Combining term I and term II together, we find that

$$\begin{aligned} F^{ij}K_{ij} \;\; &\geq \;\; F^{ij}\hat{A}_{ij,ll} + F^{ij}(-2u_{li}u_{lj} + 4u_j u_l u_{li} + (|\nabla^2 u|^2 - 2u_k u_l u_{lk}) g_{ij}) \\ &\quad + F^{ij}(2u_{li}u_{lj} + 2u_l\hat{A}_{ij,l} - 4u_i u_{jl} u_l + 2u_k u_{kl} u_l g_{ij}) - C\sum_i F^{ii}\eta^{-\frac{1}{2}}(1 + |\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

Now comes the key observation of the proof: three terms from I cancel out three terms from II. Thus, after the cancellations we arrive at

$$F^{ij}K_{ij} \geq F^{ij}\hat{A}_{ij,ll} + F^{ij}|\nabla^2 u|^2 g_{ij} + F^{ij} 2u_l \hat{A}_{ij,l} - C\sum_i F^{ii}\eta^{-\frac{1}{2}}(1 + |\nabla^2 u|^{\frac{3}{2}}). \tag{12}$$

Now returning to (11), applying η on both sides and using (12) produces

$$
\begin{aligned}
0 &\geq \eta^2 F^{ij} K_{ij} - C \sum_i F^{ii} \eta K \\
&\geq \eta^2 F^{ij} \hat{A}_{ij,ll} + \eta^2 F^{ij} |\nabla^2 u|^2 g_{ij} + \eta^2 F^{ij} (2u_l \hat{A}_{ij,l}) - C \sum_i F^{ii} (1 + \eta |\nabla^2 u| + \eta^{\frac{3}{2}} |\nabla^2 u|^{\frac{3}{2}}),
\end{aligned}
$$

where we have used the fact that $K \leq C|\nabla^2 u| + C$ and (9). By the concavity of F, we have $F^{ij}\hat{A}_{ij,ll} \geq (fe^{-2u})_{ll}$. Hence,

$$
\begin{aligned}
0 &\geq \eta^2 \sum_i F^{ii} |\nabla^2 u|^2 + \eta^2 (fe^{-2u})_{ll} + 2\eta^2 u_l (fe^{-2u})_l - C \sum_i F^{ii} (1 + \eta |\nabla^2 u| + \eta^{\frac{3}{2}} |\nabla^2 u|^{\frac{3}{2}}) \\
&\geq \sum_i F^{ii} (\eta^2 |\nabla^2 u|^2 - C - C\eta |\nabla^2 u| - C\eta^{\frac{3}{2}} |\nabla^2 u|^{\frac{3}{2}}),
\end{aligned}
$$

where in deriving the second inequality we have applied the inequality $\sum_i F^{ii} \geq 1$. This gives $(\eta|\nabla^2 u|)(x_0) \leq C$. Thus, for $x \in B_{\frac{r}{2}}$, we conclude that $H = \Delta u + |\nabla u|^2$ is bounded. As a result, Δu and $|\nabla u|^2$ are both bounded. To get the Hessian bounds, consider the maximum of $\eta(\nabla^2 u + du \otimes du)$ over the set $(x, \xi) \in (B_1, \mathbb{S}^n)$. We can perform similar computations as before using the inequality $\eta|\nabla u|^2 < C$ to obtain the Hessian bounds. □

In the special case when $F = \sigma_k^{\frac{1}{k}}$, Theorem 3 was proved by Guan-Wang [26] for the case when $F = fe^{-2u}$ and later observed by Gursky-Visclovsky [30] for the case when $F = fe^{2u}$.

Now we are in the position to give a simplified proof of the main result in [11] [12]. The proof presented here combines the result in Theorem 3 and some techniques introduced by Gursky-Viaclovsky in their works [30] [31]. The following is a more general theorem than Theorem 2.1.

Theorem 4. *(Chang-Gursky-Yang [11] [12]) Let (M, g) be a compact connected four-manifold. Suppose that (M, g) is not conformally equivalent to $(\mathbb{S}_4, g_c)$, where g_c is the standard metric on the sphere. If $Y(M, [g])$ and $\int_M \sigma_2$ are both positive, then given a positive function f there exists a metric $\hat{g} \in [g]$ such that $\sigma_2(A_{\hat{g}}) = f$.*

Proof. The proof consists of two parts: in (A), we deform the metric such that the σ_2 curvature is positive; in (B), we deform it again such that $\sigma_2 = 1$.

(A). Let the background metric g be the Yamabe metric such that R_g is a positive constant. Let $A^t = A + \frac{1-t}{2}(tr_g A)g$. Under a conformal change $\hat{g} = e^{-2u} g$,

$$
\hat{A}^t :\equiv A^t_{\hat{g}} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \frac{1-t}{2}(\Delta u - |\nabla u|^2) g + A^t.
$$

We can choose a large number Θ such that

$$
A^{-\Theta} = \frac{1}{2}(Ric_g + \frac{\Theta}{6} R_g g)
$$

is positive definite. Let $f(x)=\sigma_2^{\frac{1}{2}}(A_g^{-\Theta})$. Thus, we have $A_g^{-\Theta}\in\Gamma_2^+$ and f is positive.

Consider the following path of equations for $-\Theta\le t\le 1$:

$$\sigma_2^{\frac{1}{2}}(\nabla^2u+du\otimes du-\frac{1}{2}|\nabla u|^2g+\frac{1-t}{2}(\Delta u-|\nabla u|^2)g+A_g^t)=f(x)e^{2u}. \tag{13}$$

We will use the continuity method. Let $\mathcal{S}=\{t\in[-\Theta,1]:\exists$ a solution $u\in C^{2,\alpha}(M)$ to (13) with $\hat{A}^t\in\Gamma_2^+\}$. At $t=-\Theta$, we have $u\equiv 0$ is a solution. Hence, $\mathcal{S}$ is nonempty. At a solution u, the linearized operator $\mathcal{L}^t:C^{2,\alpha}(M)\to C^{\alpha}(M)$ is invertible and the implicit function theorem implies that $\mathcal{S}$ is open. To show $\mathcal{S}$ is closed, it remains to establish a priori estimates for solutions to (13).

At the maximal point x_0 of u, we have $|\nabla u|=0$ and $\nabla^2u(x_0)$ is negative semi-definite. Hence, $\Delta u(x_0)\le 0$. By the Newton-MacLaurin inequality, $\sigma_2^{\frac{1}{2}}\le\frac{\sqrt{6}}{4}\sigma_1$. Then

$$f(x_0)e^{2u(x_0)}=\sigma_2^{\frac{1}{2}}(g^{-1}\hat{A}^t)\le\frac{\sqrt{6}}{4}\sigma_1(g^{-1}\hat{A}^t)=\frac{\sqrt{6}}{4}(3-2t)(\Delta u-|\nabla u|^2)+\frac{\sqrt{6}}{4}tr_g\,A_g^t\le C.$$

Hence, u is upper bounded. Now by Theorem 3, we have $|\nabla u|\le C$. Therefore, $\sup_M u\le\inf_M u+C$. To get C^0 estimates, we only need to show that $\sup_M u$ is lower bounded. Integrating the equation gives

$$Ce^{4\sup_M u}\ge\int_M f^2e^{4u}dV_g=\int_M\sigma_2(g^{-1}\hat{A}^t)dV_g=\int_M\sigma_2(\hat{g}^{-1}\hat{A}^t)dV_{\hat{g}},$$

where in the second equality we use $dV_{\hat{g}}=e^{-4u}dV_g$. Note that $\sigma_2(\hat{g}^{-1}\hat{A}^t)=\sigma_2(\hat{A})+\frac{3}{2}(1-t)(2-t)\sigma_1^2(\hat{A})$ and by assumption, the conformal invariant $\int_M\sigma_2$ is positive. Thus, the above formula becomes

$$Ce^{4\sup_M u}\ge\int_M(\sigma_2(\hat{A})+\frac{3}{2}(1-t)(2-t)\sigma_1^2(\hat{A}))dV_{\hat{g}}\ge\int_M\sigma_2(\hat{A})dV_{\hat{g}}=\int_M\sigma_2(A_g)dV_g>0.$$

This gives a lower bound of $\sup_M u$.

Once we have C^0 bounds, by Theorem 3, we get C^2 estimates. The equation becomes uniformly elliptic and concave. Higher order regularity then follows from standard elliptic theories. As a result, at $t=1$, there exists a solution such that $\sigma_2(\hat{A}^1)=\sigma_2(\hat{A})>0$.

(B). We still denote the metric obtained in (A) by g. Let $\hat{A}=\nabla^2u+du\otimes du-\frac{1}{2}|\nabla u|^2g+A_g$. Now we want to solve

$$\sigma_2^{\frac{1}{2}}(\nabla^2u+du\otimes du-\frac{1}{2}|\nabla u|^2g+A_g)=f(x)\,e^{-2u}$$

with $\hat{A}\in\Gamma_2^+$. Let $\zeta(t)\in C^1[0,1]$ satisfies $0\le\zeta\le1$, $\zeta(0)=0$, and $\zeta=1$ for $t\ge\frac{1}{2}$. Consider the following path of equations for $0\le t\le1$ with $\hat{A}+S_g\in\Gamma_2^+$:

$$\sigma_2^{\frac{1}{2}}(\nabla^2u+du\otimes du-\frac{1}{2}|\nabla u|^2g+A_g+S_g)=(1-t)(\int_M e^{-5u})^{\frac{2}{5}}+\zeta(t)f\,e^{-2u}, \tag{14}$$

where $S_g = (1-\zeta(t))(\frac{1}{\sqrt{6}}V_g^{\frac{2}{5}}g - A_g)$. We will use the Leray-Schauder degree theory. If the degree is nonzero at $t=0$ and we have a priori estimates for (14), by homotopy-invariance, there exists a solution at $t=1$.

It is not hard to show that the degree is nonzero at $t=0$. In fact, by maximal principle $u \equiv 0$ is the unique solution to

$$\sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \frac{1}{\sqrt{6}}V^{\frac{2}{5}}g) = (\int_M e^{-5u})^{\frac{2}{5}}.$$

The linearized operator $\mathcal{P} : C^{2,\alpha}(M) \to C^{\alpha}(M)$ at $u=0$ is

$$\mathcal{P}(\phi) = \frac{\sqrt{6}}{4}\Delta\phi + 2V_g^{-\frac{2}{5}}\int_M \phi,$$

which is invertible. Consequently, the problem reduces to establishing a priori estimates. Again using Theorem 3, we only need to derive C^0 estimates.

We begin by proving the boundedness of the integral term in (14). At the maximum point x_0, then $\nabla u = 0$, and $\nabla^2 u$ is negative semi-definite. Therefore,

$$\begin{aligned}(1-t)(\int_M e^{-5u})^{\frac{2}{5}} &\le (1-t)(\int_M e^{-5u})^{\frac{2}{5}} + \zeta(t)f(x_0)\,e^{-2u} \\ &= \sigma_2^{\frac{1}{2}}(\nabla^2 u(x_0) + A_g(x_0) + S_g(x_0)) \le \sigma_2^{\frac{1}{2}}(A_g(x_0) + S_g(x_0)) < C. \qquad (15)\end{aligned}$$

Now we prove that $\inf_M u > -C$. Since the proof is easier for $t \in [0, 1-\varepsilon]$ for any small fixed ε, we present here the proof when $t \to 1$.

We will apply the blow-up argument. Suppose on the contrary there is a sequence of solutions $\{u^i\}$ with $t_i \to 1$ such that $u^i(p_i) = \inf u^i \to -\infty$ and $p_i \to p_0$. Let $\epsilon_i = e^{\inf u^i} \to +0$ and ι be the injectivity radius of (M, g). For simplicity, we denote the metric $e^{-2u^i}g$ by $\hat{g}_i$ and the Schouten tensor $A_{\hat{g}_i}$ by $\hat{A}_i$.

Using the normal coordinates at p_i, we define the mapping

$$\begin{aligned}\mathcal{T}_i : B(0, \tfrac{\iota}{\epsilon_i}) \subset \mathbb{R}^4 \to\ & M \\ x \to\ & \exp_{p_i}(\epsilon_i x) = y,\end{aligned}$$

where exp is the exponential map. On $\mathbb{R}^4$, define the metric $g_i = \epsilon_i^{-2}\mathcal{T}_i^* g$ and the function $\tilde{u}^i = u^i(\mathcal{T}_i(x)) - \ln\epsilon_i$. Denote the metric $e^{-2\tilde{u}^i}g_i$ by $\tilde{g}_i$. Then $\tilde{u}^i(0) = u^i(p_i) - \ln\epsilon_i = 0$ and $\tilde{u}^i(x) \ge 0$. Moreover, since $t_i \to 1$, we have $\zeta(t_i) = 1$. Therefore, $\tilde{u}^i$ satisfies

$$\sigma_2^{\frac{1}{2}}(\nabla_{g_i}^2\tilde{u}^i + d\tilde{u}^i \otimes_{g_i} d\tilde{u}^i - \frac{1}{2}|\nabla_{g_i}\tilde{u}^i|^2 g_i + A_{g_i}) = \epsilon_i^2(1-t_i)(\int_M e^{-5u^i})^{\frac{2}{5}} + f(\mathcal{T}_i(x))\,e^{-2\tilde{u}^i}$$

on $B(0, \frac{\iota}{\epsilon_i})$ in $\mathbb{R}^4$. Note that g_i tends to the Euclidean metric ds^2. By (15), the integral term in the above equation is bounded. Hence, by Theorem 3 and the fact that $\tilde{u}^i \ge 0$, we get

$$\sup_{B(0,r)}(|\nabla_{g_i}\tilde{u}^i| + |\nabla_{g_i}^2\tilde{u}^i|) < C(r).$$

Integrating from zero, we have

$$\sup_{B(0,r)} (|\tilde{u}^i| + |\nabla_{g_i} \tilde{u}^i|^2 + |\nabla^2_{g_i} \tilde{u}^i|) < C(r).$$

Since $f(\mathcal{T}_i(x)) \to f(p_0)$, the equation is uniformly elliptic and concave. Thus, we have C^∞ bounds. Then $\{\tilde{u}^i\}$ converges uniformly on compact sets to a solution $u \in C^\infty$ of

$$\sigma_2^{\frac{1}{2}}(\nabla^2 u + du \otimes u - \frac{1}{2}|\nabla u|^2 ds^2) = f(p_0)\, e^{-2u},$$

where the derivatives are with respect to ds^2. By the uniqueness theorem [11], the metric $e^{-2u} ds^2$ must come from the pulling-back of the standard metric g_c on the sphere. Hence,

$$4\pi^2 \leftarrow \int_{B(0,\frac{d_i}{\epsilon_i})} \sigma_2(A_{\tilde{g}_i}) dV_{\tilde{g}_i} = \int_{B(p_i,d_i)} \sigma_2(\hat{A}_i) dV_{\hat{g}_i} \le \int_M \sigma_2(\hat{A}_i) dV_{\hat{g}_i} < 4\pi^2.$$

This gives a contradiction as $\int_M \sigma_2(A) dV_g < 4\pi^2$ when (M, g) is not conformally equivalent to $(\mathbb{S}^4, g_c)$.

Once u is lower bounded, by Theorem 3, $\sup_M u \le \inf_M u + C$. At the minimum point x_0, we have $\nabla u = 0$, and $\nabla^2 u$ is positive semi-definite. Therefore,

$$Ce^{-2\inf u} \ge (1-t)(\int_M e^{-5u})^{\frac{2}{5}} + \zeta(t) f(x_0)\, e^{-2u} = \sigma_2^{\frac{1}{2}}(\nabla^2 u(x_0) + A_g(x_0)) \ge \sigma_2^{\frac{1}{2}}(A_g(x_0)) > 0.$$

This gives an upper bound of $\inf_M u$. The proof is complete. □

4 Yamabe-type problem on manifolds with boundary

In this section, we first recall Escobar's work on the Yamabe problem on manifolds with boundary. Then we discuss some recent progress on a nonlinear version of the problem which generalizes Escobar's results.

Recall that the Yamabe constant for compact manifolds with boundary is a conformal invariant, defined as

$$Y(M, \partial M, [g]) = \inf_{\hat{g} \in [g], Vol_{\hat{g}}=1} (\int_M R_{\hat{g}} + \oint_{\partial M} h_{\hat{g}}),$$

where h is the mean curvature. The Yamabe problem on manifolds with boundary consists in finding a conformal metric $\hat{g} = e^{-2u} g$ such that the scalar curvature is constant on the manifold and the mean curvature is zero on the boundary. Recall that the boundary is umbilic if the second fundamental form $L_{\alpha\beta} = \mu(x) g_{\alpha\beta}$, which is a conformal invariant property.

Theorem 5. *(Escobar [18]) Let (M, g) be a compact manifold with boundary. Suppose that (M, g) does not satisfy the following condition:*

$$n \ge 6, W|_{\partial M} = 0, W \ne 0 \text{ on } M \text{ and } (\partial M, g) \text{ is umbilic.} \tag{16}$$

Then there exists $\hat{g} \in [g]$ such that $R_{\hat{g}}$ is constant and $h_{\hat{g}} = 0$.

The proof of Theorem 5 consists in finding a testing function whose "energy" is smaller than that of the standard model case, i.e., the round upper-hemisphere. When $n \geq 6$, if either ∂M is nonumbilic or W is nonzero at some boundary point, one can construct a local testing function. While in the cases when $n = 3, 4, 5$ or when (M, g) is conformally flat, one constructs a global testing function by using properties the Green's function.

It has been conjectured for a long time whether the result in Theorem 5 still holds when (M, g) satisfies the condition (16). The difficulty lies in the fact that when $W = 0$ on the boundary, the local testing function used in the original proof by Aubin [2] on closed manifolds does not have an energy smaller than that of the standard model case. For recent progress for this remaining case, the readers are referred to Brendle and Chen [5].

Now we return to the study of a nonlinear version of the Yamabe problem on manifolds with boundary. The first fundamental question we ask is the following:

Question 1: Given a curvature polynomial like that of the $\sigma_k(A_g)$ on the manifold, what are the natural matching curvatures on the boundary?

When the curvature polynomial is the scalar curvature, the mean curvature is a natural matching one on the boundary. To study the conformal deformation problem of the $\sigma_k(A)$ curvature on manifolds with boundary, we need to give a criterion to define the matching curvatures on the boundary. We also have to find some relevant conformal invariants (like that of the Yamabe constant), which in general involve a more complicated curvature polynomial than the mean curvature.

To begin with, we consider four dimensional compact manifolds with boundary. Recall the Gauss-Bonnet formula:

$$32\pi^2\chi(M, \partial M) = \int_M |W|^2 + 16\Big(\int_M \sigma_2(A_g) + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g\Big), \tag{17}$$

where $\mathcal{B}_g = \frac{1}{2}Rh - R_{nn}h - R_{\gamma\alpha\gamma\beta}L^{\alpha\beta} + \frac{1}{3}h^3 - h|L|^2 + \frac{2}{3}trL^3$. Similar to the reasoning in the previous section, we have that $\int_M \sigma_2 + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g$ is a conformal invariant. As a generalization of Theorem 1, it is natural to study the conformal deformation of a metric to constant $\sigma_2(A_g)$ curvature with vanishing $\mathcal{B}$ curvature. The first result in this direction was proved by Chen [17] [16] in her thesis.

Theorem 6. *(Chen [16]) Let (M, g) be a compact connected four-manifold with umbilic boundary. If $Y(M, \partial M, [g])$ and $\int_M \sigma_2(A_g) + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g$ are both positive, then there exists a metric $\hat{g} \in [g]$ such that $\sigma_2(A_{\hat{g}})$ is a positive constant and $\mathcal{B}_{\hat{g}}$ is zero.*

The metric $\hat{g}$ found in Theorem 6 satisfies $Ric_{\hat{g}} > 0$ and totally geodesic boundary. Therefore, it gives a topological constraint. Moreover, Theorem 6 can be applied to the setting of conformally compact Einstein manifolds; see [16].

The proof of Theorem 6 relies on some delicate boundary estimates. The first key ingredient is to prove that the maximum of the second derivatives can not happen on the boundary. This is achieved by deriving uniform estimates of the third derivatives on the boundary. The proof is "non-traditional" in the sense that we do not use the method of constructing a barrier function. Instead, we use an idea similar to the Hopf lemma. Once we have determined that the maximum of the second derivatives happens in the interior, the second key ingredient is to use the method of skipping gradient estimates as in Section 3 to prove boundary C^2 estimates directly from boundary C^0 estimates.

Now we come back to Question 1 above. The question is partially answered when $k = 2, n = 4$, i.e., $\mathcal{B}$ curvature is a natural matching boundary curvature for $\sigma_2(A)$. However, the answer is not completely satisfactory as $\sigma_2(A)$ is a symmetric function of the eigenvalues of A, while $\mathcal{B}$ does not seem to be as "symmetric" as $\sigma_2(A)$. To this end, in [16] the following boundary curvatures are defined. Let

$$\mathcal{B}^2 = \begin{cases} \frac{2}{n-2}\sigma_{2,1}(A^T, L) + \frac{2}{(n-2)(n-3)}\sigma_{3,0}(A^T, L) & n \geq 4 \\ 2\sigma_{2,1}(A^T, L) + \frac{1}{3}h^3 - \frac{1}{2}h|L|^2 & n = 3, \end{cases} \tag{18}$$

where A^T is the tangential part of A and $\sigma_{i,j}$'s are the mixed symmetric functions; see [39]. The notion $\sigma_{i,j}$ can be viewed as a polarization of σ_i such that $\sigma_{i,j}(A, A) = \sigma_i(A)$; therefore this is a natural generalization of σ_i. For $k \geq 3$, let

$$\mathcal{B}^k = \textstyle\sum_{i=0}^{k-1} C(n, k, i)\sigma_{2k-i-1,i}(A^T, L) \quad n \geq 2k. \tag{19}$$

The above definition breaks down when $n < 2k$ since in this case we can not define σ_{2k-1} on the boundary with dimension less than $2k - 1$.

The definitions of $\mathcal{B}^k$'s are motivated by the Gauss-Bonnet formulas. More specifically, when $n = 2k$, $\mathcal{B}^k$ is the boundary term in the Gauss-Bonnet formula plus some local conformal invariant. In other words, let $\mathcal{F}_k(g) = \int_M \sigma_k(A) + \oint_{\partial M} \mathcal{B}^k$. For $n = 4$, we have $\mathcal{F}_2 = 2\pi^2\chi(M, \partial M) - \frac{1}{16}\int |\mathcal{W}|^2 + \frac{1}{4}\oint \mathcal{L}_4$, where $\mathcal{L}_4$ is some local conformal invariant. When $n = 2k$, suppose M is locally conformally flat. Then $\mathcal{F}_{\frac{n}{2}} = \frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2})!}\chi(M, \partial M)$ and the Gauss-Bonnet formulas can be written as

$$\frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2})!}\chi(M, \partial M) = \int_M \sigma_{\frac{n}{2}}(A_g) + \oint_{\partial M} \sum_{i=0}^{\frac{n}{2}-1} C(n, \frac{n}{2}, i)\sigma_{n-i-1,i}(A^T, L);$$

see [16].

By using variational characterizations, we can justify that for $\sigma_k(A)$, the $\mathcal{B}^k$'s are natural curvatures on the boundary.

Theorem 7. *(Chen [16]) Let (M, g_0) be a compact manifold of dimension $n \geq 3$ with boundary.*
(a) Suppose $n \neq 4$. Then g is a critical point of $\mathcal{F}_2 |_{\mathcal{M}}$ if and only if g satisfies

$$\begin{cases} \sigma_2(A_g) = constant & in\, M \\ \mathcal{B}_g^2 = 0 & on\, \partial M. \end{cases}$$

(b) Suppose $n > 2k$ and M is locally conformally flat. Then g is a critical point of $\mathcal{F}_k |_{\mathcal{M}}$ if and only if g satisfies

$$\begin{cases} \sigma_k(A_g) = constant & in\, M \\ \mathcal{B}_g^k = 0 & on\, \partial M. \end{cases}$$

The above variational properties still hold if we add some local conformal invariants on the boundary to $\mathcal{B}^k$. More precisely, let $\mathcal{L}$ be a local conformal invariant with an appropriate weight. If we consider a conformal variation of $(\mathcal{F}_k + \oint \mathcal{L})\,|_{\mathcal{M}}$, then a critical metric g satisfies $\sigma_k(A_g) =$ constant on M and $\mathcal{B}^k_g + \mathcal{L} = 0$ on the boundary [16]. For closed manifolds, such variational properties were proved in [45].

A generalization of the Yamabe problem on manifolds with boundary can be stated as:

Question 2: Can we find a conformal metric $\hat{g}$ such that

$$\begin{cases} \sigma_k(A_{\hat{g}}) = 1 & \textit{in } M \\ \mathcal{B}^k_{\hat{g}} = 0 & \textit{on } \partial M? \end{cases} \tag{20}$$

Again, the existence results are in two different categories: (A) the result starting from the sign of some integral conformal invariants, and (B) the result starting from the assumption that g is already in Γ_k^+.

For (A), when $k = 2, n = 4$, assume further that the boundary is umbilic. Theorem 6 says that (20) is solvable if $Y(M, \partial M, [g]) > 0$ and $\int_M \sigma_2(A_g) + \frac{1}{2}\oint_{\partial M} \mathcal{B}_g > 0$. In [16], for locally conformally flat compact manifolds with umbilic boundary, Theorem 6 is generalized to $k > 2$ under some conformal invariant condition.

In the case of (B), for locally conformally flat compact manifolds with umbilic boundary, Chen [15] proved that if $h_g \geq 0$, then there exists a metric $\hat{g} \in [g]$ such that $\sigma_k(A_{\hat{g}}) = 1$ and the boundary is totally geodesic. Note that in this case, $\mathcal{B}^k = 0$ if and only if the boundary is totally geodesic.

In general, Question 2 remains largely open.

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