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PRINCIPAL CURVATURE ESTIMATES FOR THE CONVEX LEVEL SETS OF SEMILINEAR ELLIPTIC EQUATIONS

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Dedicated to Professor Nirenberg on his 85th birthday

Abstract. We give a positive lower bound for the principal curvature of the strict convex level sets of harmonic functions in terms of the principal curvature of the domain boundary and the norm of the boundary gradient. We also extend this result to a class of semi-linear elliptic partial differential equations under certain structure condition.

1. **Introduction.** The convexity of the level sets of the solutions of elliptic partial differential equations is a classical subject. For instance, Ahlfors [1] contains the well-known result that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. In 1931, Gergen [8] proved the star-shapeness of the level sets of Green function on 3-dimensional star-shaped domain. In 1956, Shiffman [20] studied the minimal annulus in $\mathbb{R}^3$ whose boundary consists of two closed convex curves in parallel planes P_1, P_2. He proved that the intersection of the surface with any parallel plane P, between P_1 and P_2, is a convex Jordan curve. In 1957, Gabriel [7] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex, see also the book by Hormander [9]. Lewis [13] extended Gabriel's result to p-harmonic functions in higher dimensions. Caffarelli-Spruck [6] generalized the results [13] to a class of semilinear elliptic partial differential equations. Using the idea of Caffarelli-Friedman [4], Korevaar [12] gave a new proof on the results of [13, 6] by applying the constant rank theorem of the second fundamental form of the convex level sets of p-harmonic function. A survey of this subject is given by Kawohl [11]. For more

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recent related extensions, please see the papers by Bianchini-Longinetti-Salani [3] and Bian-Guan-Ma-Xu [2].

Now we turn to the curvature estimates of the level sets of the solutions of elliptic partial differential equations. For 2-dimensional harmonic function and minimal surface with convex level curves, Ortel-Schneider [19], Longinetti [14] and [15] proved that the curvature of the level curves attains its minimum on the boundary. Jost-Ma-Ou [10] and Ma-Ye-Ye [18] proved that the Gaussian curvature and the principal curvature of the convex level sets of 3-dimensional harmonic function attains its minimum on the boundary.

In this paper, using the strong maximum principle, we obtain a principal curvature estimates for the strictly convex level set of higher dimensional harmonic function and a class of semilinear elliptic partial differential equations. Our curvature estimate is in terms of the principal curvature of the boundary and the boundary gradient of the solution of elliptic partial differential equations.

Now we state our result.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $a \leq u \leq b$ and $u \in C^4(\Omega)\bigcap C^2(\bar{\Omega})$ be a solution for*

$$\Delta u = f(x,u) \geq 0 \quad in \quad \Omega. \tag{1.1}$$

Assume $|\nabla u| \neq 0$ in Ω. If the level sets of u are strictly convex with respect to normal Du, and let k_1 be the least principal curvature of the level sets. Consider the following two assertions (A_1) and (A_2):

(A_1) The function $|Du|k_1$ attains it minimum on the boundary;

(A_2) The function $|Du|^{-2}k_1$ attains it minimum on the boundary.

Then we have the following:

Case 1*: Suppose $f = 0$, then (A_1) is valid.*

Case 2*: Suppose $f = f(u)$. If $f_u \leq 0$, then (A_1) is valid; if $f_u \geq 0$, then (A_2) is valid.*

Case 3*: Suppose $f = f(x)$. If $F(t,x) := t^3 f(x)$ is a convex function for $(t,x) \in (0,+\infty)\times\Omega$ (or for $f > 0$ and $f^{-\frac{1}{2}}$ is concave), then (A_1) is valid.*

Case 4*: Suppose $f = f(x,u)$. If $f_u \leq 0$ and $F_u(t,x) := t^3 f(x,u)$ is a convex function for $(t,x) \in (0,+\infty)\times\Omega$ for every choice of $u \in (a,b)$, then (A_1) is valid.*

If the level sets of the solution u in the above Theorem 1.1 are strictly convex with respect to normal Du, then it is proved in [16, 17] that the norm of gradient $|\nabla u|$ attains its maximum and minimum on the boundary. Combining this fact with Gabriel and Lewis theorem [7, 13], we have the following consequence.

Corollary 1.2. *Let $u \in C^\infty(\bar{\Omega})$ satisfy*

$$\begin{cases} \Delta u = 0 & in \quad \Omega = \Omega_0\backslash\bar{\Omega}_1, \\ u = 0 & on \quad \partial\Omega_0, \\ u = 1 & on \quad \partial\Omega_1, \end{cases} \tag{1.2}$$

where Ω_0 and Ω_1 are bounded convex smooth domains in $\mathbb{R}^n, n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$. Let k_1 be the least principal curvature of the level sets of u in Ω, then we have the following estimates

$$\min_{\Omega} k_1 \geq \min_{\partial\Omega} k_1 \frac{\min_{\partial\Omega_0}|Du|}{\max_{\partial\Omega_1}|Du|}.$$

We give an application to the following semilinear elliptic boundary value problem, its strict convexity of the level sets had been obtained in Caffarelli-Spruck [6] and Korevaar [12].

Corollary 1.3. *Let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ satisfy*

$$\begin{cases} \Delta u = f(u) & in \quad \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on \quad \partial \Omega_0, \\ u = 1 & on \quad \partial \Omega_1, \end{cases} \tag{1.3}$$

where Ω_0 and Ω_1 are bounded convex smooth domains in $\mathbb{R}^n, n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$. We assume f is C^2 increasing function and $f(0) = 0$, and let k_1 be the least principal curvature of the level sets of u in Ω, then we have the following estimates.

$$\min_{\Omega} k_1 \geq \min_{\partial \Omega} k_1 (\frac{\min_{\partial \Omega_0} |Du|}{\max_{\partial \Omega_1} |Du|})^2.$$

Assuming $|\nabla u| \neq 0$, Bianchini-Longinetti-Salani [3] proved the convexity of the level sets of solution u for some semilinear elliptic equation in convex ring with Dirichlet boundary conditions as in (1.3). It follows from the constant rank theorem of the second fundamental form of the convex level sets in [12], that the level sets are strictly convex. For the Poisson equation, our structure condition is the same as theirs.

Now we outline the proof of the Theorem 1.1. Let $\{a_{ij}\}$ be the symmetry curvature matrix on the strict convex level sets defined in (2.4), and let $\{a^{ij}\}$ be its inverse matrix. We consider the auxiliary function

$$\varphi(x, \xi) := |Du|^{\theta} a^{ij} \xi_i \xi_j, \quad where \quad \xi = (\xi_1, ...\xi_{n-1}) \in \mathbb{R}^{n-1}, \quad |\xi| = 1.$$

For suitable choice θ, we shall derive the following elliptic inequality

$$\Delta \varphi \geq 0 \qquad \text{mod} \qquad \nabla \varphi \quad \text{in} \quad \Omega, \tag{1.4}$$

here we have suppressed the terms containing the gradient of φ with locally bounded coefficients, then we apply the strong maximum principle to obtain the results.

In section 2, we first give brief definition on the convexity of the level sets, then obtain the curvature matrix a_{ij} of the level sets of a function, which appeared in [2, 5]. In section 3, we treat the semilinear elliptic partial differential equation and complete the proof of Theorem 1.1. The main technique in the proof of theorems consists in rearranging the third derivatives terms using the equation and the first derivatives condition for φ.

2. **The curvature matrix of level sets.** In this section, we shall give the brief definition on the convexity of the level sets, then introduce the curvature matrix (a_{ij}) of the level sets of a function, which appeared in [2]. Firstly, we recall some fundamental notations in classical surface theory. Assume a surface $\Sigma \subset \mathbb{R}^n$ is given by the graph of a function v in a domain in $\mathbb{R}^{n-1}$:

$$x_n = v(x'), x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

Definition 2.1. We define the graph of function $x_n = v(x')$ is convex with respect to the upward normal $\vec{\nu} = \frac{1}{W}(-v_1, -v_2, \cdots, -v_{n-1}, 1)$ if the second fundamental form $b_{ij} = \dfrac{v_{ij}}{W}$ of the graph $x_n = v(x')$ is nonnegative definite, where $W = \sqrt{1 + |\nabla v|^2}$.

The principal curvature $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ of the graph of v, being the eigenvalues of the second fundamental form relative to the first fundamental form. We have the following well-known formula.

Lemma 2.2. *([5]) The principal curvature of the graph $x_n = v(x')$ with respect to the upward normal $\vec{\nu}$ are the eigenvalues of the symmetric curvature matrix*

$$a_{il} = \frac{1}{W}\Big\{v_{il} - \frac{v_i v_j v_{jl}}{W(1+W)} - \frac{v_l v_k v_{ki}}{W(1+W)} + \frac{v_i v_l v_j v_k v_{jk}}{W^2(1+W)^2}\Big\}, \tag{2.1}$$

where the summation convention over repeated indices is employed.

Now we give the definition of the convex level sets of the function u. Let Ω be a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$, its level sets can be usually defined in the following sense.

Definition 2.3. Assume $|\nabla u| \neq 0$ in Ω, we define the level set of u passing through the point $x_o \in \Omega$ as $\Sigma^{u(x_o)} = \{x \in \Omega | u(x) = u(x_o)\}$.

Now we shall work near the point x_o where $|\nabla u(x_o)| \neq 0$. By the implicit function theorem, locally the level set $\Sigma^{u(x_o)}$ can be represented as a graph

$$x_n = v(x'), x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1},$$

and $v(x')$ satisfies the following equation

$$u(x_1, x_2, \cdots, x_{n-1}, v(x_1, x_2, \cdots, x_{n-1})) = u(x_o).$$

Then the first fundamental form of the level set is $g_{ij} = \delta_{ij} + \frac{u_i u_j}{u_n^2}$, and $W = (1 + |\nabla v|^2)^{\frac{1}{2}} = \frac{|\nabla u|}{|u_n|}$. The upward normal direction of the level set is

$$\vec{\nu} = \frac{|u_n|}{|\nabla u| u_n}(u_1, u_2, \cdots, u_{n-1}, u_n). \tag{2.2}$$

Let

$$h_{ij} = u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}, \tag{2.3}$$

then the second fundamental form of the level set of function u is $b_{ij} = \frac{v_{ij}}{W} = -\frac{|u_n| h_{ij}}{|\nabla u| u_n^3}$.

Definition 2.4. For the function $u \in C^2(\Omega)$ we assume $|\nabla u| \neq 0$ in Ω. Without loss of generality we can let $u_n(x_o) \neq 0$ for $x_o \in \Omega$. We define locally the level set $\Sigma^{u(x_o)} = \{x \in \Omega | u(x) = u(x_o)\}$ is convex with respect to the upward normal direction $\vec{\nu}$ if the second fundamental form b_{ij} is nonnegative definite.

Remark 2.5. If we let ∇u be the upward normal of the level set $\Sigma^{u(x_o)}$ at x_o, then $u_n(x_o) > 0$ by (2.2). Acccording to the definition 2.4, if the level set $\Sigma^{u(x_o)}$ is convex with respect to the normal direction ∇u, then the matrix $(h_{ij}(x_o))$ is nonpositive definite.

Now we obtain the representation of the curvature matrix (a_{ij}) of the level sets of the function u with the derivative of the function u,

$$a_{ij} = \frac{1}{|\nabla u| u_n^2}\Big\{ - h_{ij} + \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} + \frac{u_j u_l h_{il}}{W(1+W)u_n^2} - \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4}\Big\}. \tag{2.4}$$

From now on we denote

$$B_{ij} = \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} + \frac{u_j u_l h_{il}}{W(1+W)u_n^2}, \quad C_{ij} = \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4}, \tag{2.5}$$

and

$$A_{ij} = -h_{ij} + B_{ij} - C_{ij}, \tag{2.6}$$

then the symmetric curvature matrix of the level sets of u can be represented as

$$a_{ij} = \frac{1}{|\nabla u| u_n^2} \big[-h_{ij} + B_{ij} - C_{ij} \big] = \frac{1}{|\nabla u| u_n^2} A_{ij}. \tag{2.7}$$

We end this section with the following Codazzi condition which will be used in the next sections.

Proposition 2.6. *(see [2]) Denote $a_{ij,k} = \frac{\partial a_{ij}}{\partial x_k}$ for $1 \le i,j,k \le n-1$, then at the point where $u_n = |\nabla u| > 0$, $u_i = 0$, $a_{ij,k}$ is commutative in "i,j,k", i.e.*

$$a_{ij,k} = a_{ik,j}.$$

Proof. Direct calculation shows

$$a_{ij,k} = -u_n^{-1} u_{ijk} + u_n^{-2} (u_{ij} u_{kn} + u_{ik} u_{jn} + u_{jk} u_{in}). \tag{2.8}$$

The right hand side of (2.8) is obviously commutative in "i,j,k". □

3. **Principal curvature estimates of level set of Poisson equation.** In this section, we prove the Theorem 1.1. We study the following equation

$$\Delta u = f(x,u) \ge 0 \quad in \quad \Omega. \tag{3.1}$$

Proof of Theorem 1.1: Since the level sets of u are strictly convex with respect to normal Du, the curvature matrix a_{ij} of the level sets is positive definite in Ω. Let a^{ij} be the inverse matrix of a_{ij}.

We consider the auxiliary function

$$\varphi(x,\xi) := |Du|^{\theta} a^{ij}(x) \xi_i \xi_j, \quad where \quad \xi = (\xi_1, ... \xi_{n-1}) \in \mathbb{R}^{n-1}, \quad |\xi| = 1.$$

For suitable choice θ, we shall derive the following elliptic inequality

$$\Delta \varphi \ge 0 \qquad \text{mod} \qquad \nabla \varphi \quad \text{in} \quad \Omega, \tag{3.2}$$

where we modify the terms of the gradient of φ with locally bounded coefficients. Then by the standard strong maximum principle, we get the result immediately.

In order to prove (3.2) at an arbitrary point $x_o \in \Omega$, as in Caffarelli-Friedman [4], we choose the normal coordinate at x_o. We have mentioned in remark 2.5, since the level sets of u are strictly convex with respect to normal Du, by rotating the coordinate system suitably by T_{x_o}, we may assume that $u_i(x_o) = 0$, $1 \le i \le n-1$ and $u_n(x_o) = |\nabla u| > 0$. And we can further assume $\xi = e_1$, the matrix $\{u_{ij}\}(x_o)$ $(1 \le i < j \le n-1)$ is diagonal and $u_{ii}(x_o) < 0$. Consequently we can choose T_{x_o} to vary smoothly with x_o. If we can establish (3.2) at x_o under the above assumption, then go back to the original coordinates we find that (3.2) remain valid with new locally bounded coefficients on $\nabla\varphi$ in (3.2), depending smoothly on the independent variable. Thus it remains to establish (3.2) under the above assumption.

Now we write

$$\varphi(x) := |Du|^{\theta} a^{11}.$$

From now on, *all the calculations will be done at the fixed point x_0.*

Step1: we first compute the formula (3.19)

Taking first derivative of φ, we get

$$\varphi_\alpha = \frac{\theta}{2} |Du|^{\theta - 2} |Du|^2_\alpha a^{11} + |Du|^{\theta} a^{11}_\alpha, \tag{3.3}$$

since

$$a^{11}_\alpha = -\sum_{k,l=1}^{n-1} a^{1k}a^{1l}a_{kl,\alpha}, \tag{3.4}$$

it follows that

$$a_{11,\alpha} = \theta\frac{u_{n\alpha}}{u_n}a_{11} - u_n^{-\theta}a_{11}^2\varphi_\alpha. \tag{3.5}$$

From now on, we follow the convention: the Greek indices $1 \le \alpha, \beta, \gamma \le n$, the Latin indices $1 \le i, j, k, l \le n-1$.

Since

$$\begin{aligned} a^{11}_{\alpha\alpha} &= \sum_{k,l,r,s=1}^{n-1}[a^{1r}a^{ks}a^{1l}a_{kl,\alpha}a_{rs,\alpha} + a^{1k}a^{1r}a^{ls}a_{kl,\alpha}a_{rs,\alpha}] - \sum_{k,l=1}^{n-1}a^{1k}a^{1l}a_{kl,\alpha\alpha} \\ &= 2(a^{11})^2\sum_{k=1}^{n-1}a^{kk}a^2_{1k,\alpha} - (a^{11})^2a_{11,\alpha\alpha}, \end{aligned}$$

Taking derivative of equation (3.3) once more, and using (3.5), it follows that

$$\begin{aligned} a_{11}^2\varphi_{\alpha\alpha} &= -|Du|^\theta a_{11,\alpha\alpha} + 2|Du|^\theta\sum_{k=1}^{n-1}a^{kk}a^2_{1k,\alpha} + \theta|Du|^{\theta-1}a_{11}u_{\alpha\alpha n} \\ &\quad -\theta(\theta+2)|Du|^{\theta-2}u^2_{n\alpha}a_{11} + \theta|Du|^{\theta-2}a_{11}\sum_{\gamma=1}^n u^2_{\alpha\gamma} + 2\theta a^2_{11}u_n^{-1}u_{n\alpha}\varphi_\alpha. \end{aligned}$$

From the equation (3.1),

$$\begin{aligned} u_n^{2-\theta}a_{11}^2\Delta\varphi &= -u_n^2\sum_{\alpha=1}^n a_{11,\alpha\alpha} + 2u_n^2\sum_{k=1}^{n-1}\sum_{\alpha=1}^n a^{kk}a^2_{1k,\alpha} + 2\theta a^2_{11}u_n^{1-\theta}\sum_{\alpha=1}^n u_{n\alpha}\varphi_\alpha \\ &\quad +[\theta\sum_{\alpha,\gamma=1}^n u^2_{\alpha\gamma} + \theta u_nD_nf - \theta(\theta+2)\sum_{\alpha=1}^n u^2_{n\alpha}]a_{11}. \end{aligned} \tag{3.6}$$

Now we use (3.5) and the Codazzi identity (2.6) to treat the following term in (3.6).

$$\begin{aligned} \sum_{k=1}^{n-1}\sum_{\alpha=1}^n a^{kk}a^2_{1k,\alpha} &= a^{11}\sum_{\alpha=1}^n a^2_{11,\alpha} + \sum_{k=2}^{n-1}a^{kk}a^2_{kk,1} + \sum_{k=2}^{n-1}\sum_{\alpha=1,\alpha\ne k}^n a^{kk}a^2_{1k,\alpha} \\ &= \sum_{k=2}^{n-1}\sum_{\alpha=1,\alpha\ne k}^n a^{kk}a^2_{1k,\alpha} + \sum_{k=2}^{n-1}a^{kk}a^2_{kk,1} + \theta^2u_n^{-2}a_{11}\sum_{\alpha=1}^n u^2_{n\alpha} \\ &\quad +a^3_{11}u_n^{-2\theta}\sum_{\alpha=1}^n\varphi^2_\alpha - 2\theta a^2_{11}u_n^{-1-\theta}\sum_{\alpha=1}^n u_{n\alpha}\varphi_\alpha. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), it follows that

$$\begin{aligned} u_n^{2-\theta}a_{11}^2\Delta\varphi &= -u_n^2\sum_{\alpha=1}^{n}a_{11,\alpha\alpha}+2u_n^2\sum_{k=2}^{n-1}a^{kk}a_{kk,1}^2+2u_n^2\sum_{k=2,k\neq\alpha}^{n-1}\sum_{\alpha=1}^{n}a^{kk}a_{1k,\alpha}^2 \\ &\quad +[\theta\sum_{\alpha,\gamma=1}^{n}u_{\alpha\gamma}^2+\theta u_nD_nf+\theta(\theta-2)\sum_{\alpha=1}^{n}u_{n\alpha}^2]a_{11} \\ &\quad +2a_{11}^3u_n^{2-2\theta}\sum_{\alpha=1}^{n}\varphi_\alpha^2-2\theta a_{11}^2u_n^{1-\theta}\sum_{\alpha=1}^{n}u_{n\alpha}\varphi_\alpha. \end{aligned} \quad (3.8)$$

Now we calculate the term

$$-u_n^2\sum_{\alpha=1}^{n}a_{11,\alpha\alpha}.$$

Let $D=|\nabla u|u_n^2$, then at x_o, we get

$$\begin{aligned} D_\alpha &= 3u_n^2u_{n\alpha}, \\ D_{\alpha\alpha} &= 5u_nu_{n\alpha}^2+3u_n^2u_{\alpha\alpha n}+u_n\sum_{\gamma=1}^{n}u_{\alpha\gamma}^2. \end{aligned}$$

It follows that

$$\sum_{\alpha=1}^{n}D_{\alpha\alpha}=5u_n\sum_{\alpha=1}^{n}u_{n\alpha}^2+3u_n^2D_nf+u_n\sum_{\alpha,\gamma=1}^{n}u_{\alpha\gamma}^2. \quad (3.9)$$

By (2.4), we have

$$A_{11}=|\nabla u|u_n^2a_{11}. \quad (3.10)$$

Taking derivative of equation (3.10), it follows that

$$\begin{aligned} A_{11,\alpha} &= a_{11}D_\alpha+Da_{11,\alpha}, \\ A_{11,\alpha\alpha} &= Da_{11,\alpha\alpha}+2a_{11,\alpha}D_\alpha+a_{11}D_{\alpha\alpha}. \end{aligned} \quad (3.11)$$

By (2.6)

$$A_{11,\alpha\alpha}=-h_{11,\alpha\alpha}+B_{11,\alpha\alpha}-C_{11,\alpha\alpha}. \quad (3.12)$$

By (2.5), at x_o we have

$$\sum_{\alpha=1}^{n}C_{11,\alpha\alpha}=0. \quad (3.13)$$

We get

$$\begin{aligned} -u_n^2\sum_{\alpha=1}^{n}a_{11,\alpha\alpha} &= u_n^{-1}\sum_{\alpha=1}^{n}h_{11,\alpha\alpha}-u_n^{-1}\sum_{\alpha=1}^{n}B_{11,\alpha\alpha} \\ &\quad +u_n^{-1}\sum_{\alpha=1}^{n}[a_{11}D_{\alpha\alpha}+2D_\alpha a_{11,\alpha}]. \end{aligned} \quad (3.14)$$

Taking first and second derivatives of equation (2.5) on B_{ij}, we have

$$B_{11,\alpha\alpha}=4\sum_{l=1}^{n-1}\frac{u_{1\alpha}u_{l\alpha}h_{1l}}{W(1+W)u_n^2}. \quad (3.15)$$

Hence using $u_{jj} = -u_n a_{jj}$ and $W(x_o) = 1$, we get

$$\begin{aligned} -u_n^{-1}\sum_{\alpha=1}^{n} B_{11,\alpha\alpha} &= -4\sum_{l=1}^{n-1}\sum_{\alpha=1}^{n} \frac{u_{1\alpha}^2 h_{11}}{W(1+W)u_n^3} \\ &= -\frac{2u_{11}}{u_n}\sum_{\alpha=1}^{n} u_{1\alpha}^2 = 2a_{11}^3 u_n^2 + 2a_{11}u_{n1}^2. \end{aligned} \tag{3.16}$$

By (3.5), it follows that

$$2u_n^{-1}\sum_{\alpha=1}^{n} D_\alpha a_{11,\alpha} = 6\theta a_{11}\sum_{\alpha=1}^{n} u_{n\alpha}^2 - 6a_{11}^2 u_n^{1-\theta}\sum_{\alpha=1}^{n} u_{n\alpha}\varphi_\alpha. \tag{3.17}$$

Combining (3.14) with (3.9) and (3.16)-(3.17), we get

$$\begin{aligned} -u_n^2\sum_{\alpha=1}^{n} a_{11,\alpha\alpha} &= u_n^{-1}\sum_{\alpha=1}^{n} h_{11,\alpha\alpha} + [(5+6\theta)\sum_{\alpha=1}^{n} u_{n\alpha}^2 + 3u_n D_n f + \sum_{\alpha,\gamma=1}^{n} u_{\alpha\gamma}^2]a_{11} \\ &\quad + 2a_{11}^3 u_n^2 + 2a_{11}u_{n1}^2 - 6a_{11}^2 u_n^{1-\theta}\sum_{\alpha=1}^{n} u_{n\alpha}\varphi_\alpha. \end{aligned} \tag{3.18}$$

From (3.8) and (3.18), it follows that

$$\begin{aligned} u_n^{2-\theta}a_{11}^2\Delta\varphi &= u_n^{-1}\sum_{\alpha=1}^{n} h_{11,\alpha\alpha} + 2u_n^2\sum_{k=2,k\neq\alpha}^{n-1}\sum_{\alpha=1}^{n} a^{kk}a_{1k,\alpha}^2 + 2u_n^2\sum_{k=2}^{n-1} a^{kk}a_{kk,1}^2 \\ &\quad + [(\theta+1)\sum_{\alpha,\gamma=1}^{n} u_{\alpha\gamma}^2 + (3+\theta)u_n D_n f \\ &\quad + (\theta^2+4\theta+5)\sum_{\alpha=1}^{n} u_{n\alpha}^2 + 2u_{n1}^2]a_{11} \\ &\quad + 2a_{11}^3 u_n^2 + 2a_{11}^3 u_n^{2-2\theta}\sum_{\alpha=1}^{n}\varphi_\alpha^2 \\ &\quad - (6+2\theta)a_{11}^2 u_n^{1-\theta}\sum_{\alpha=1}^{n} u_{n\alpha}\varphi_\alpha. \end{aligned} \tag{3.19}$$

STEP 2: In this step we calculate the following term in (3.19)

$$u_n^{-1}\sum_{\alpha=1}^{n} h_{11,\alpha\alpha},$$

in order to derive the formula (3.32).

By (2.3), we have

$$\begin{aligned} h_{11,\alpha} &= 2u_n u_{n\alpha}u_{11} + u_n^2 u_{11\alpha} + u_{nn\alpha}u_1^2 + 2u_{nn}u_1 u_{1\alpha} \\ &\quad - 2u_{n\alpha}u_1 u_{1n} - 2u_n u_{1\alpha}u_{1n} - 2u_n u_1 u_{1n\alpha}, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} h_{11,\alpha\alpha} &= 2u_n u_{n\alpha\alpha}u_{11} + 4u_n u_{n\alpha}u_{11\alpha} - 2u_n u_{1\alpha\alpha}u_{1n} + u_n^2 u_{11\alpha\alpha} \\ &\quad + 2u_{11}u_{n\alpha}^2 + 2u_{nn}u_{1\alpha}^2 - 4u_{n\alpha}u_{1\alpha}u_{1n} - 4u_n u_{1\alpha}u_{1n\alpha}. \end{aligned} \tag{3.21}$$

From the equation (3.1), we find

$$\begin{aligned} u_n^{-1}\sum_{\alpha=1}^n h_{11,\alpha\alpha} &= u_n\sum_{\alpha=1}^n u_{11\alpha\alpha}+2u_{11}\sum_{\alpha=1}^n u_{\alpha\alpha n}-2u_{1n}\sum_{\alpha=1}^n u_{\alpha\alpha 1}\\ &\quad +4\sum_{\alpha=1}^n u_{n\alpha}u_{11\alpha}-4\sum_{\alpha=1}^n u_{1\alpha}u_{1n\alpha}-2a_{11}\sum_{\alpha=1}^n u_{n\alpha}^2\\ &\quad +2u_n^{-1}u_{nn}\sum_{\alpha=1}^n u_{1\alpha}^2-4u_n^{-1}u_{1n}\sum_{\alpha=1}^n u_{1\alpha}u_{n\alpha}\\ &= u_nD_{11}f+2u_{11}D_nf-2u_{1n}D_1f+4\sum_{\alpha=1}^n u_{n\alpha}u_{11\alpha}\\ &\quad -4\sum_{\alpha=1}^n u_{1\alpha}u_{1n\alpha}-2a_{11}\sum_{\alpha=1}^n u_{n\alpha}^2+2u_n^{-1}u_{nn}\sum_{\alpha=1}^n u_{1\alpha}^2\\ &\quad -4u_n^{-1}u_{1n}\sum_{\alpha=1}^n u_{1\alpha}u_{n\alpha}. \end{aligned} \tag{3.22}$$

Since

$$A_{jj}=Da_{jj}\quad\text{where}\quad D:=|\nabla u|u_n^2. \tag{3.23}$$

Taking derivative of equation (3.23), we find that

$$A_{jj,\alpha}=a_{jj}D_\alpha+Da_{jj,\alpha}, \tag{3.24}$$

Similar using (3.20), at x_o,

$$A_{jj,\alpha}=-h_{jj,\alpha}=2u_nu_{j\alpha}u_{nj}-2u_nu_{n\alpha}u_{jj}-u_n^2u_{jj\alpha}. \tag{3.25}$$

From (3.24)- (3.25) and (3.9), for $1\le j\le n-1$, we have

$$u_{jj\alpha}=-u_na_{jj,\alpha}+2u_n^{-1}u_{jn}u_{j\alpha}-a_{jj}u_{n\alpha}. \tag{3.26}$$

Then for $2\le j\le n-1$, we get

$$u_{jj1}=-u_na_{jj,1}-u_{1n}a_{jj}. \tag{3.27}$$

From (3.5) and (3.26), it follows that

$$\begin{aligned} u_{11\alpha} &= -u_na_{11,\alpha}+2u_n^{-1}u_{1n}u_{1\alpha}-u_{n\alpha}a_{11}\\ &= -(1+\theta)u_{n\alpha}a_{11}+2u_n^{-1}u_{1n}u_{1\alpha}+u_n^{1-\theta}a_{11}^2\varphi_\alpha, \end{aligned} \tag{3.28}$$

so

$$\begin{aligned} u_{111} &= -(3+\theta)u_{1n}a_{11}+u_n^{1-\theta}a_{11}^2\varphi_1,\\ u_{11n} &= -(1+\theta)u_{nn}a_{11}+2u_n^{-1}u_{1n}^2+u_n^{1-\theta}a_{11}^2\varphi_n. \end{aligned} \tag{3.29}$$

Using (3.28), we have

$$\begin{aligned} 4\sum_{\alpha=1}^n u_{n\alpha}u_{11\alpha} &= -4(1+\theta)a_{11}\sum_{\alpha=1}^n u_{n\alpha}^2\\ &\quad +8u_n^{-1}u_{1n}\sum_{\alpha=1}^n u_{1\alpha}u_{n\alpha}+4a_{11}^2u_n^{1-\theta}\sum_{\alpha=1}^n u_{n\alpha}\varphi_\alpha. \end{aligned} \tag{3.30}$$

By (3.27), (3.29) and the equation (3.1), we treat the other third derivative term,

$$\begin{aligned}
-4\sum_{\alpha=1}^{n}u_{1\alpha}u_{1n\alpha} &= -4u_{11}u_{11n}-4u_{1n}u_{nn1}\\
&= -4u_{11}u_{11n}-4u_{1n}D_1f+4u_{1n}\sum_{j=1}^{n-1}u_{jj1}\\
&= 4u_{1n}u_{111}-4u_{11}u_{11n}-4u_{1n}D_1f+4u_{1n}\sum_{j=2}^{n-1}u_{jj1}\\
&= -4u_nu_{1n}\sum_{k=2}^{n-1}a_{kk,1}-4u_{1n}D_1f-4(1+\theta)a_{11}u_{1n}^2\\
&\quad -4(1+\theta)u_na_{11}^2u_{nn}-4u_{1n}^2\sum_{i=2}^{n-1}a_{ii}\\
&\quad +4a_{11}^2u_n^{1-\theta}u_{n1}\varphi_1+4a_{11}^3u_n^{2-\theta}\varphi_n. \qquad (3.31)
\end{aligned}$$

Combining (3.22), (3.30)-(3.31), we have

$$\begin{aligned}
u_n^{-1}\sum_{\alpha=1}^{n}h_{11,\alpha\alpha} &= -4u_nu_{1n}\sum_{k=2}^{n-1}a_{kk,1}-6u_{1n}D_1f+u_nD_{11}f-2u_na_{11}D_nf\\
&\quad -(6+4\theta)a_{11}\sum_{\alpha=1}^{n}u_{n\alpha}^2-(4+4\theta)a_{11}u_{1n}^2+6u_n^{-1}u_{nn}u_{n1}^2\\
&\quad -(2+4\theta)u_nu_{nn}a_{11}^2-4u_{n1}^2\sum_{i=1}^{n-1}a_{ii}+4a_{11}^3u_n^{2-\theta}\varphi_n\\
&\quad +4a_{11}^2u_n^{1-\theta}u_{n1}\varphi_1+4a_{11}^2u_n^{1-\theta}\sum_{\alpha=1}^{n}u_{n\alpha}\varphi_\alpha. \qquad (3.32)
\end{aligned}$$

STEP 3: The conclusion of the proof.
Now we combine the (3.19) and (3.32), it follows that

$$\begin{aligned}
u_n^{2-\theta}a_{11}^2\Delta\varphi &= 2u_n^2\sum_{k=2,k\neq\alpha}^{n-1}\sum_{\alpha=1}^{n}a^{kk}a_{1k,\alpha}^2+2u_n^2\sum_{k=2}^{n-1}a^{kk}a_{kk,1}^2-4u_nu_{1n}\sum_{k=2}^{n-1}a_{kk,1}\\
&\quad -6u_{1n}D_1f+u_nD_{11}f+(1+\theta)u_na_{11}D_nf\\
&\quad +(\theta^2-1)a_{11}\sum_{\alpha=1}^{n}u_{n\alpha}^2-(2+4\theta)a_{11}u_{n1}^2-4u_{n1}^2\sum_{i=1}^{n-1}a_{ii}+2a_{11}^3u_n^2\\
&\quad +6u_n^{-1}u_{nn}u_{n1}^2-(2+4\theta)u_nu_{nn}a_{11}^2+(\theta+1)a_{11}\sum_{\alpha,\gamma=1}^{n}u_{\alpha\gamma}^2\\
&\quad +4a_{11}^2u_n^{1-\theta}u_{n1}\varphi_1+4a_{11}^3u_n^{2-\theta}\varphi_n\\
&\quad +2a_{11}^3u_n^{2-2\theta}\sum_{\alpha=1}^{n}\varphi_\alpha^2-(2+2\theta)a_{11}^2u_n^{1-\theta}\sum_{\alpha=1}^{n}u_{n\alpha}\varphi_\alpha. \qquad (3.33)
\end{aligned}$$

Since

$$2u_n^2\sum_{k=2}^{n-1}a^{kk}a_{kk,1}^2-4u_nu_{1n}\sum_{k=2}^{n-1}a_{kk,1}\geq 2u_{1n}^2(a_{11}-\sum_{i=1}^{n-1}a_{ii}). \tag{3.34}$$

It follows that

$$\begin{aligned} u_n^{2-\theta}a_{11}^2\Delta\varphi \quad \geq \quad & u_nD_{11}f-6u_{1n}D_1f+(1+\theta)u_na_{11}D_nf+6u_n^{-1}u_{nn}u_{n1}^2 \\ & +(\theta^2-1)a_{11}\sum_{\alpha=1}^{n}u_{n\alpha}^2-4\theta a_{11}u_{n1}^2-6u_{n1}^2\sum_{i=1}^{n-1}a_{ii}+2a_{11}^3u_n^2 \\ & -(2+4\theta)u_nu_{nn}a_{11}^2+(\theta+1)a_{11}\sum_{\alpha,\gamma=1}^{n}u_{\alpha\gamma}^2 \quad \mathrm{mod}\nabla\varphi, \end{aligned} \tag{3.35}$$

where we modify the terms of the gradient of φ with locally bounded coefficients, from now on we omit $\nabla\varphi$.

From the equation, at x_o, we have

$$u_{nn}=f-\sum_{i=1}^{n-1}u_{ii}=f+u_n\sum_{i=1}^{n-1}a_{ii}. \tag{3.36}$$

Now we let $\lambda_i=a_{ii}>0,\quad 1\leq i\leq n-1$, use the following abbreviation

$$\sigma_k(\lambda)=\sum_{1\leq i_1<\cdots<i_k\leq n-1}\lambda_{i_1}\cdots\lambda_{i_k},$$

and let $\sigma_k(\lambda|i)$ denote the summation in which the terms involving λ_i are deleted.

Combining (3.35)- (3.36), it follows that

$$\begin{aligned} u_n^{2-\theta}a_{11}^2\Delta\varphi \quad \geq \quad & u_nD_{11}f+(1+\theta)u_n\lambda_1D_nf-6u_{1n}D_1f+6u_n^{-1}u_{n1}^2f \\ & +(\theta^2+\theta)\lambda_1f^2+2(\theta^2+\theta)u_n\lambda_1\sigma_1(\lambda|1)f+2(\theta^2-\theta-1)u_n{\lambda_1}^2f \\ & +\lambda_1u_n^2[(\theta-1)^2\lambda_1^2+(\theta+1)^2{\sigma_1}^2(\lambda|1) \\ & \qquad +2(\theta^2-\theta-1)\lambda_1\sigma_1(\lambda|1)-2(1+\theta)\sigma_2(\lambda|1)] \\ & +(\theta-1)^2\lambda_1u_{n1}^2+(\theta+1)^2\lambda_1\sum_{j=2}^{n-1}u_{nj}^2. \end{aligned} \tag{3.37}$$

Now we make the following choice of θ to complete the proof.

Case 1: if $f=0$, for $n=3$, we let $\theta=0$ then for $\varphi:=a^{11}$, we get

$$u_n^2a_{11}^2\Delta\varphi\geq\lambda_1(u_{13}^2+u_{23}^2)+\lambda_1u_3^2(\lambda_1-\lambda_2)^2\geq 0. \tag{3.38}$$

If $n\geq 3$, we can choose $\theta=-1$. Then from (3.37), for $\varphi:=|Du|^{-1}a^{11}$ satisfies

$$u_n^3a_{11}^2\Delta\varphi\geq 4\lambda_1u_{n1}^2+2\lambda_1^2u_n^2[\lambda_1+\sigma_1(\lambda)]\geq 0. \tag{3.39}$$

Case 2: if $f=f(x)\geq 0$, we can choose $\theta=-1$. Then from (3.37), for $\varphi:=|Du|^{-1}a^{11}$ satisfies

$$\begin{aligned} u_n^3a_{11}^2\Delta\varphi \quad \geq \quad & u_nf_{11}-6u_{1n}f_1+6u_n^{-1}u_{n1}^2f \\ & +2u_n{\lambda_1}^2f+4\lambda_1u_{n1}^2+2\lambda_1^2u_n^2[\lambda_1+\sigma_1(\lambda)]. \end{aligned} \tag{3.40}$$

If the function $(t,x)\longrightarrow t^3f(x)$ is a convex function for $x\in\Omega$ and $t\in(0,+\infty)$ (or for $f>0$ and $f^{-\frac{1}{2}}$ is a concave function). So the matrix $\{2ff_{ij}-3f_if_j\}$ is nonnegative definite. Then

$$u_nf_{11}-6u_{1n}f_1+6u_n^{-1}u_{n1}^2f\geq 0. \tag{3.41}$$

It follows that

$$u_n^3 a_{11}^2 \Delta\varphi \ge 2u_n {\lambda_1}^2 f + 4\lambda_1 u_{n1}^2 + 2\lambda_1^2 u_n^2[\lambda_1 + \sigma_1(\lambda)] \ge 0. \tag{3.42}$$

Case 3: for $f = f(u) \ge 0$, so

$$D_{11} f = f_u u_{11} = -u_n \lambda_1 f_u,$$

in this case we have

$$u_n D_{11} f + (1+\theta) u_n \lambda_1 D_n f - 6u_{1n} D_1 f = \theta u_n^2 \lambda_1 f_u. \tag{3.43}$$

Using (3.37) and (3.43), we can make the following choice.

When $f_u \le 0$, we let $\theta = -1$. Then from (3.37), for $\varphi := |Du|^{-1} a^{11}$ satisfies

$$\begin{aligned} u_n^3 a_{11}^2 \Delta\varphi &\ge -u_n^2 \lambda_1 f_u + 6u_n^{-1} u_{n1}^2 f \\ &+ 2u_n {\lambda_1}^2 f + 4\lambda_1 u_{n1}^2 + 2\lambda_1^2 u_n^2 [\lambda_1 + \sigma_1(\lambda)] \\ &\ge 0. \end{aligned} \tag{3.44}$$

When $f_u \ge 0$, we let $\theta = 2$. From (3.37), for $\varphi := |Du|^2 a^{11}$ satisfies

$$\begin{aligned} a_{11}^2 \Delta\varphi &\ge 2u_n^2 \lambda_1 f_u + 6u_n^{-1} u_{n1}^2 f + 6\lambda_1 f^2 \\ &\quad +12 u_n \lambda_1 \sigma_1(\lambda|1) f + 2u_n {\lambda_1}^2 f + \lambda_1 u_{n1}^2 + 9\lambda_1 \sum_{j=2}^{n-1} u_{nj}^2 \\ &\quad +\lambda_1 u_n^2 [\lambda_1^2 + 9{\sigma_1}^2(\lambda|1) + 2\lambda_1 \sigma_1(\lambda|1) - 6\sigma_2(\lambda|1)] \\ &\ge \lambda_1 u_n^2 [\lambda_1^2 + 9{\sigma_1}^2(\lambda|1) + 2\lambda_1\sigma_1(\lambda|1) - 6\sigma_2(\lambda|1)]. \end{aligned} \tag{3.45}$$

For $n = 2$ or $n = 3$, $\sigma_2(\lambda|1) = 0$. For $n \ge 4$ we use the following Maclaurin inequalities,

$$[\frac{\sigma_2(\lambda|1)}{C_{n-2}^2}]^{\frac{1}{2}} \le \frac{\sigma_1(\lambda|1)}{n-2},$$

i.e. for $n \ge 4$, we have

$$\sigma_2(\lambda|1) \le \frac{n-3}{2(n-2)} \sigma_1^2(\lambda|1)$$

Then we have

$$9{\sigma_1}^2(\lambda|1) - 6\sigma_2(\lambda|1) \ge 0. \tag{3.46}$$

By (3.45)-(3.46), it follows that

$$a_{11}^2 \Delta\varphi \ge 0.$$

Case 4: for $f = f(x,u) \ge 0$. Since

$$D_1 f = f_1, \quad D_{11} f = f_{11} - u_n \lambda_1 f_u, \tag{3.47}$$

so for $\theta = -1$, we get

$$u_n D_{11} f + (1+\theta) u_n \lambda_1 D_n f - 6u_{1n} D_1 f = u_n f_{11} - 6u_{1n} f_1 - u_n^2 \lambda_1 f_u.$$

If $f_u \le 0$ and for any $u \in (a,b)$ the function $(t,x) \longrightarrow t^3 f(x,u)$ is a convex function for $x \in \Omega$ and $t \in (0,+\infty)$ (or for $f > 0$ and $f^{-\frac{1}{2}}$ is a concave function for

x), then we also have (3.41). Now let $\theta = -1$. From (3.37), for $\varphi := |Du|^{-1}a^{11}$, it follows that

$$\begin{aligned} u_n^3 a_{11}^2 \Delta\varphi &\geq u_n f_{11} - 6u_{1n} f_1 + 6u_n^{-1} u_{1n}^2 f - u_n^2 \lambda_1 f_u \\ &+ 2u_n {\lambda_1}^2 f + 4\lambda_1 u_{n1}^2 + 2\lambda_1^2 u_n^2 [\lambda_1 + \sigma_1(\lambda)] \\ &\geq 2u_n {\lambda_1}^2 f + 4\lambda_1 u_{n1}^2 + 2\lambda_1^2 u_n^2 [\lambda_1 + \sigma_1(\lambda)] \\ &\geq 0. \end{aligned} \tag{3.48}$$

Then we complete the proof of the Theorem 1.1. □

Remark 3.7. Recently Ma-Ou-Zhang [17] obtained the Gaussian curvature lower bound estimate for the convex level sets of harmonic function on convex ring.

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