GLOBAL SOLUTIONS OF CERTAIN PLASMA FLUID MODELS IN 3D

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Abstract. We consider several dispersive time-reversible plasma fluid models in 3 dimensions: the Euler-Poisson 2-fluid model, the relativistic Euler–Maxwell 1-fluid model, and the relativistic Euler–Maxwell 2-fluid model. In all of these models, we prove global stability of the constant background solutions, in the sense that small, smooth, and irrotational perturbations lead to smooth global solutions that decay as $t \to \infty$.

Contents

1. Introduction

A plasma is a collection of fast-moving charged particles. It is believed that more than 90% of the matter in the universe is in the form of plasma, from sparse intergalactic plasma, to the interior of stars to neon signs. In addition, understanding of the instability formation in plasma is one of the main challenges for nuclear fusion, in which charged particles are accelerated at high speed to create energy. We refer to [2, 9] for physics references in book form.

At high temperature and velocity, ions and electrons in a plasma tend to become two separate fluids due to their different physical properties (inertia, charge). The dynamics of these charged particles can be described by so-called "two-fluid models" in plasma physics, in which ions and electrons are governed by two compressible Euler equations separately, while the electromagnetic field is governed by the Maxwell system. Since the relaxation time is extremely long in the context of a plasma, the momentum relaxation is usually ignored. These two-fluid models captures complex dynamics of a plasma due to electromagnetic interactions. Even at the linear level, there are new ion-acoustic waves, Langmuir waves, as well as light waves etc. At the nonlinear level, these two-fluid models form the origin of many well-known dispersive PDE, such as KdV [20], KP [34, 38], Zakharov [41], Zakharov-Kuznetsov [34, 38] and NLS, which can be derived from (1.2) and (1.3) via different scaling and asymptotic expansions. We also refer to [3, 10, 11]

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for a derivation of the cold-ion and quasi-neutral equations and to [5] for a study of a similar model for semiconductors.

In this paper we consider the question of the dynamic stability of the flat neutral equilibrium. From a PDE viewpoint, these two-fluid models can be classified as systems of nonlinear hyperbolic conservation laws with *no dissipation and no relaxation effects*[1]. It is well-known that shock waves (i.e., discontinuities) will generically develop even from small smooth initial data (see e.g. John [30]). Even worse, a classical result of Sideris [40] demonstrates that, for the compressible Euler equation for a neutral gas, shock waves will develop even for smooth irrotational initial data with small amplitude. This shock formation was recently further described in [8] (see also [1]). The result of blow-up of Sideris for the pure compressible Euler equations [40] can be understood from the fact that small and irrotational perturbations of a constant background for the pure compressible Euler equations obey a *quasilinear wave equation without null-structure* of the form

$$(\partial_{tt} - \Delta)\,\alpha = \mathcal{Q}(\alpha, \nabla\alpha, \nabla^2\alpha) \tag{1.1}$$

where α is related to the unknown and the right-hand side denotes a quadratic nonlinearity in up to two derivatives of α. This type of equation has slow decay of linear waves and strong resonances and therefore blow-up or formation of shocks is expected.

On the other hand, it was observed [17] that the electromagnetic interaction in these two-fluid models could create stronger dispersive effects, enhance linear decay rates and prevent formation of shock waves with small amplitude[2]. Several positive results have been established in recent years along this direction, for various nonlinear wave equations with null structure [7, 32, 33], dispersive scalar equations or systems [14, 22, 23, **?**], water waves [15, 16, 27] and one and two-fluid models, see [12, 13, 18, 19, 24, 25, 28, 29, 35]. In [18], global smooth solutions with small amplitude have been constructed for the "full[3]" Euler-Maxwell system, which describes the dynamical evolution of electron (resp. ion) densities $n_e, n_i : \mathbb{R}^3 \to \mathbb{R}$, electron (resp. ion) velocities v_e, v_i and electromagnetic field $E, B : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\begin{aligned}
&\partial_t n_e + \operatorname{div}(n_e v_e) = 0,\\
&n_e m_e\left[\partial_t v_e + v_e\cdot\nabla v_e\right] + \nabla p_e = -n_e e\left[E + \frac{v_e}{c}\times B\right],\\
&\partial_t n_i + \operatorname{div}(n_i v_i) = 0,\\
&n_i M_i\left[\partial_t v_i + v_i\cdot\nabla v_i\right] + \nabla p_i = Z n_i e\left[E + \frac{v_i}{c}\times B\right],\\
&\partial_t B + c\nabla\times E = 0,\\
&\partial_t E - c\nabla\times B = 4\pi e\left[n_e v_e - Z n_i v_i\right],
\end{aligned} \tag{1.2}$$

together with the elliptic equations

$$\operatorname{div}(B) = 0, \quad \operatorname{div}(E) = 4\pi e(Zn_i - n_e). \tag{1.3}$$

These equations describe a plasma composed of electrons and one species of ions. The speed of light is denoted by c, the electrons have charge $-e$, density n_e, mass m_e, velocity v_e, and pressure p_e, and the ions have charge Ze, density n_i, mass M_i, velocity v_i, and pressure p_i. In addition, in [18], the pressure laws were assumed quadratic. The two equations (1.3) are propagated by the dynamic flow, provided that we assume that they are satisfied at the initial time. The absence of shock waves for such a "master system" in the two-fluid models reveals an exciting and deep contrast between a neutral gas and a charged plasma due to subtle dispersive effects. More importantly, a general and robust mathematical

[1]When dissipation or relaxation is present, one expects stronger decay, even at the level of the L^2-norm, see e.g. [5, 37] and the references therein. In our case however, the evolution is time-reversible and we need a different mechanism of decay based on dispersion.

[2]For large perturbations, formation of shock in expected, even with an electromagnetic field [21].

[3]In the sense that all the previous works addressed simplifications of (1.2)-(1.3).

approach has been developed in [18] (see also the earlier work [25]) to construct global smooth solutions for hyperbolic systems in the presence of different characteristic speeds.

The model (1.2)-(1.3) in the case of quadratic pressure laws studied in [18] is the simplest model which contains all the key aspects in the theory of two-fluid plasma physics, at least for small perturbations. The purpose of the present paper is to justify this assertion and complement the results in [18] by showing *i*) how to consider more general pressure laws in (1.2)-(1.3) and more importantly *ii*) how the analysis developed can be adapted to apply to three different natural plasma fluid models.

One of the main motivation for the models we consider comes from the invariance properties of the equation. The Euler-Maxwell two-fluid model couples classical fluid models which are Galilean-invariant to the Maxwell equations which are Lorentz-invariant. The full system then loses both kind of invariances. A way to restore Galilean invariance is to replace Maxwell theory by the electrostatic theory. This gives the classical Euler-Poisson equation for 2 fluids, which is the first model we consider. If one instead wants to keep the Lorentz invariance, one may replace the model of classical fluids by models of relativistic fluids. This gives the (special) relativistic Euler-Maxwell model, which corresponds to the two other models we study. More precisely, we start with the study of the one-fluid relativistic Euler-Maxwell system for electrons and generalize this to the case of the two-fluid relativistic Euler-Maxwell system.

Let us however mention that the models we consider in this paper are also important in applications: the Euler-Poisson model is used e.g. to study plasma extension between electrodes [6], while relativistic plasmas are relevant in the study of strong laser-plasma interactions or in astrophysics e.g. in pulsars or in the solar atmosphere [36, 42].

The analysis presented here extends the work [18] and our proof relies on some key estimates from this paper. We also refer to the introduction of [18] for a more extensive presentation of the strategy and more references about previous results on quasilinear dispersive equations and conservation laws.

In Section 2 we introduce our main models and state our global existence results for each of them. In Section 3 we provide the proof of Theorem 2.3 which is the most difficult theorem.

2. The main models

In this section we derive our three main models and state the main theorems.

2.1. **The Euler-Poisson model.**

One of the basic fluid models for describing plasma dynamics is the Euler–Poisson model, in which two compressible ion and electron fluids interact with their own self-consistent electrostatic field. The system describes the dynamical evolution of the functions $n_e, n_i : \mathbb{R}^3 \to \mathbb{R}$ and $v_e, v_i : \mathbb{R}^3 \to \mathbb{R}^3$, which evolve according to the quasi-linear coupled system,

$$\begin{aligned} \partial_t n_e + \operatorname{div}(n_e v_e) &= 0, \\ n_e m_e \left[\partial_t v_e + v_e \cdot \nabla v_e\right] + \nabla p_e &= n_e e \nabla\phi, \\ \partial_t n_i + \operatorname{div}(n_i v_i) &= 0, \\ n_i M_i \left[\partial_t v_i + v_i \cdot \nabla v_i\right] + \nabla p_i &= -Z n_i e \nabla\phi, \\ -\Delta\phi &= 4\pi e(Z n_i - n_e). \end{aligned} \tag{2.1}$$

These equations describe a plasma composed of electrons and ions. The electrons have charge $-e$, density n_e, mass m_e, velocity v_e, and pressure p_e, and the ions have charge Ze, density n_i, mass M_i, velocity v_i, and pressure p_i. The two fluids interact through the self-consistent electric field $E = -\nabla\phi$.

We will assume that the pressures p_e and p_i depend only on the densities n_e and n_i respectively, i.e. $p_e = p_e(n_e)$, $p_i = p_i(n_i)$.

The system corresponds to formally taking $c \to \infty$, $B \equiv 0$, $E = -\nabla\phi$ in the more general Euler–Maxwell 2-fluid system, see for example [18, Section 1].

2.1.1. *Galilean invariance.* The Euler–Poisson system is invariant under a Galilean change of unknowns. More precisely, let $V \in \mathbb{R}^3$ be a fixed vector and let

$$\begin{aligned} & x \to x' := x + Vt, \quad t \to t' := t \\ & n_e \to n_e' := n_e, \quad v_e \to v_e' := v_e - V, \quad n_i \to n_i' := n_i, \quad v_i \to v_i' := v_i - V, \quad \phi \to \phi' := \phi, \end{aligned}$$

then, we observe that $(n_e, v_e, n_i, v_i, \phi)(x,t)$ solves (2.1) if and only if $(n_e', v_e', n_i', v_i', \phi')(x', t')$ does.

2.1.2. *Normalizations.* We will study our system in a neighborhood of the constant solution

$$(n_e, v_e, n_i, v_i) = (n_0, 0, n_0/Z, 0),$$

where $n_0 \in (0, \infty)$ is fixed. In order to state our main result, we normalize the Euler–Poisson system. Assume that the pressure fields are given by the barotropic pressure laws

$$\begin{aligned} \nabla p_e(x,t) &= p_e'(n_e(x,t)) \nabla n_e(x,t), \\ \nabla p_i(x,t) &= p_i'(n_i(x,t)) \nabla n_i(x,t), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \frac{p_e'(y)}{y} &= P_e + P_e^1 \cdot (y - n_0) + (y - n_0)^2 \cdot c_e(y - n_0), \\ \frac{p_i'(y)}{Z^2 y} &= P_i + P_i^1 \cdot (y - n_0/Z) + (y - n_0/Z)^2 \cdot c_i(y - n_0/Z), \end{aligned} \tag{2.3}$$

for some constants $P_e, P_i \in (0, \infty)$, $P_e^1, P_i^1 \in \mathbb{R}$, and some sufficiently smooth functions c_e, c_i defined in a small neighborhood of the origin.

The Euler–Poisson system can be adimensionalized to depend only on two parameters at the linear level: the ratio of the electron to ion masses (per charge)

$$\varepsilon := Z m_e / M_i, , \tag{2.4}$$

and the ratio of the temperatures

$$T := P_e / P_i. \tag{2.5}$$

More precisely, let

$$\lambda := \sqrt{\frac{4\pi e^2}{P_i}}, \qquad \beta := \sqrt{\frac{4\pi n_0 Z e^2}{M_i}},$$

and

$$\begin{aligned} & n_e(x,t) = n_0 \big[n(\lambda x, \beta t) + 1 \big], && n_i(x,t) = (n_0/Z) \big[\rho(\lambda x, \beta t) + 1 \big], \\ & v_e(x,t) = (\beta/\lambda) v(\lambda x, \beta t), && v_i(x,t) = (\beta/\lambda) u(\lambda x, \beta t), \\ & \phi(x,t) = (n_0 P_i / e) \widetilde{\phi}(\lambda x, \beta t). \end{aligned} \tag{2.6}$$

The parameter β is the ion plasma frequency and β/λ is the ion thermal velocity. In terms of $n, v, \rho, u, \widetilde{\phi}$ the system (2.1) becomes

$$\begin{aligned} \partial_t n + \operatorname{div}((n+1)v) &= 0, \\ \varepsilon \left(\partial_t v + v \cdot \nabla v \right) + T \nabla n - \nabla \widetilde{\phi} + \widetilde{P_e^1} n \nabla n + n^2 \widetilde{c}_e(n) \nabla n &= 0, \\ \partial_t \rho + \operatorname{div}((\rho+1)u) &= 0, \\ (\partial_t u + u \cdot \nabla u) + \nabla \rho + \nabla \widetilde{\phi} + \widetilde{P_i^1} \rho \nabla \rho + \rho^2 \widetilde{c}_i(\rho) \nabla \rho &= 0, \\ -\Delta \widetilde{\phi} - \rho + n &= 0, \end{aligned} \tag{2.7}$$

where

$$\widetilde{P_e^1} := \frac{n_0 T P_e^1}{P_e}, \quad \widetilde{c_e}(y) := \frac{n_0^2 T}{P_e} c_e(n_0 y), \quad \widetilde{P_i^1} := \frac{n_0 P_i^1}{Z P_i}, \quad \widetilde{c_i}(y) := \frac{n_0^2}{Z^2 P_e} c_i(n_0 y/Z).$$

2.1.3. *The main theorem.* We can now state the main result we prove concerning global stability of the Euler–Poisson system in the case of irrotational perturbations.

Theorem 2.1. *Assume $\varepsilon \leq 10^{-3}$ and $T \in [1, 100]$. Let $N_0 = 10^4$ and assume that*

$$\begin{aligned} &\|(n^0, v^0, \rho^0, u^0, |\nabla|^{-1}(n^0 - \rho^0))\|_{H^{N_0}} + \|(n^0, v^0, \rho^0, u^0, |\nabla|^{-1}(n^0 - \rho^0))\|_Z = \delta_0 \leq \overline{\delta}, \\ &\nabla \times v^0 = \nabla \times u^0 = 0, \end{aligned} \tag{2.8}$$

where $\overline{\delta} > 0$ is sufficiently small, and the Z norm is defined in Definition 3.2. Then there exists a unique global solution $(n, v, \rho, u) \in C([0, \infty) : H^{N_0})$ of the system (2.7) with initial data $(n(0), v(0), \rho(0), u(0)) = (n^0, v^0, \rho^0, u^0)$. Moreover, for any $t \in [0, \infty)$,

$$\nabla \times v(t) = \nabla \times u(t) \equiv 0, \ \textit{(irrotationality)} \tag{2.9}$$

and, for any $t \in [0, \infty)$ and any $h(t) \in \{n(t), v(t), \rho(t), u(t), |\nabla|^{-1}(n(t) - \rho(t))\}$

$$\|h(t)\|_{H^{N_0}} + \sup_{|\alpha| \leq 4} (1 + t)^{1 + \beta/2} \|D_x^\alpha h(t)\|_{L^\infty} \lesssim \delta_0. \tag{2.10}$$

where $\beta := 1/100$.

We note that, for initial data with $n^0 - \rho^0 \in L^1(\mathbb{R}^3)$, assumption (2.8) implies global neutrality

$$\int_{\mathbb{R}^3} (n^0(x) - \rho^0(x)) dx = 0.$$

This is consistent with usual properties of plasmas.

2.1.4. *Derivation of the main dispersive system.* The proof of this theorem is similar to the proof of the main theorem in [18]. One starts by constructing local solutions, using the energy method and the higher order energies

$$\mathcal{E}_N := \sum_{|\gamma| \leq N} \int_{\mathbb{R}^3} \left[q_e(n) |D_x^\gamma n|^2 + \varepsilon(1 + n)|D_x^\gamma v|^2 + q_i(\rho)|D_x^\gamma \rho|^2 + (1 + \rho)|D_x^\gamma u|^2 + |D_x^\gamma \nabla \phi|^2 \right] dx,$$

where q_e and q_i are such that

$$\begin{aligned} &[x^2 q_e(x)]'' = 2T + 2\widetilde{P_e^1} x + 2x^2 \widetilde{c_e}(x), \qquad q_e(0) = T, \, q_e'(0) = \widetilde{P_e^1}/3, \\ &[x^2 q_i(x)]'' = 2 + 2\widetilde{P_i^1} x + 2x^2 \widetilde{c_i}(x), \qquad q_i(0) = 1, \, q_i'(0) = \widetilde{P_i^1}/3. \end{aligned}$$

Note that $q_e(x) > 0$ and $q_i(x) > 0$ for $|x| \ll 1$. A standard argument using these higher order energies and integration by parts gives local existence of smooth solutions.

We also notice that

$$\partial_t \left[\nabla \times v \right] = \nabla \times \left[v \times \left[\nabla \times v \right] \right], \qquad \partial_t \left[\nabla \times u \right] = \nabla \times \left[u \times \left[\nabla \times u \right] \right].$$

This shows the consistency of the irrotationality assumption in Theorem 2.1, i.e. smooth solutions with irrotational initial data remain irrotational for all time.

We can therefore write

$$v_\alpha = R_\alpha h, \ u_\alpha = R_\alpha g, \quad \text{or} \quad h := -|\nabla|^{-1} \mathrm{div}(v), \ g := -|\nabla|^{-1} \mathrm{div}(u),$$

where R_α and $|\nabla|$ are defined by the Fourier multipliers $R_\alpha(\xi) := i\xi_\alpha/|\xi|$ and $|\nabla|(\xi) := |\xi|$. The system (2.7) becomes

$$\begin{aligned} \partial_t n - |\nabla| h &= -\partial_\alpha \left[n R_\alpha h \right], \\ \partial_t \rho - |\nabla| g &= -\partial_\alpha \left[\rho R_\alpha g \right], \\ \partial_t h + |\nabla|^{-1} H_\varepsilon^2 n - \varepsilon^{-1} |\nabla|^{-1} \rho &= -(1/2)|\nabla| \left[R_\alpha h R_\alpha h \right] - \varepsilon^{-1} |\nabla| \left[(\widetilde{P_e^1}/2) n^2 + n^2 C_e(n) \right], \\ \partial_t g - |\nabla|^{-1} n + |\nabla|^{-1} H_1^2 \rho &= -(1/2)|\nabla| \left[R_\alpha g R_\alpha g \right] - |\nabla| \left[(\widetilde{P_i^1}/2) \rho^2 + \rho^2 C_i(\rho) \right], \end{aligned} \tag{2.11}$$

where H_1 and H_ε are given by the Fourier multipliers

$$H_1(\xi) := \sqrt{1 + |\xi|^2}, \qquad H_\varepsilon(\xi) := \varepsilon^{-1/2} \sqrt{1 + T|\xi|^2},$$

and C_i and C_e are smooth functions related to $\widetilde{c_i}$ and $\widetilde{c_e}$ by

$$C_e(y) = \frac{1}{y^2} \int_0^y s^2 \widetilde{c_e}(s) ds, \qquad C_i(y) = \frac{1}{y^2} \int_0^y s^2 \widetilde{c_i}(s) ds.$$

In particular, they satisfy that $C_e(0) = C_i(0) = 0$.

As in [18, Section 3], we make linear changes of variables to diagonalize the system (2.11). Let

$$\begin{aligned} \Lambda_e &:= \varepsilon^{-1/2} \sqrt{\frac{(1+\varepsilon) - (T+\varepsilon)\Delta + \sqrt{((1-\varepsilon) - (T-\varepsilon)\Delta)^2 + 4\varepsilon}}{2}}, \\ \Lambda_i &:= \varepsilon^{-1/2} \sqrt{\frac{(1+\varepsilon) - (T+\varepsilon)\Delta - \sqrt{((1-\varepsilon) - (T-\varepsilon)\Delta)^2 + 4\varepsilon}}{2}}, \end{aligned} \tag{2.12}$$

such that

$$(\Lambda_e^2 - H_\varepsilon^2)(H_\varepsilon^2 - \Lambda_i^2) = \varepsilon^{-1}, \qquad \Lambda_e^2 - H_1^2 = H_\varepsilon^2 - \Lambda_i^2.$$

Let

$$R := \sqrt{\frac{\Lambda_e^2 - H_\varepsilon^2}{H_\varepsilon^2 - \Lambda_i^2}}, \tag{2.13}$$

and notice that

$$\Lambda_e^2 - H_\varepsilon^2 = H_1^2 - \Lambda_i^2 = \varepsilon^{-1/2} R, \qquad H_\varepsilon^2 - \Lambda_i^2 = \Lambda_e^2 - H_1^2 = \varepsilon^{-1/2} R^{-1}.$$

Let

$$\begin{aligned} U_e &:= \frac{1}{2\sqrt{1+R^2}} \big[-\varepsilon^{1/2} |\nabla|^{-1} \Lambda_e n + R |\nabla|^{-1} \Lambda_e \rho - i\varepsilon^{1/2} h + iRg \big], \\ U_i &:= \frac{1}{2\sqrt{1+R^2}} \big[\varepsilon^{1/2} R |\nabla|^{-1} \Lambda_i n + |\nabla|^{-1} \Lambda_i \rho + i\varepsilon^{1/2} R h + ig \big]. \end{aligned} \tag{2.14}$$

Using the system (2.11) and the identities above, it is easy to check that the complex variables U_e and U_i satisfy the identities

$$\begin{aligned} (\partial_t + i\Lambda_e) U_e &= \mathcal{N}_e + \mathcal{N}_e', \\ (\partial_t + i\Lambda_i) U_i &= \mathcal{N}_i + \mathcal{N}_i', \end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
\Re(\mathcal{N}_e) &= \frac{\Lambda_e R_\alpha}{2\sqrt{1+R^2}}\big[\varepsilon^{1/2}(nR_\alpha h) - R(\rho R_\alpha g)\big],\\
\Im(\mathcal{N}_e) &= \frac{|\nabla|}{4\sqrt{1+R^2}}\big[\varepsilon^{1/2}R_\alpha h R_\alpha h - R(R_\alpha g R_\alpha g) + \varepsilon^{-1/2}\widetilde{P_e^1}n^2 - \widetilde{P_i^1}R(\rho^2)\big],\\
\Re(\mathcal{N}_i) &= \frac{-\Lambda_i R_\alpha}{2\sqrt{1+R^2}}\big[\varepsilon^{1/2}R(nR_\alpha h) + \rho R_\alpha g\big],\\
\Im(\mathcal{N}_i) &= \frac{-|\nabla|}{4\sqrt{1+R^2}}\big[\varepsilon^{1/2}R(R_\alpha h R_\alpha h) + R_\alpha g R_\alpha g + \varepsilon^{-1/2}\widetilde{P_e^1}R(n^2) + \widetilde{P_i^1}\rho^2\big],
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\mathcal{N}_e' &= i\frac{|\nabla|}{2\sqrt{1+R^2}}\left[\varepsilon^{-1/2}n\cdot n\cdot C_e(n) - R[\rho\cdot\rho\cdot C_i(\rho)]\right],\\
\mathcal{N}_i' &= -i\frac{|\nabla|}{2\sqrt{1+R^2}}\left[\varepsilon^{-1/2}R[n\cdot n\cdot C_e(n)] + \rho\cdot\rho\cdot C_i(\rho)\right].
\end{aligned} \tag{2.17}$$

The nonlinear terms $\mathcal{N}_e$ and $\mathcal{N}_i$ are quadratic in the main variables n, ρ, h, g, while $\mathcal{N}_e'$ and $\mathcal{N}_i'$ are cubic nonlinearities.

The system (2.15) can be analyzed as in [18]: the quadratic nonlinearities can be estimated as in the proof of [18, Proposition 4.3] (see, in particular, the more general [18, Proposition 5.1]), while the cubic nonlinearities fall under the scope of Proposition 3.6 below. The argument in [18, Section 4] can then be easily adapted to complete the proof of Theorem 2.1.

2.2. **The relativistic Euler–Maxwell 1-fluid model.** We consider now relativistic models. As a starter, we introduce a simple one-fluid relativistic Euler-Maxwell model, namely the model for the electrons. This is the relativistic counterpart of the (classical) Euler-Maxwell model for electrons already discussed in [12, 25].

We consider the Minkowski space $(\mathbb{R}^{1+3}, g_{\alpha\beta})$ with $g_{00} = -1$, $g_{ij} = \delta_{ij}$ and $g_{0j} = g_{j0} = 0$.[4] Its inverse is denoted $g^{\mu\nu}$ where $g^{00} = -1$, $g^{ij} = \delta_{ij}$ and $g^{0j} = 0 = g^{j0}$. We use the Einstein convention that repeated up-down indices be summed and we raise and lower indices using the metric. Latin indices $i, j \dots$ vary from 1 to 3, while greek indices $\mu, \nu \dots$ vary from 0 to 3.

We denote by $\mathcal{T}^d(\mathbb{R}^{1+3})$ the set of contravariant d-tensors on the Minkowski space. We model the electron fluid by a scalar function $n \in \mathcal{T}^0(\mathbb{R}^{1+3})$ and a velocity function $u = (u^\alpha)_{0\le\alpha\le3} \in \mathcal{T}^1(\mathbb{R}^{1+3})$ that satisfies the normalization

$$u^\alpha u_\alpha = -1. \tag{2.18}$$

Below, we let $\gamma_e = u^0$ so that $u^\nu = (\gamma_e, v^1, v^2, v^3)$ with $\gamma_e = \sqrt{1+|v|^2}$.

In addition, we also consider an electromagnetic field $F = \{F^{\mu\nu}\}_{0\le\mu,\nu\le3} \in \mathcal{T}^2(\mathbb{R}^{1+3})$. We assume that this field is skew-symmetric, $F^{\mu\nu} = -F^{\nu\mu}$. Finally, we also assume the presence of a uniform flat positively charged background of density of charge $n_0 e$ and velocity $\partial_t = (1,0,0,0) \in \mathcal{T}^1(\mathbb{R}^{1+3})$.

We can introduce the energy-momentum tensor associated to the fluid under consideration:

$$T^{\mu\nu} = nh(n)u^\mu u^\nu + p(n)g^{\mu\nu} \in \mathcal{T}^2(\mathbb{R}^{1+3}),$$

where $p = p(n)$ is a smooth function and $h = h(n)$, the *specific enthalpy*, is a function satisfying

$$h'(x) = \frac{p'(x)}{x}, \quad h(n_0) > 0, \quad h'(n_0) > 0. \tag{2.19}$$

[4] In this subsection, for simplicity, we set the speed of light c equal to 1.

where n_0 is the rest density. We can also consider the energy-momentum tensor of the electromagnetic field,

$$\mathcal{E}^{\mu\nu} = -(4\pi)^{-1}\left[F^{\mu\alpha}F^{\beta\nu}g_{\alpha\beta} + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}\right] \in \mathcal{T}^2(\mathbb{R}^{1+3}).$$

The dynamics are then given by three equations: the Maxwell equations, the continuity of matter, and the balance of energy-momentum. The Maxwell equations give

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu, \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \tag{2.20}$$

where the total relativistic current is defined by

$$J^\nu = en_0\partial_t^\nu - enu^\nu. \tag{2.21}$$

The continuity of matter gives

$$\partial_\nu(nu^\nu) = 0. \tag{2.22}$$

The balance of energy-momentum is then

$$\partial_\nu T^{\mu\nu} = enu_\alpha F^{\mu\alpha}.$$

After using (2.22), this reduces to

$$nu^\nu\partial_\nu\left[hu^\mu\right] + g^{\mu\nu}\partial_\nu p = enu_\alpha F^{\mu\alpha}. \tag{2.23}$$

Projecting onto the direction of u (multiplying by u_μ) gives

$$-nu^\nu\partial_\nu h + u^\nu\partial_\nu p = 0$$

which is always satisfied as a consequence of the definition (2.19). Therefore, (2.23) only contains three nontrivial equations which can be obtained by projecting onto $\mathbb{R}^3$, the orthogonal of ∂_t.

2.2.1. *Lorentz Covariance.* Consider a Lorentz-transformation L, i.e. a (fixed) 2-tensor L satisfying $L_{\alpha\beta}L^{\alpha\gamma} = \delta_\beta^\gamma$ and define

$$(X')^\alpha = L^{\alpha\beta}X_\beta, \quad n'(X') = n(X), \quad (u')^\alpha(X') = L^{\alpha\beta}u_\beta(X), \quad (F')^{\alpha\beta}(X') = L^{\alpha\gamma}L^{\beta\delta}F_{\gamma\delta}(X),$$
$$(J')^\alpha(X') = L^{\alpha\beta}J_\beta(X).$$

Then, we see that (n,u,J,F) satisfy (2.20)–(2.23) if and only if (n',u',J',F') satisfy the same equations.

2.2.2. *Irrotational flows.* We introduce the (generalized) vorticity defined by

$$\omega_{\alpha\beta} = \partial_\alpha(hu_\beta) - \partial_\beta(hu_\alpha) + eF_{\alpha\beta}.$$

This is transported by the flow in the following sense:

$$u^\nu\partial_\nu\omega_{\alpha\beta} = (\partial_\alpha u^\nu)\omega_{\beta\nu} - (\partial_\beta u^\nu)\omega_{\alpha\nu}. \tag{2.24}$$

Indeed, we may simply compute

$$\begin{aligned}
u^\nu\partial_\nu\omega_{\alpha\beta} &= \partial_\alpha(u^\nu\partial_\nu(hu_\beta)) - \partial_\nu(hu_\beta)\partial_\alpha u^\nu - \partial_\beta(u^\nu\partial_\nu(hu_\alpha)) + \partial_\nu(hu_\alpha)\partial_\beta u^\nu + eu^\nu\partial_\nu F_{\alpha\beta}\\
&= -\partial_\alpha\Big(\frac{1}{n}\partial_\beta p - eu^\gamma F_{\beta\gamma}\Big) + \partial_\beta\Big(\frac{1}{n}\partial_\alpha p - eu^\gamma F_{\alpha\gamma}\Big) - eu^\nu(\partial_\alpha F_{\beta\nu} + \partial_\beta F_{\nu\alpha})\\
&\quad - (\partial_\alpha u^\nu)\omega_{\nu\beta} - (\partial_\alpha u^\nu)\partial_\beta(hu_\nu) + e(\partial_\alpha u^\nu)F_{\nu\beta} + (\partial_\beta u^\nu)\omega_{\nu\alpha} + (\partial_\beta u^\nu)\partial_\alpha(hu_\nu) - e(\partial_\beta u^\nu)F_{\nu\alpha}\\
&= e\{\partial_\alpha(u^\gamma F_{\beta\gamma}) - \partial_\beta(u^\gamma F_{\alpha\gamma}) - u^\nu\partial_\alpha F_{\beta\nu} - u^\nu\partial_\beta F_{\nu\alpha} + (\partial_\alpha u^\nu)F_{\nu\beta} - (\partial_\beta u^\nu)F_{\nu\alpha}\}\\
&\quad - (\partial_\alpha u^\nu)\omega_{\nu\beta} + (\partial_\beta u^\nu)\omega_{\nu\alpha} - \{(\partial_\alpha u^\nu)\partial_\beta(hu_\nu) - (\partial_\beta u^\nu)\partial_\alpha(hu_\nu)\}\\
&= -(\partial_\alpha u^\nu)\omega_{\nu\beta} + (\partial_\beta u^\nu)\omega_{\nu\alpha}.
\end{aligned}$$

Moreover, using (2.23),

$$nu^\alpha\omega_{\alpha\beta} = nu^\alpha\partial_\alpha(hu_\beta) - nu^\alpha\partial_\beta(hu_\alpha) + enu^\alpha F_{\alpha\beta} = (-\partial_\beta p + enu^\alpha F_{\beta\alpha}) + n\partial_\beta h + enu^\alpha F_{\alpha\beta} = 0,$$

so that
$$u^\alpha \omega_{\alpha\beta} = 0. \tag{2.25}$$
Therefore initial data for which $\omega_{jk} = 0$ at the initial time lead to solutions which remain irrotational, i.e. $\omega_{\alpha\beta} = 0$ as long as the solution remains smooth.

2.2.3. *The main theorem.* We are now ready to state our main theorem in this subsection.

Theorem 2.2. *Assume h, n_0, and e are fixed as before and let $N_0 = 10^4$. Assume ${}^{(0)}v$ is a vector-field on $\mathbb{R}^3$, ${}^{(0)}F$ is an antisymmetric 2-tensor, and ${}^{(0)}n$ is a real-valued function, satisfying*
$$\frac{1}{4\pi e}\partial_j {}^{(0)}F^{j0} = n_0 - {}^{(0)}n\sqrt{1+|{}^{(0)}v|^2}, \qquad e^{(0)}F_{jk} = \partial_k(h({}^{(0)}n){}^{(0)}v_j) - \partial_j(h({}^{(0)}n){}^{(0)}v_k), \tag{2.26}$$
and
$$\sum_{j=1}^{3}\|({}^{(0)}v^j, {}^{(0)}F^{j0})\|_{H^{N_0+2}} + \sum_{j=1}^{3}\|((1-\Delta){}^{(0)}v^j, (1-\Delta){}^{(0)}F^{j0})\|_Z = \varepsilon_0 \le \overline{\varepsilon},$$
where $\overline{\varepsilon} > 0$ is sufficiently small, and the Z norm is defined in Definition 3.2. Then there exists a unique global solution (n, u, F), satisfying $u = (\sqrt{1+|v|^2}, v^1, v^2, v^3)$ and $(n - n_0, v, F) \in C([0,\infty) : H^{N_0+1})$, of the system
$$\begin{aligned} &\partial_\mu F^{\mu\nu} = 4\pi(en_0\partial_t^\nu - enu^\nu), \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0,\\ &\partial_\nu(nu^\nu) = 0,\\ &nu^\nu\partial_\nu\left[hu^\mu\right] + g^{\mu\nu}\partial_\nu p = enu_\alpha F^{\mu\alpha}, \end{aligned} \tag{2.27}$$
with initial data $(n(0), v(0), F(0)) = ({}^{(0)}n, {}^{(0)}v, {}^{(0)}F)$. Moreover, for any $t \ge 0$,
$$\frac{1}{4\pi e}\partial_j F^{j0}(t) = n_0 - n(t)\sqrt{1+|v(t)|^2}, \qquad eF_{jk}(t) = \partial_k[h(n(t))v_j(t)] - \partial_j[h(n(t))v_k(t)],$$
and, with $\beta = 1/100$,
$$\sup_{t\in[0,\infty)}\|(v(t), F(t))\|_{H^{N_0+1}} + \sup_{t\in[0,\infty)}\sup_{|\rho|\le 4}(1+t)^{1+\beta}\|(D_x^\rho v(t), D_x^\rho F(t))\|_{L^\infty} \lesssim \varepsilon_0.$$

Note that the second constraint in (2.26) corresponds to a (generalized) irrotationality condition, while the first implies that the plasma is neutral.

Qualitatively, the theorem states that small, smooth, localized, and irrotational perturbations of the rest solution $(n, v, F) = (n_0, 0, 0)$ lead to global solutions that scatter.

2.2.4. *Outline of the proof.* The proof of the main theorem follows the same strategy as in [25]. Using simple changes of variables, the system (2.27) can be rewritten as a quasi-linear evolution system. More precisely, we consider the variables[5]
$$\mu_e^j := hu^j, \qquad E^j := F^{j0}, \qquad B^j := -(1/2)\in^{jkl} F_{kl}, \tag{2.28}$$
for $j \in \{1, 2, 3\}$. Notice that
$$F^{jk} = -\in^{jkl} B_l.$$

[5]This choice of variables is motivated from the choice of variables in the non-relativistic case, see [25].

In terms of the new variables, the system (2.27) becomes

$$
\begin{aligned}
\partial_0(n\gamma_e) + \partial_k\Big(\frac{n}{h}\mu_e^k\Big) &= 0,\\
\partial_0(\mu_e^j) + eE^j + \frac{\partial_j h}{\gamma_e} + \frac{1}{h\gamma_e}\mu_e^k\partial_k\mu_e^j + \frac{e\in^{jkl}(\mu_e)_kB_l}{h\gamma_e} &= 0,\\
\partial_0 E^j - \in^{jkl}\partial_k B_l - \frac{4\pi e n\mu_e^j}{h} &= 0,\\
\partial_0 B^j + \in^{jkl}\partial_k E_l &= 0,\\
\partial_j B^j &= 0,\\
\partial_j E^j + 4\pi e(n\gamma_e - n_0) &= 0.
\end{aligned}
$$

We make linear changes of variables to simplify this system:

$$
\begin{aligned}
&n_e(x,t) = n_0(1+\widetilde{n}(\lambda x,\lambda t)),\\
&\mu_e(x,t) = h(n_0)\widetilde{\mu}(\lambda x,\lambda t),\\
&E(x,t) = Z\widetilde{E}(\lambda x,\lambda t),\\
&B(x,t) = Z\widetilde{B}(\lambda x,\lambda t),\\
&\gamma_e(x,t) = \widetilde{\gamma}(\lambda x,\lambda t),\\
&\lambda := \sqrt{\frac{4\pi e^2 n_0}{h(n_0)}}, \qquad Z := \frac{\lambda h(n_0)}{e} = \sqrt{4\pi n_0 h(n_0)}.
\end{aligned}
$$

Let also $\widetilde{h}(\alpha) := \frac{h(n_0(1+\alpha))}{h(n_0)}$. In terms of the new variables, the system becomes

$$
\begin{aligned}
\partial_t(\widetilde{\gamma} + \widetilde{n}\widetilde{\gamma}) + \partial_k\Big(\frac{(1+\widetilde{n})}{\widetilde{h}(\widetilde{n})}\widetilde{\mu}^k\Big) &= 0,\\
\partial_t\widetilde{\mu}^j + \widetilde{E}^j + \frac{\widetilde{h}'(\widetilde{n})}{\widetilde{\gamma}}\partial_j\widetilde{n} + \frac{\widetilde{\mu}^k\partial_k\widetilde{\mu}^j}{\widetilde{\gamma}\widetilde{h}(\widetilde{n})} + \frac{\in^{jkl}\widetilde{\mu}_k\widetilde{B}_l}{\widetilde{\gamma}\widetilde{h}(\widetilde{n})} &= 0,\\
\partial_t\widetilde{E}^j - \in^{jkl}\partial_k\widetilde{B}_l - \frac{(1+\widetilde{n})\widetilde{\mu}^j}{\widetilde{h}(\widetilde{n})} &= 0,\\
\partial_t\widetilde{B}^j + \in^{jkl}\partial_k\widetilde{E}_l &= 0,
\end{aligned}
\tag{2.29}
$$

together with the elliptic constraints

$$
\partial_j\widetilde{B}^j = 0, \qquad \partial_j\widetilde{E}^j + (\widetilde{\gamma} + \widetilde{n}\widetilde{\gamma} - 1) = 0. \tag{2.30}
$$

The irrotationality assumption, at time $t = 0$, in Theorem 2.2 is equivalent to

$$
\widetilde{B}^j - \in^{jkl}\partial_k\widetilde{\mu}_l = 0. \tag{2.31}
$$

Notice also that

$$
\widetilde{\gamma} = \sqrt{1 + \frac{|\widetilde{\mu}|^2}{\widetilde{h}(\widetilde{n})^2}}. \tag{2.32}
$$

Therefore, using the second equation in (2.29),

$$\begin{aligned}\widetilde{\gamma}\partial_t\widetilde{\gamma} &= \frac{\widetilde{\mu}_j\partial_t\widetilde{\mu}^j}{\widetilde{h}(\widetilde{n})^2} - \frac{|\widetilde{\mu}|^2\widetilde{h}'(\widetilde{n})}{\widetilde{h}(\widetilde{n})^3}\partial_t\widetilde{n} \\ &= -\frac{\widetilde{\mu}_j}{\widetilde{h}(\widetilde{n})^2}\Big[\widetilde{E}^j + \frac{\widetilde{h}'(\widetilde{n})}{\widetilde{\gamma}}\partial_j\widetilde{n} + \frac{\widetilde{\mu}^k\partial_k\widetilde{\mu}^j}{\widetilde{\gamma}\widetilde{h}(\widetilde{n})}\Big] - \frac{|\widetilde{\mu}|^2\widetilde{h}'(\widetilde{n})}{\widetilde{h}(\widetilde{n})^3}\partial_t\widetilde{n}.\end{aligned}$$

The first equation in (2.29) then gives

$$\begin{aligned}\partial_t\widetilde{n}\Big[\widetilde{\gamma} - \frac{(1+\widetilde{n})|\widetilde{\mu}|^2\widetilde{h}'(\widetilde{n})}{\widetilde{\gamma}\widetilde{h}(\widetilde{n})^3}\Big] + \partial_k\Big(\frac{(1+\widetilde{n})}{\widetilde{h}(\widetilde{n})}\widetilde{\mu}^k\Big) \\ - \frac{(1+\widetilde{n})\widetilde{\mu}^j\widetilde{E}^j}{\widetilde{\gamma}\widetilde{h}(\widetilde{n})^2} - \frac{(1+\widetilde{n})\widetilde{h}'(\widetilde{n})\widetilde{\mu}^j}{\widetilde{\gamma}^2\widetilde{h}(\widetilde{n})^2}\partial_j\widetilde{n} - \frac{(1+\widetilde{n})\widetilde{\mu}^j\widetilde{\mu}^k\partial_k\widetilde{\mu}^j}{\widetilde{\gamma}^2\widetilde{h}(\widetilde{n})^3} = 0.\end{aligned} \tag{2.33}$$

This equation is equivalent to the first equation in the system (2.29).

We can define now (modified) higher order energy functionals,

$$\mathcal{E}_N := \sum_{|\gamma|\leq N}\int_{\mathbb{R}^3} F\cdot|D_x^\gamma\widetilde{n}|^2 + G_{kj}D_x^\gamma\widetilde{\mu}^j D_x^\gamma\widetilde{\mu}^k + |D_x^\gamma\widetilde{E}|^2 + |D_x^\gamma\widetilde{B}|^2\,dx$$

for $N\geq 0$, where

$$\begin{aligned}F &:= \widetilde{h}'(\widetilde{n})\Big[\widetilde{\gamma} - \frac{(1+\widetilde{n})|\widetilde{\mu}|^2\widetilde{h}'(\widetilde{n})}{\widetilde{\gamma}\widetilde{h}(\widetilde{n})^3}\Big], \\ G_{kj} &:= \frac{\widetilde{\gamma}(1+\widetilde{n})}{\widetilde{h}(\widetilde{n})}\Big[\delta_{kj} - \frac{\widetilde{\mu}^k\widetilde{\mu}^j}{\widetilde{\gamma}^2\widetilde{h}(\widetilde{n})^2}\Big].\end{aligned}$$

Using the formulas (2.29), (2.32), and (2.33), it is easy to verify that, if $N\geq 4$ and $\mathcal{E}_N(t)\ll 1$, then

$$\mathcal{E}_N(t)\approx\|\widetilde{n}(t)\|_{H^N}^2 + \|\widetilde{\mu}(t)\|_{H^N}^2 + \|\widetilde{E}(t)\|_{H^N}^2 + \|\widetilde{B}(t)\|_{H^N}^2,$$

and

$$\frac{d}{dt}\mathcal{E}_N(t)\lesssim\mathcal{E}_N(t)\cdot\sup_{|\rho|\leq 4}\|(D_x^\rho\widetilde{n}(t), D_x^\rho\widetilde{\mu}(t), D_x^\rho\widetilde{E}(t), D_x^\rho\widetilde{B}(t))\|_{L^\infty}.$$

In other words, the modified energy functionals $\mathcal{E}_N$ are coercive, and their increment is controlled by the $W^{4,\infty}$ norm of the solution. The standard theory of quasi-linear symmetric hyperbolic systems (see [31]) applies to construct local smooth solutions of the system (2.29), (2.32); moreover the (elliptic) relations (2.30) and (2.31) are propagated by the flow, provided that they are satisfied at the initial time.

To complete the proof of the main theorem it suffices to prove integrable decay in time of the $\|(\widetilde{n}(t),\widetilde{\mu}(t),\widetilde{E}(t),\widetilde{B}(t))\|_{W^{4,\infty}}$ norm. This is only possible in the case of irrotational flows, using dispersion. More precisely, assuming irrotationality, the dynamical equations in the system (2.29) become

$$\begin{aligned}\partial_t\widetilde{\mu}^j + \widetilde{E}^j + \frac{\widetilde{h}'(\widetilde{n})}{\widetilde{\gamma}}\partial_j\widetilde{n} + \frac{\partial_j(|\widetilde{\mu}|^2)}{2\widetilde{\gamma}\widetilde{h}(\widetilde{n})} &= 0, \\ \partial_t\widetilde{E}^j + \partial_k(\partial_k\widetilde{\mu}^j - \partial_j\widetilde{\mu}^k) - \frac{(1+\widetilde{n})\widetilde{\mu}^j}{\widetilde{h}(\widetilde{n})} &= 0,\end{aligned} \tag{2.34}$$

where $\widetilde{n}$ and $\widetilde{\gamma}$ are defined (implicitly) by the formulas

$$\partial_j\widetilde{E}^j + (\widetilde{\gamma} + \widetilde{n}\widetilde{\gamma} - 1) = 0, \qquad \widetilde{\gamma} = \sqrt{1 + \frac{|\widetilde{\mu}|^2}{\widetilde{h}(\widetilde{n})^2}}. \tag{2.35}$$

Recall that $\widetilde{h}(0) = 1$ and let $T := \widetilde{h}'(0)$. The system (2.34) can be rewritten in the form

$$\partial_t \widetilde{\mu}^j + \widetilde{E}^j - T\partial_j\partial_k \widetilde{E}^k = \mathcal{N}_1^j,$$
$$\partial_t \widetilde{E}^j + \Delta\widetilde{\mu}^j - \partial_j\partial_k\widetilde{\mu}^k - \widetilde{\mu}^j = \mathcal{N}_2^j,$$

where

$$\mathcal{N}_1^j = \Big[\frac{\widetilde{h}'(\widetilde{n})}{\widetilde{\gamma}} - T\Big]\partial_j\partial_k\widetilde{E}^k + \frac{\widetilde{h}'(\widetilde{n})}{\widetilde{\gamma}}\partial_j(\widetilde{\gamma} + \widetilde{n}\widetilde{\gamma} - \widetilde{n} - 1) - \frac{\partial_j(|\widetilde{\mu}|^2)}{2\widetilde{\gamma}\widetilde{h}(\widetilde{n})},$$
$$\mathcal{N}_2^j = \Big[\frac{1+\widetilde{n}}{\widetilde{h}(\widetilde{n})} - 1\Big]\widetilde{\mu}^j.$$

Let

$$Q = |\nabla|^{-1}\mathrm{curl}, \quad P = -\nabla(-\Delta)^{-1}\mathrm{div}, \quad P^2 + Q^2 = Id, \quad PQ = QP = 0, \quad P^2 = P, \quad Q^3 = Q.$$

We define

$$\Lambda_e^2 := 1 - T\Delta, \qquad \Lambda_b^2 := 1 - \Delta,$$

and introduce the dispersive unknowns

$$U_e := P\widetilde{\mu} - i\Lambda_e P\widetilde{E}, \qquad U_b := Q\widetilde{\mu} - i\Lambda_b^{-1}Q\widetilde{E}. \tag{2.36}$$

These unknowns satisfy the system

$$\begin{aligned} (\partial_t + i\Lambda_e)\,U_e &= P\mathcal{N}_1 - i\Lambda_e P\mathcal{N}_2,\\ (\partial_t + i\Lambda_b)\,U_b &= Q\mathcal{N}_1 - i\Lambda_b^{-1}Q\mathcal{N}_2. \end{aligned} \tag{2.37}$$

Notice also that the variables $\widetilde{E}$ and $\widetilde{\mu}$ can be expressed in terms of U_e and U_b, more precisely

$$P\widetilde{\mu} = \Re(U_e), \qquad Q\widetilde{\mu} = \Re(U_b), \qquad P\widetilde{E} = -\Im(\Lambda_e^{-1}U_e), \qquad Q\widetilde{E} = -\Im(\Lambda_b U_b).$$

The semilinear analysis is a direct adaptation of [25, Section 3 and 4], using also a variant of Proposition 3.6 below to deal with the contribution of cubic nonlinearities. We will not provide further details since we are giving a complete proof in the more general 2-fluid model.

2.3. **The relativistic Euler-Maxwell 2-fluid model.** We are now ready to consider the full relativistic Euler–Maxwell 2-fluid model. We consider again the standard Minkowski space $(\mathbb{R}^{1+3}, g)$, as in the previous section.

The main unknowns are two densities n_i and n_e, two velocity fields v_i and v_e (both of which satisfy (2.18)) and an electromagnetic field F. We are also given smooth pressure laws p_i and p_e and enthalpies h_i and h_e satisfying

$$h_i'(x) = \frac{p_i'(x)}{x}, \qquad h_e'(x) = \frac{p_e'(x)}{x}, \qquad h_i(n_0/Z) > 0,\ h_i'(n_0/Z) > 0,\ h_e(n_0) > 0,\ h_e'(n_0) > 0, \tag{2.38}$$

where $Z > 0$. The Maxwell equations (2.20) remain essentially the same, with a new formula for the relativistic current, i.e.

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu = 4\pi(Zen_iu_i^\nu - en_eu_e^\nu), \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{2.39}$$

Both species are independently conserved so that

$$\partial_\nu(n_iu_i^\nu) = 0 = \partial_\nu(n_eu_e^\nu), \tag{2.40}$$

and we have two forms of balance of momentum:

$$\begin{aligned} n_iu_i^\nu\partial_\nu\left[h_iu_i^\mu\right] + g^{\mu\nu}\partial_\nu p_i &= -Zen_i(u_i)_\alpha F^{\mu\alpha},\\ n_eu_e^\nu\partial_\nu\left[h_eu_e^\mu\right] + g^{\mu\nu}\partial_\nu p_e &= en_e(u_e)_\alpha F^{\mu\alpha}. \end{aligned} \tag{2.41}$$

In particular, we recover the fact that the stress-energy tensor is divergence free,

$$\partial_\nu \left[T_i^{\mu\nu} + T_e^{\mu\nu} + \mathcal{E}^{\mu\nu}\right] = 0,$$

$$T_i^{\mu\nu} = n_i h_i u_i^\mu u_i^\nu + p_i g^{\mu\nu}, \quad T_e^{\mu\nu} = n_e h_e u_e^\mu u_e^\nu + p_e g^{\mu\nu}, \quad \mathcal{E}^{\mu\nu} = -(4\pi)^{-1}\left[F^{\mu\alpha}F^{\beta\nu}g_{\alpha\beta} + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}\right].$$

Again, we have two naturally transported (generalized) vorticities:

$$\omega^i_{\alpha\beta} = \partial_\alpha\left[h_i(u_i)_\beta\right] - \partial_\beta\left[h_i(u_i)_\alpha\right] - ZeF_{\alpha\beta},$$
$$\omega^e_{\alpha\beta} = \partial_\alpha\left[h_e(u_e)_\beta\right] - \partial_\beta\left[h_e(u_e)_\alpha\right] + eF_{\alpha\beta},$$

which satisfy the identities

$$\begin{aligned} u_i^\nu\partial_\nu\omega^i_{\alpha\beta} &= -(\partial_\alpha u_i^\nu)\omega^i_{\nu\beta} + (\partial_\beta u_i^\nu)\omega^i_{\nu\alpha},\\ u_e^\nu\partial_\nu\omega^e_{\alpha\beta} &= -(\partial_\alpha u_e^\nu)\omega^e_{\nu\beta} + (\partial_\beta u_e^\nu)\omega^e_{\nu\alpha}. \end{aligned} \tag{2.42}$$

and

$$u_i^\nu\omega^i_{\nu\beta} = 0 = u_e^\nu\omega^e_{\nu\beta}. \tag{2.43}$$

Therefore initial data for which $\omega^i_{jk} = 0 = \omega^e_{jk}$ at the initial time lead to solutions which remain irrotational, i.e. $\omega^i_{\alpha\beta} = 0 = \omega^e_{\alpha\beta}$ as long as the solution remains smooth.

2.3.1. *The main theorem.* We state now our main theorem in the paper.

Theorem 2.3. *Assume h_i, h_e, n_0, Z, and e are fixed as before and let $N_0 = 10^4$. Let*

$$\varepsilon := \frac{Zh_e(n_0)}{h_i(n_0)}, \qquad T := \frac{Z^2h_e'(n_0)}{h_i'(n_0/Z)}, \qquad C_b := \frac{Z^2h_e(n_0)}{n_0h_i'(n_0/Z)}, \tag{2.44}$$

and assume that

$$\varepsilon \le 10^{-3}, \qquad T \in [1, 100], \qquad C_b \ge 6T. \tag{2.45}$$

Assume ${}^{(0)}v_i, {}^{(0)}v_e$ are vector-fields on $\mathbb{R}^3$, ${}^{(0)}F$ is an antisymmetric 2-tensor, and ${}^{(0)}n_i, {}^{(0)}n_e$ are real-valued functions, satisfying

$$\partial_j{}^{(0)}F^{j0} + 4\pi e\big[-Z^{(0)}n_i\sqrt{1+|{}^{(0)}v_i|^2} + {}^{(0)}n_e\sqrt{1+|{}^{(0)}v_e|^2}\big] = 0, \tag{2.46}$$

$$\begin{aligned} e^{(0)}F_{jk} &= Z^{-1}\{\partial_j[h_i({}^{(0)}n_i)({}^{(0)}v_i)_k] - \partial_k[h_i({}^{(0)}n_i)({}^{(0)}v_i)_j]\}\\ &= -\partial_j[h_e({}^{(0)}n_e)({}^{(0)}v_e)_k] + \partial_k[h_e({}^{(0)}n_e)({}^{(0)}v_e)_j], \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} &\|({}^{(0)}n_i - Z^{-1}n_0, {}^{(0)}v_i, {}^{(0)}n_e - n_0, {}^{(0)}v_e, {}^{(0)}F)\|_{H^{N_0+2}}\\ &+ \|(1-\Delta)({}^{(0)}n_i - Z^{-1}n_0, {}^{(0)}v_i, {}^{(0)}n_e - n_0, {}^{(0)}v_e, {}^{(0)}F)\|_Z = \varepsilon_0 \le \overline{\varepsilon}, \end{aligned} \tag{2.48}$$

where $\overline{\varepsilon} > 0$ is sufficiently small, and the Z norm is defined in Definition 3.2. Then there exists a unique global solution (n_i, u_i, n_e, u_e, F) of the system

$$\begin{aligned} &\partial_\mu F^{\mu\nu} = 4\pi(Zen_iu_i^\nu - en_eu_e^\nu), \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0,\\ &\partial_\nu(n_iu_i^\nu) = \partial_\nu(n_eu_e^\nu) = 0,\\ &u_i^\nu\partial_\nu\left[h_iu_i^\mu\right] + g^{\mu\nu}\partial_\nu h_i = -Ze(u_i)_\alpha F^{\mu\alpha}, \qquad u_e^\nu\partial_\nu\left[h_eu_e^\mu\right] + g^{\mu\nu}\partial_\nu h_e = e(u_e)_\alpha F^{\mu\alpha}, \end{aligned} \tag{2.49}$$

satisfying

$$\begin{aligned} &u_i = (\sqrt{1+|v_i|^2}, v_i^1, v_i^2, v_i^3), \qquad u_e = (\sqrt{1+|v_e|^2}, v_e^1, v_e^2, v_e^3),\\ &(n_i - Z^{-1}n_0, v_i, n_e - n_0, v_e, F) \in C([0,\infty) : H^{N_0}),\\ &(n_i, v_i, n_e, v_e, F)(0) = ({}^{(0)}n_i, {}^{(0)}v_i, {}^{(0)}n_e, {}^{(0)}v_e, {}^{(0)}F). \end{aligned}$$

Moreover, for any $t \in [0,\infty)$,

$$\begin{aligned} eF_{jk}(t) &= Z^{-1}\{\partial_j[h_i(n_i(t))(v_i(t))_k] - \partial_k[h_i(n_i(t))(v_i(t))_j]\} \\ &= -\partial_j[h_e(n_e(t))(v_e(t))_k] + \partial_k[h_e(n_e(t))(v_e(t))_j], \end{aligned}$$

and, with $\beta := 1/100$,

$$\begin{aligned} &\|(n_i(t) - Z^{-1}n_0, v_i(t), n_e(t) - n_0, v_e(t), F(t))\|_{H^{N_0}} \\ &+ \sup_{|\rho|\le 4}(1+t)^{1+\beta/2}\|D_x^\rho[n_i(t) - Z^{-1}n_0, v_i(t), n_e(t) - n_0, v_e(t), F(t)]\|_{L^\infty} \lesssim \varepsilon_0. \end{aligned}$$

Once again, note that (2.47) correspond to (generalized) irrationality conditions.

3. Proof of Theorem 2.3

3.1. **New variables and local existence.** In this section we prove Theorem 2.3. As before, we start by defining the new variables

$$\mu_i^j = h_i u_i^j, \qquad \mu_e^j = h_e u_e^j, \qquad E^j = F^{j0}, \qquad B^j = -(1/2) \in^{jkl} F_{kl}, \tag{3.1}$$

for $j \in \{1,2,3\}$. Notice that $F^{jk} = - \in^{jkl} B_l$.

Let $\gamma_i = u_i^0 = \sqrt{1+|v_i|^2}$ and $\gamma_e = u_e^0 = \sqrt{1+|v_e|^2}$. We can rewrite our evolution system (2.49) in the form

$$\begin{aligned} &\partial_t(n_i\gamma_i) + \partial_k\Big(\frac{n_i}{h_i}\mu_i^k\Big) = 0, \\ &\partial_t(n_e\gamma_e) + \partial_k\Big(\frac{n_e}{h_e}\mu_e^k\Big) = 0, \\ &\partial_t(\mu_i^j) - ZeE^j + \frac{\partial_j h_i}{\gamma_i} + \frac{\mu_i^k\partial_k\mu_i^j}{h_i\gamma_i} - \frac{Ze \in^{jkl} (\mu_i)_k B_l}{h_i\gamma_i} = 0, \\ &\partial_t(\mu_e^j) + eE^j + \frac{\partial_j h_e}{\gamma_e} + \frac{\mu_e^k\partial_k\mu_e^j}{h_e\gamma_e} + \frac{e \in^{jkl} (\mu_e)_k B_l}{h_e\gamma_e} = 0, \\ &\partial_t E^j - \in^{jkl} \partial_k B_l + 4\pi e\big[Z\frac{n_i}{h_i}\mu_i - \frac{n_e}{h_e}\mu_e\big] = 0, \\ &\partial_t B^j + \in^{jkl} \partial_k E_l = 0, \\ &\partial_j B^j = 0, \\ &\partial_j E^j + 4\pi e(-Zn_i\gamma_i + n_e\gamma_e) = 0. \end{aligned}$$

We now set

$$\begin{aligned} H_i := h_i(n_0/Z), \quad P_i := Z^{-2}h_i'(n_0/Z), \quad H_e := h_e(n_0), \quad P_e := h_e'(n_0), \\ \beta := \sqrt{\frac{4\pi n_0 Ze^2}{H_i}}, \qquad \lambda := \sqrt{\frac{4\pi e^2}{P_i}}, \qquad \mu := \sqrt{n_0 Z P_i H_i}, \end{aligned}$$

and

$$\varepsilon := \frac{ZH_e}{H_i}, \qquad T := \frac{P_e}{P_i}, \qquad C_b := \frac{H_e}{n_0 P_i}. \tag{3.2}$$

We make linear changes of variables to further simplify the system. More precisely, we define new functions $\rho, \widetilde{\rho}, \widetilde{\gamma}_i, n, \widetilde{n}, \widetilde{\gamma}_e, \widetilde{h}_i, \widetilde{h_e}$ and new vector-fields $u, v, \widetilde{E}, \widetilde{B}$ such that

$$\begin{aligned} \gamma_i(x,t) &= \widetilde{\gamma}_i(\lambda x, \beta t), & \gamma_e(x,t) &= \widetilde{\gamma}_e(\lambda x, \beta t), \\ n_i(x,t)\gamma_i(x,t) &= (n_0/Z)[\rho(\lambda x, \beta t) + 1], & n_e(x,t)\gamma_e(x,t) &= n_0[n(\lambda x, \beta t) + 1], \\ n_i(x,t) &= (n_0/Z)[\widetilde{\rho}(\lambda x, \beta t) + 1], & n_e(x,t) &= n_0[\widetilde{n}(\lambda x, \beta t) + 1], \\ \mu_i(x,t) &= \mu u(\lambda x, \beta t), & \mu_e(x,t) &= (\varepsilon\mu/Z) v(\lambda x, \beta t), \\ E(x,t) &= e^{-1} n_0 \lambda P_i \widetilde{E}(\lambda x, \beta t), & B(x,t) &= (Ze)^{-1}\lambda\mu \widetilde{B}(\lambda x, \beta t), \\ \widetilde{h}_i(\alpha) &= \frac{h_i((n_0/Z)(\alpha+1))}{h_i(n_0/Z)}, & \widetilde{h}_e(\alpha) &= \frac{h_e(n_0(\alpha+1))}{h_e(n_0)}. \end{aligned}$$

The new variables solve the normalized system

$$\begin{aligned} \partial_t \rho + \partial_k \Big[\frac{1+\widetilde{\rho}}{\widetilde{h}_i(\widetilde{\rho})} u^k\Big] &= 0, \\ \partial_t n + \partial_k \Big[\frac{1+\widetilde{n}}{\widetilde{h}_e(\widetilde{n})} v^k\Big] &= 0, \\ \partial_t u^j - \widetilde{E}^j + \frac{\widetilde{h}_i'(\widetilde{\rho})\partial_j \widetilde{\rho}}{\widetilde{h}_i'(0)\widetilde{\gamma}_i} + \frac{u^k \partial_k u^j - \in^{jkl} u_k \widetilde{B}_l}{\widetilde{\gamma}_i \widetilde{h}_i(\widetilde{\rho})} &= 0, \\ \varepsilon \partial_t v^j + \widetilde{E}^j + \frac{T\widetilde{h}_e'(\widetilde{n})\partial_j \widetilde{n}}{\widetilde{h}_e'(0)\widetilde{\gamma}_e} + \frac{\varepsilon v^k \partial_k v^j + \in^{jkl} v_k \widetilde{B}_l}{\widetilde{\gamma}_e \widetilde{h}_e(\widetilde{n})} &= 0, \\ \partial_t \widetilde{E}^j - \frac{C_b}{\varepsilon} \in^{jkl} \partial_k \widetilde{B}_l + \Big[\frac{1+\widetilde{\rho}}{\widetilde{h}_i(\widetilde{\rho})} u^j - \frac{1+\widetilde{n}}{\widetilde{h}_e(\widetilde{n})} v^j\Big] &= 0, \\ \partial_t \widetilde{B}^j + \in^{jkl} \partial_k \widetilde{E}_l &= 0, \end{aligned} \tag{3.3}$$

together with the elliptic equations

$$\partial_j \widetilde{B}^j = 0, \qquad \partial_j \widetilde{E}^j + n - \rho = 0. \tag{3.4}$$

Moreover, the functions $\widetilde{\rho}, \widetilde{\gamma}_i, \widetilde{n}, \widetilde{\gamma}_e$ can be expressed (implicitly) in terms of $\rho, |u|^2, n, |v|^2$,

$$\widetilde{\gamma}_i = \frac{\rho+1}{\widetilde{\rho}+1} = \sqrt{1 + \frac{\varepsilon|u|^2}{C_b(\widetilde{h}_i(\widetilde{\rho}))^2}}, \qquad \widetilde{\gamma}_e = \frac{n+1}{\widetilde{n}+1} = \sqrt{1 + \frac{\varepsilon|v|^2}{C_b(\widetilde{h}_e(\widetilde{n}))^2}}. \tag{3.5}$$

Finally, the irrotationality condition in (2.47) is equivalent to

$$\widetilde{B}^j + \in^{jkl} \partial_k u_l = \widetilde{B}^j - \varepsilon \in^{jkl} \partial_k v_l = 0. \tag{3.6}$$

Notice also that

$$-\partial_k \Big[\frac{1+\widetilde{\rho}}{\widetilde{h}_i(\widetilde{\rho})} u^k\Big] = \partial_t \rho = (\widetilde{\rho}+1)\partial_t \widetilde{\gamma}_i + \widetilde{\gamma}_i \partial_t \widetilde{\rho}$$

and

$$\begin{aligned} \widetilde{\gamma}_i \partial_t \widetilde{\gamma}_i &= \frac{\varepsilon u^j}{C_b(\widetilde{h}_i(\widetilde{\rho}))^2} \partial_t u^j - \frac{\varepsilon |u|^2 \widetilde{h}_i'(\widetilde{\rho})}{C_b(\widetilde{h}_i(\widetilde{\rho}))^3} \partial_t \widetilde{\rho} \\ &= -\frac{\varepsilon u^j}{C_b(\widetilde{h}_i(\widetilde{\rho}))^2} \Big[-\widetilde{E}^j + \frac{\widetilde{h}_i'(\widetilde{\rho})\partial_j \widetilde{\rho}}{\widetilde{h}_i'(0)\widetilde{\gamma}_i} + \frac{u^k \partial_k u^j}{\widetilde{\gamma}_i \widetilde{h}_i(\widetilde{\rho})}\Big] - \frac{\varepsilon |u|^2 \widetilde{h}_i'(\widetilde{\rho})}{C_b(\widetilde{h}_i(\widetilde{\rho}))^3} \partial_t \widetilde{\rho}. \end{aligned}$$

Therefore

$$\partial_t\widetilde{\rho}\Big[\widetilde{\gamma}_i-\frac{\varepsilon|u|^2\widetilde{h}_i'(\widetilde{\rho})(1+\widetilde{\rho})}{C_b(\widetilde{h}_i(\widetilde{\rho}))^3\widetilde{\gamma}_i}\Big]+\partial_k\Big[\frac{1+\widetilde{\rho}}{\widetilde{h}_i(\widetilde{\rho})}u^k\Big]-\frac{\varepsilon u^j(1+\widetilde{\rho})}{C_b(\widetilde{h}_i(\widetilde{\rho}))^2\widetilde{\gamma}_i}\Big[-\widetilde{E}^j+\frac{\widetilde{h}_i'(\widetilde{\rho})\partial_j\widetilde{\rho}}{\widetilde{h}_i'(0)\widetilde{\gamma}_i}+\frac{u^k\partial_k u^j}{\widetilde{\gamma}_i\widetilde{h}_i(\widetilde{\rho})}\Big]=0. \tag{3.7}$$

Similarly

$$\partial_t\widetilde{n}\Big[\widetilde{\gamma}_e-\frac{\varepsilon|v|^2\widetilde{h}_e'(\widetilde{n})(1+\widetilde{n})}{C_b(\widetilde{h}_i(\widetilde{n}))^3\widetilde{\gamma}_e}\Big]+\partial_k\Big[\frac{1+\widetilde{n}}{\widetilde{h}_e(\widetilde{n})}v^k\Big]-\frac{v^j(1+\widetilde{n})}{C_b(\widetilde{h}_e(\widetilde{n}))^2\widetilde{\gamma}_e}\Big[\widetilde{E}^j+\frac{T\widetilde{h}_e'(\widetilde{n})\partial_j\widetilde{n}}{\widetilde{h}_e'(0)\widetilde{\gamma}_e}+\frac{\varepsilon v^k\partial_k v^j}{\widetilde{\gamma}_e\widetilde{h}_e(\widetilde{n})}\Big]=0. \tag{3.8}$$

Notice that the system (3.3)–(3.6) is similar to the Euler–Maxwell two-fluid system analyzed in [18, Theorem 1.1], at least up to linear and quadratic terms. The local existence theory is based on energy estimates. For any $N\geq 0$ we define

$$\begin{aligned}\mathcal{E}_N:=\sum_{|\gamma|\leq N}\int_{\mathbb{R}^3}\big[&F^1|D^\gamma_x\widetilde{\rho}|^2+TF^2|D^\gamma_x\widetilde{n}|^2\\&+G^1_{jk}D^\gamma_x u^jD^\gamma_x u^k+\varepsilon G^2_{jk}D^\gamma_x v^jD^\gamma_x v^k+|D^\gamma_x\widetilde{E}|^2+\frac{C_b}{\varepsilon}|D^\gamma_x\widetilde{B}|^2\big]\,dx,\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}F^1&:=\frac{\widetilde{h}_i'(\widetilde{\rho})}{\widetilde{h}_i'(0)}\Big[\widetilde{\gamma}_i-\frac{\varepsilon(1+\widetilde{\rho})|u|^2\widetilde{h}_i'(\widetilde{\rho})}{C_b\widetilde{\gamma}_i\widetilde{h}(\widetilde{\rho})^3}\Big], &\quad G^1_{kj}&:=\frac{\widetilde{\gamma}_i(1+\widetilde{\rho})}{\widetilde{h}_i(\widetilde{\rho})}\Big[\delta_{kj}-\frac{\varepsilon u^ku^j}{C_b\widetilde{\gamma}_i^2\widetilde{h}_i(\widetilde{\rho})^2}\Big],\\F^2&:=\frac{\widetilde{h}_e'(\widetilde{n})}{\widetilde{h}_e'(0)}\Big[\widetilde{\gamma}_e-\frac{\varepsilon(1+\widetilde{n})|v|^2\widetilde{h}_e'(\widetilde{n})}{C_b(\widetilde{h}_i(\widetilde{n}))^3\widetilde{\gamma}_e}\Big], &\quad G^2_{kj}&:=\frac{\widetilde{\gamma}_e(1+\widetilde{n})}{\widetilde{h}_e(\widetilde{n})}\Big[\delta_{kj}-\frac{\varepsilon v^kv^j}{C_b\widetilde{\gamma}_e^2\widetilde{h}(\widetilde{n})^2}\Big].\end{aligned} \tag{3.10}$$

Notice that

$$\mathcal{E}_N\approx\|(\rho,u,n,v,\widetilde{E},\widetilde{B})\|^2_{H^N}\qquad\text{if}\qquad\|(\rho,u,n,v,\widetilde{E},\widetilde{B})\|_{H^4}\ll 1. \tag{3.11}$$

The following proposition is our main local regularity result:

Proposition 3.1. *(i) There is $\delta_1\in(0,1]$ such that if*

$$\|(\rho^0,u^0,n^0,v^0,\widetilde{E}^0,\widetilde{B}^0)\|_{H^4}\leq\delta_1 \tag{3.12}$$

then there is a unique solution $(\rho,u,n,v,\widetilde{E},\widetilde{B})\in C([0,1]:H^4)$ of the system (3.3) *with*

$$(\rho(0),u(0),n(0),v(0),\widetilde{E}(0),\widetilde{B}(0))=(\rho^0,u^0,n^0,v^0,\widetilde{E}^0,\widetilde{B}^0).$$

Moreover,

$$\sup_{t\in[0,1]}\|(\rho(t),u(t),n(t),v(t),\widetilde{E}(t),\widetilde{B}(t))\|_{H^4}\lesssim\|(\rho^0,u^0,n^0,v^0,\widetilde{E}^0,\widetilde{B}^0)\|_{H^4}.$$

(ii) If $N\geq 4$ and $(\rho^0,u^0,n^0,v^0,\widetilde{E}^0,\widetilde{B}^0)\in H^N$ satisfies (3.12) *then $(\rho,u,n,v,\widetilde{E},\widetilde{B})\in C([0,1]:H^N)$, and*

$$\mathcal{E}_N(t')-\mathcal{E}_N(t)\lesssim\int_t^{t'}\mathcal{A}(s)\mathcal{E}_N(s)\,ds \tag{3.13}$$

for any $t\leq t'\in[0,1]$, where

$$\mathcal{A}(s):=\sum_{|\gamma|\leq 2}\|D^\gamma_x[\rho(s),u(s),n(s),v(s),\widetilde{E}(s),\widetilde{B}(s)]\|_{L^\infty}. \tag{3.14}$$

(iii) If $(\rho^0,u^0,n^0,v^0,\widetilde{E}^0,\widetilde{B}^0)\in H^4$ satisfies (3.12)*, and, in addition,*

$$div(\widetilde{E}^0)+n^0-\rho^0=0,\qquad\widetilde{B}^0=\varepsilon\nabla\times v^0=-\nabla\times u^0,$$

then, for any $t\in[0,1]$,

$$div(\widetilde{E})(t)+n(t)-\rho(t)=0,\qquad\widetilde{B}(t)=\varepsilon\nabla\times v(t)=-\nabla\times u(t). \tag{3.15}$$

The proof of this proposition is very similar to the proof of the corresponding result in [18], namely Proposition 2.1. For the local existence part, one can rewrite the system as a quasi-linear symmetric hyperbolic system, and use the main theorems in [31]. The energy inequality (3.13) follows using the equations and integration by parts. Finally, it is easy to see that the identities in (3.15) are transported by the nonlinear flow.

3.2. **The dispersive system and the main bootstrap argument.** Given Proposition 3.1, the main remaining step is to prove the global integrability of the function A appearing in (3.13). For this we need to use the dispersive effect of the flow, and take advantage of the irrotationality assumption $\widetilde{B}(t) = \varepsilon\nabla\times v(t) = -\nabla\times u(t)$ in (3.15).

We proceed as in [18, Section 3]. For $\xi\in\mathbb{R}^3$ we define

$$\begin{aligned}&|\nabla|(\xi) := |\xi|, \qquad R_j(\xi) := i\xi_j/|\xi|, \qquad Q_{jk}(\xi) := i \in_{jlk} \xi_l/|\xi|,\\ &H_1(\xi) := \sqrt{1+|\xi|^2}, \qquad H_\varepsilon(\xi) := \varepsilon^{-1/2}\sqrt{1+T|\xi|^2}, \qquad \Lambda_b(\xi) := \varepsilon^{-1/2}\sqrt{1+\varepsilon+C_b|\xi|^2}.\end{aligned} \tag{3.16}$$

By a slight abuse of notation we also let $|\nabla|, R_j, Q, H_1, H_\varepsilon, \Lambda_b$ denote the operators on $\mathbb{R}^3$ defined by the corresponding Fourier multipliers. Notice that

$$Q^3 = Q \qquad \text{and} \qquad QA = |\nabla|^{-1}(\nabla\times A) \text{ for any vector-field } A.$$

We define

$$2U_b := \Lambda_b|\nabla|^{-1}Q\widetilde{B} - iQ^2\widetilde{E}, \qquad g := -|\nabla|^{-1}\mathrm{div}(u), \qquad h := -|\nabla|^{-1}\mathrm{div}(v).$$

Let

$$A_j = 2\Lambda_b^{-1}\mathrm{Re}(U_b)_j, \qquad q_i(\alpha) := \frac{1+\alpha-\widetilde{h}_i(\alpha)}{\widetilde{h}_i(\alpha)}, \qquad q_e(\alpha) := \frac{1+\alpha-\widetilde{h}_e(\alpha)}{\widetilde{h}_e(\alpha)}.$$

Recalling that $\widetilde{B} = \varepsilon\nabla\times v = -\nabla\times u$ and $\mathrm{div}(\widetilde{E}) = \rho - n$, the functions U_b, h, g together with n, ρ allow us to recover all the physical unknowns, i.e.

$$\begin{aligned}&u_j = R_j g - A_j, \qquad v_j = R_j h + \varepsilon^{-1}A_j,\\ &\widetilde{E}_j = -|\nabla|^{-1}R_j\left[\rho - n\right] - 2\mathrm{Im}(U_b)_j, \qquad \widetilde{B} = |\nabla|Q(A).\end{aligned} \tag{3.17}$$

In terms of ρ, g, n, h, U_b the system (3.3)-(3.6) becomes

$$\begin{aligned}\partial_t\rho - |\nabla|g &= -\partial_j\big[u_j q_i(\widetilde{\rho})\big],\\ \partial_t n - |\nabla|h &= -\partial_j\big[v_j q_e(\widetilde{n})\big],\\ \partial_t g - |\nabla|^{-1}n + |\nabla|^{-1}H_1^2\rho &= R_j\Big[\frac{\partial_j(|u|^2)}{2\widetilde{\gamma}_i\widetilde{h}_i(\widetilde{\rho})}\Big] + R_j\Big[\frac{\widetilde{h}_i'(\widetilde{\rho})\partial_j\widetilde{\rho}}{\widetilde{h}_i'(0)\widetilde{\gamma}_i} - \partial_j\rho\Big],\\ \partial_t h + |\nabla|^{-1}H_\varepsilon^2 n - \varepsilon^{-1}|\nabla|^{-1}\rho &= R_j\Big[\frac{\partial_j(|v|^2)}{2\widetilde{\gamma}_e\widetilde{h}_e(\widetilde{n})}\Big] + \frac{T}{\varepsilon}R_j\Big[\frac{\widetilde{h}_e'(\widetilde{n})\partial_j\widetilde{n}}{\widetilde{h}_e'(0)\widetilde{\gamma}_e} - \partial_j n\Big],\\ \partial_t(U_b)_j + i\Lambda_b(U_b)_j &= (i/2)(Q^2)_{jk}\big[q_i(\widetilde{\rho})u_k - q_e(\widetilde{n})v_k\big],\end{aligned} \tag{3.18}$$

where the left-hand sides of the equations above are linear in the variables n, h, ρ, g, U_b, and the right-hand sides are at least quadratic.

We make linear changes of variables to diagonalize this system. Let

$$\begin{aligned}\Lambda_e &:= \varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon)-(T+\varepsilon)\Delta+\sqrt{((1-\varepsilon)-(T-\varepsilon)\Delta)^2+4\varepsilon}}{2}},\\ \Lambda_i &:= \varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon)-(T+\varepsilon)\Delta-\sqrt{((1-\varepsilon)-(T-\varepsilon)\Delta)^2+4\varepsilon}}{2}},\end{aligned}\tag{3.19}$$

such that

$$(\Lambda_e^2-H_\varepsilon^2)(H_\varepsilon^2-\Lambda_i^2)=\varepsilon^{-1},\qquad \Lambda_e^2-H_1^2=H_\varepsilon^2-\Lambda_i^2.\tag{3.20}$$

Let

$$R:=\sqrt{\frac{\Lambda_e^2-H_\varepsilon^2}{H_\varepsilon^2-\Lambda_i^2}},\tag{3.21}$$

and notice that

$$\Lambda_e^2-H_\varepsilon^2=H_1^2-\Lambda_i^2=\varepsilon^{-1/2}R,\qquad H_\varepsilon^2-\Lambda_i^2=\Lambda_e^2-H_1^2=\varepsilon^{-1/2}R^{-1}.$$

As in [18] let

$$\begin{aligned}U_i &:= \frac{1}{2\sqrt{1+R^2}}\big[|\nabla|^{-1}\Lambda_i\rho+\varepsilon^{1/2}R|\nabla|^{-1}\Lambda_i n+ig+i\varepsilon^{1/2}Rh\big],\\ U_e &:= \frac{1}{2\sqrt{1+R^2}}\big[R|\nabla|^{-1}\Lambda_e\rho-\varepsilon^{1/2}|\nabla|^{-1}\Lambda_e n+iRg-i\varepsilon^{1/2}h\big].\end{aligned}\tag{3.22}$$

Using the system (3.18) it is easy to check that the complex variables U_e, U_i and U_b satisfy the identities

$$\begin{aligned}&(\partial_t+i\Lambda_i)U_i=\mathcal{N}_i,\\ &(\partial_t+i\Lambda_e)U_e=\mathcal{N}_e,\\ &(\partial_t+i\Lambda_b)(U_b)_j=(\mathcal{N}_b)_j,\end{aligned}\tag{3.23}$$

where

$$\begin{aligned}\Re(\mathcal{N}_i) &= \frac{-\Lambda_i R_j}{2\sqrt{1+R^2}}\big[u_j q_i(\widetilde{\rho})+\varepsilon^{1/2}R[v_j q_e(\widetilde{n})]\big],\\ \Im(\mathcal{N}_i) &= \frac{R_j}{2\sqrt{1+R^2}}\Big[\Big(\frac{\partial_j(|u|^2)}{2\widetilde{\gamma}_i\widetilde{h}_i(\widetilde{\rho})}+\frac{\widetilde{h}_i'(\widetilde{\rho})\partial_j\widetilde{\rho}}{\widetilde{h}_i'(0)\widetilde{\gamma}_i}-\partial_j\rho\Big)+\varepsilon^{1/2}R\Big(\frac{\partial_j(|v|^2)}{2\widetilde{\gamma}_e\widetilde{h}_e(\widetilde{n})}+\frac{T}{\varepsilon}\frac{\widetilde{h}_e'(\widetilde{n})\partial_j\widetilde{n}}{\widetilde{h}_e'(0)\widetilde{\gamma}_e}-\frac{T}{\varepsilon}\partial_j n\Big)\Big],\\ \Re(\mathcal{N}_e) &= \frac{\Lambda_e R_j}{2\sqrt{1+R^2}}\big[-R[u_j q_i(\widetilde{\rho})]+\varepsilon^{1/2}v_j q_e(\widetilde{n})\big],\\ \Im(\mathcal{N}_e) &= \frac{R_j}{2\sqrt{1+R^2}}\Big[R\Big(\frac{\partial_j(|u|^2)}{2\widetilde{\gamma}_i\widetilde{h}_i(\widetilde{\rho})}+\frac{\widetilde{h}_i'(\widetilde{\rho})\partial_j\widetilde{\rho}}{\widetilde{h}_i'(0)\widetilde{\gamma}_i}-\partial_j\rho\Big)-\varepsilon^{1/2}\Big(\frac{\partial_j(|v|^2)}{2\widetilde{\gamma}_e\widetilde{h}_e(\widetilde{n})}+\frac{T}{\varepsilon}\frac{\widetilde{h}_e'(\widetilde{n})\partial_j\widetilde{n}}{\widetilde{h}_e'(0)\widetilde{\gamma}_e}-\frac{T}{\varepsilon}\partial_j n\Big)\Big],\\ \Re(\mathcal{N}_b)_j &= 0,\\ \Im(\mathcal{N}_b)_j &= (1/2)(Q^2)_{jk}\big[q_i(\widetilde{\rho})u_k-q_e(\widetilde{n})v_k\big].\end{aligned}\tag{3.24}$$

The system (3.23) is our main dispersive system, which is diagonalized at the linear level. To analyze it we have to express the nonlinearities $\mathcal{N}_e$, $\mathcal{N}_i$, and $\mathcal{N}_b$ in terms of the complex variables U_e, U_i, and U_b.

Indeed, it follows from (3.22) that

$$\begin{aligned}
\rho &= \frac{|\nabla|}{\sqrt{1+R^2}\Lambda_i}(U_i+\overline{U_i}) + \frac{|\nabla|R}{\sqrt{1+R^2}\Lambda_e}(U_e+\overline{U_e}),\\
n &= \frac{|\nabla|\varepsilon^{-1/2}R}{\sqrt{1+R^2}\Lambda_i}(U_i+\overline{U_i}) - \frac{|\nabla|\varepsilon^{-1/2}}{\sqrt{1+R^2}\Lambda_e}(U_e+\overline{U_e}),\\
g &= -\frac{i}{\sqrt{1+R^2}}(U_i-\overline{U_i}) - \frac{iR}{\sqrt{1+R^2}}(U_e-\overline{U_e}),\\
h &= -\frac{i\varepsilon^{-1/2}R}{\sqrt{1+R^2}}(U_i-\overline{U_i}) + \frac{i\varepsilon^{-1/2}}{\sqrt{1+R^2}}(U_e-\overline{U_e}),\\
A &= \Lambda_b^{-1}(U_b+\overline{U_b}).
\end{aligned} \tag{3.25}$$

The variables u, v can then be recovered using (3.17), and the remaining variables $\widetilde{\rho}, \widetilde{\gamma}_i, \widetilde{n}, \widetilde{\gamma}_e$ can be recovered (implicitly) using (3.5).

3.2.1. *The Z norm.* To analyze the system (3.23) we use the Fourier transform method and a special norm called the Z norm. We recall our main function spaces, used also in [25] and [18]. We fix $\varphi : \mathbb{R} \to [0,1]$ an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For simplicity of notation, we also let $\varphi : \mathbb{R}^d \to [0,1]$ denote the corresponding radial function on $\mathbb{R}^d$, $d = 2, 3$. For $d \in \{1, 2, 3\}$ let

$$\begin{aligned}
&\varphi_k(x) = \varphi_{k,(d)}(x) := \varphi(|x|/2^k) - \varphi(|x|/2^{k-1}) \qquad \text{for any } k \in \mathbb{Z},\ x \in \mathbb{R}^d,\\
&\varphi_I := \sum_{m\in I\cap\mathbb{Z}} \varphi_m \text{ for any } I \subseteq \mathbb{R}.
\end{aligned}$$

Let

$$\mathcal{J} := \{(k,j) \in \mathbb{Z}\times\mathbb{Z}_+ : k+j \geq 0\}.$$

For any $(k,j) \in \mathcal{J}$ let

$$\widetilde{\varphi}^{(k)}_j(x) := \begin{cases} \varphi_{(-\infty,-k]}(x) & \text{if } k+j=0 \text{ and } k\leq 0,\\ \varphi_{(-\infty,0]}(x) & \text{if } j=0 \text{ and } k\geq 0,\\ \varphi_j(x) & \text{if } k+j\geq 1 \text{ and } j\geq 1. \end{cases}$$

and notice that, for any $k \in \mathbb{Z}$ fixed,

$$\sum_{j\geq -\min(k,0)} \widetilde{\varphi}^{(k)}_j = 1.$$

For any interval $I \subseteq \mathbb{R}$ let

$$\widetilde{\varphi}^{(k)}_I(x) := \sum_{j\in I,\,(k,j)\in\mathcal{J}} \widetilde{\varphi}^{(k)}_j(x).$$

Let P_k, $k \in \mathbb{Z}$, denote the operator on $\mathbb{R}^3$ defined by the Fourier multiplier $\xi \to \varphi_k(\xi)$. Similarly, for any $I \subseteq \mathbb{R}$ let P_I denote the operator on $\mathbb{R}^3$ defined by the Fourier multiplier $\xi \to \varphi_I(\xi)$.

Definition 3.2. *Let*

$$\beta := 1/100, \qquad \alpha := \beta/2, \qquad \gamma := 3/2 - 4\beta. \tag{3.26}$$

We define

$$Z := \{f \in L^2(\mathbb{R}^3) : \|f\|_Z := \sup_{(k,j)\in\mathcal{J}} \|\widetilde{\varphi}^{(k)}_j(x)\cdot P_k f(x)\|_{B_{k,j}} < \infty\}, \tag{3.27}$$

where, with $\widetilde{k} := \min(k,0)$ *and* $k_+ := \max(k,0)$,

$$\|g\|_{B_{k,j}} := \inf_{g=g_1+g_2} \big[\|g_1\|_{B^1_{k,j}} + \|g_2\|_{B^2_{k,j}}\big], \tag{3.28}$$

$$\|h\|_{B^1_{k,j}} := (2^{\alpha k} + 2^{10k})\big[2^{(1+\beta)j}\|h\|_{L^2} + 2^{(1/2-\beta)\widetilde{k}}\|\widehat{h}\|_{L^\infty}\big], \tag{3.29}$$

and

$$\|h\|_{B^2_{k,j}} := 2^{10|k|}(2^{\alpha k} + 2^{10k})\big[2^{(1-\beta)j}\|h\|_{L^2} + \|\widehat{h}\|_{L^\infty} + 2^{\gamma j}\sup_{R\in[2^{-j},2^k],\,\xi_0\in\mathbb{R}^3} R^{-2}\|\widehat{h}\|_{L^1(B(\xi_0,R))}\big]. \tag{3.30}$$

The Z norm is our main tool to prove L^∞ decay of solutions. In a slightly different form, it has been introduced by two of the authors in [25], in the context of Klein–Gordon system with different speeds. Here we use the same Z norm as in [18].

The Z norm can be used to bound other norms, which is important in nonlinear estimates. The following lemma, which is a consequence of Lemma A.5 in [18], summarizes some of these bounds.

Lemma 3.3. *Assume $\|f\|_Z \leq 1$, $t \in \mathbb{R}$, $(k,j) \in \mathcal{J}$, and let $\widetilde{k} = \min(k,0)$ and*

$$f_{k,j} := P_{[k-2,k+2]}[\widetilde{\varphi}^{(k)}_j \cdot P_k f].$$

(i) Then

$$\|f_{k,j}\|_{L^2} \lesssim (2^{\alpha k} + 2^{10k})^{-1} \cdot 2^{2\beta\widetilde{k}} 2^{-(1-\beta)j} \tag{3.31}$$

and

$$\sup_{\xi\in\mathbb{R}^3} \big|D^\rho_\xi \widehat{f_{k,j}}(\xi)\big| \lesssim_{|\rho|} (2^{\alpha k} + 2^{10k})^{-1} \cdot 2^{-(1/2-\beta)\widetilde{k}} 2^{|\rho| j}. \tag{3.32}$$

Moreover

$$\sum_{j\geq \max(-k,0)} \|f_{k,j}\|_{L^2} \lesssim \min(2^{(1+\beta-\alpha)k}, 2^{-10k}) \tag{3.33}$$

and, for any $\sigma \in \{i,e,b\}$,

$$\sum_{j\geq \max(-k,0)} \big\|e^{it\Lambda_\sigma} f_{k,j}\big\|_{L^\infty} \lesssim \min(2^{(1/2-\beta-\alpha)k}, 2^{-6k})(1+|t|)^{-1-\beta}. \tag{3.34}$$

3.2.2. *The main bootstrap proposition.* We are now ready to state our main proposition.

Proposition 3.4. *Assume $N_0 = 10^4$, $T_0 \geq 0$, and $U = (U_i, U_e, U_b) \in C([0,T_0] : H^{N_0})$ is a solution of the system of equations* (3.23)-(3.24)*. Assume that*

$$\sup_{t\in[0,T_0]} \sup_{\sigma\in\{i,e,b\}} \|e^{it\Lambda_\sigma} U_\sigma(t)\|_{H^{N_0}\cap Z} \leq \delta_1 \leq 1. \tag{3.35}$$

Then

$$\sup_{t\in[0,T_0]} \sup_{\sigma\in\{i,e,b\}} \|e^{it\Lambda_\sigma} U_\sigma(t) - U_\sigma(0)\|_Z \lesssim \delta_1^2, \tag{3.36}$$

where the implicit constant in (3.36) *may depend only on the constants T, ε, C_b.*

A standard continuity argument, as in [18, Section 4], shows that the bootstrap estimate in Proposition 3.4 can be combined with the local existence theory in Proposition 3.1 and the energy increment bound (3.13) to complete the proof of the global regularity result in Theorem 2.3. The only additional ingredient that is needed is the dispersive bound

$$\sum_{k\in\mathbb{Z}} (1+2^{4k})\|P_k e^{it\Lambda_\sigma} f\|_{L^\infty} \lesssim (1+|t|)^{-1-\beta}\|f\|_Z, \qquad \sigma \in \{i,e,b\},\ t\in\mathbb{R},$$

which follows from (3.34).

3.3. **Proof of Proposition 3.4.** It remains to prove Proposition 3.4. For this we first decompose the nonlinearities in (3.24) into their quadratic components and their cubic (and higher order) components. More precisely, we consider the expansions around $\alpha = 0$,

$$\widetilde{h}_i(\alpha) = 1 + c_i\alpha + d_i\alpha^2/2 + h_i^{\geq 3}(\alpha), \qquad \widetilde{h}_e(\alpha) = 1 + c_e\alpha + d_e\alpha^2/2 + h_e^{\geq 3}(\alpha),$$

where $c_i, c_e \in (0,\infty)$, $d_i, d_e \in \mathbb{R}$, and $h_i^{\geq 3}, h_e^{\geq 3}$ are cubic remainders. Then

$$q_i(\alpha) = (1-c_i)\alpha + q_i^{\geq 2}(\alpha), \qquad q_e(\alpha) = (1-c_e)\alpha + q_e^{\geq 2}(\alpha).$$

Notice that, as a consequence of (3.5) and (3.25),

$$\rho - \widetilde{\rho} = \frac{\varepsilon|u|^2}{2C_b} + \mathrm{Cubic}(U_i, U_e, U_b), \qquad n - \widetilde{n} = \frac{\varepsilon|v|^2}{2C_b} + \mathrm{Cubic}(U_i, U_e, U_b).$$

We decompose

$$\mathcal{N}_i = \mathcal{N}_i^2 + \mathcal{N}_i^{\geq 3}, \qquad \mathcal{N}_e = \mathcal{N}_e^2 + \mathcal{N}_e^{\geq 3}, \qquad \mathcal{N}_b = \mathcal{N}_b^2 + \mathcal{N}_b^{\geq 3},$$

where

$$\begin{aligned}
\Re(\mathcal{N}_i^2) &= \frac{-\Lambda_i R_j}{2\sqrt{1+R^2}}\big[(1-c_i)\rho u_j + (1-c_e)\varepsilon^{1/2}R(nv_j)\big],\\
\Im(\mathcal{N}_i^2) &= \frac{-|\nabla|}{4\sqrt{1+R^2}}\Big[\Big(\Big(1-\frac{\varepsilon}{C_b}\Big)|u|^2 + \frac{d_i}{c_i}\rho^2\Big) + \varepsilon^{1/2}R\Big(\Big(1-\frac{T}{C_b}\Big)|v|^2 + \frac{Td_e}{\varepsilon c_e}n^2\Big)\Big],\\
\Re(\mathcal{N}_e^2) &= \frac{\Lambda_e R_j}{2\sqrt{1+R^2}}\big[-(1-c_i)R(\rho u_j) + (1-c_e)\varepsilon^{1/2}nv_j\big],\\
\Im(\mathcal{N}_e^2) &= \frac{-|\nabla|}{4\sqrt{1+R^2}}\Big[R\Big(\Big(1-\frac{\varepsilon}{C_b}\Big)|u|^2 + \frac{d_i}{c_i}\rho^2\Big) - \varepsilon^{1/2}\Big(\Big(1-\frac{T}{C_b}\Big)|v|^2 + \frac{Td_e}{\varepsilon c_e}n^2\Big)\Big],\\
\Re(\mathcal{N}_b^2)_j &= 0,\\
\Im(\mathcal{N}_b^2)_j &= (1/2)(Q^2)_{jk}\big[(1-c_i)\rho u_k - (1-c_e)nv_k\big].
\end{aligned} \tag{3.37}$$

The nonlinearities $\mathcal{N}_i^2, \mathcal{N}_e^2, \mathcal{N}_b^2$ are quadratic nonlinearities, while $\mathcal{N}_i^{\geq 3}, \mathcal{N}_e^{\geq 3}, \mathcal{N}_b^{\geq 3}$ are cubic (in the main variables U_i, U_e, U_b). We will estimate the contributions of the quadratic nonlinearities using the main Proposition 5.1 in [18].

We turn now to the proof of Proposition 3.4. The equations (3.23) give

$$[\partial_t + i\Lambda_\sigma(\xi)]\widehat{U_\sigma}(\xi,t) = \widehat{\mathcal{N}_\sigma}(\xi,t), \qquad \sigma \in \{i,e,b\}. \tag{3.38}$$

Let

$$V_\sigma(t) := e^{it\Lambda_\sigma}U_\sigma(t), \qquad \sigma \in \{i,e,b\},\ t \in [0,T_0].$$

The equations (3.38) are equivalent to

$$\frac{d}{dt}[\widehat{V_\sigma}(\xi,t)] = e^{it\Lambda_\sigma(\xi)}\widehat{\mathcal{N}_\sigma}(\xi,t),$$

therefore, for any $t \in [0,T_0]$ and $\sigma \in \{i,e,b\}$,

$$\widehat{V_\sigma}(\xi,t) - \widehat{V_\sigma}(\xi,0) = \int_0^t e^{is\Lambda_\sigma(\xi)}\widehat{\mathcal{N}_\sigma}(\xi,s)\,ds. \tag{3.39}$$

The desired bound (3.36) is equivalent to proving that

$$\|V_\sigma(t) - V_\sigma(0)\|_Z \lesssim \delta_1^2,$$

for any $t \in [0,T_0]$ and any $\sigma \in \{i,e,b\}$.

The contributions of the quadratic nonlinearities $\mathcal{N}_i^2, \mathcal{N}_e^2, \mathcal{N}_b^2$ can be estimated using Proposition 5.1 in [18].[6] The bound for the cubic contributions follows from Proposition 3.5 below.

Proposition 3.5. *For any $t \in [0, T_0]$ and any $\sigma \in \{i, e, b\}$,*

$$\Big\|\mathcal{F}^{-1}\Big[\int_0^t e^{is\Lambda_\sigma(\xi)}[\widehat{\mathcal{N}_\sigma}(\xi, s) - \widehat{\mathcal{N}^2_\sigma}(\xi, s)]ds\Big]\Big\|_Z \lesssim \delta_1^2.$$

To prove Proposition 3.5, given $t \in [0, T_0]$, we fix a suitable decomposition of the function $\mathbf{1}_{[0,t]}$, i.e. we fix functions $q_0, \dots, q_{L+1} : \mathbb{R} \to [0, 1]$, $|L - \log_2(2 + t)| \le 2$, with the properties

$$\begin{split}&\sum_{m=0}^{L+1} q_l(s) = \mathbf{1}_{[0,t]}(s), \qquad \operatorname{supp} q_0 \subseteq [0, 2], \qquad \operatorname{supp} q_{L+1} \subseteq [t - 2, t], \qquad \operatorname{supp} q_m \subseteq [2^{m-1}, 2^{m+1}],\\ &q_m \in C^1(\mathbb{R}) \qquad \text{and} \qquad \int_0^t |q_m'(s)|\, ds \lesssim 1 \qquad \text{for } m = 1, \dots, L.\end{split} \tag{3.40}$$

For $\sigma, \mu, \nu \in \{i, e, b\}$, $\iota_\mu, \iota_\nu \in \{-1, 1\}$, and $m \in \{0, 1, \dots, L\}$ we consider the trilinear operator $\tilde{T}_m^{\sigma;\mu,\nu}$ defined by

$$\mathcal{F}\tilde{T}_m^{\sigma;\mu,\nu}[f, g; h](\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}^3\times\mathbb{R}^3} q_m(s)e^{is[\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta-\theta)]}\widehat{f}(\xi - \eta, s)\widehat{g}(\eta - \theta, s)\widehat{h}(\theta, s)dsd\eta d\theta,$$

where $\widetilde{\Lambda}_\mu = \iota_\mu\Lambda_\mu$ and $\widetilde{\Lambda}_\nu = \iota_\nu\Lambda_\nu$.

We prove first the following trilinear estimate involving the operators $\tilde{T}_m^{\sigma;\mu,\nu}$.

Lemma 3.6. *Assume that $f^\mu, f^\nu \in C([0, T_0] : L^2)$ satisfy*

$$\sup_{s\in[0,T_0]} \big[\|f^\mu(s)\|_{H^{N_0}\cap Z} + \|f^\nu(s)\|_{H^{N_0}\cap Z}\big] \le 1$$

and decompose

$$\begin{split}f^\mu &= \sum_{k'\in\mathbb{Z}}\sum_{j'\ge\max(-k',0)} P_{[k'-2,k'+2]}(\widetilde{\varphi}^{(k')}_{j'} \cdot P_{k'}f^\mu) = \sum_{(k',j')\in\mathcal{J}} f^\mu_{k',j'},\\ f^\nu &= \sum_{k'\in\mathbb{Z}}\sum_{j'\ge\max(-k',0)} P_{[k'-2,k'+2]}(\widetilde{\varphi}^{(k')}_{j'} \cdot P_{k'}f^\nu) = \sum_{(k',j')\in\mathcal{J}} f^\nu_{k',j'}.\end{split}$$

Assume that $h \in C([0, T_0] : L^2)$ satisfies, for any $s \in [0, T_0]$ and any $k \in \mathbb{Z}$,

$$\|h(s)\|_{H^{N_0-2}} + (1 + s)^{1+\beta}\|h(s)\|_{W^{\infty,4}} \le 1. \tag{3.41}$$

Then

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J},k_3\in\mathbb{Z}} (1 + 2^{k_1} + 2^{k_2} + 2^{k_3})\big\|\widetilde{\varphi}^{(k)}_j \cdot P_k\tilde{T}_m^{\sigma;\mu,\nu}[f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2}; P_{k_3}h]\big\|_{B^1_{k,j}} \lesssim 2^{-\beta^4 m} \tag{3.42}$$

for any fixed

$$\sigma, \mu, \nu \in \{i, e, b\}, \quad (k, j) \in \mathcal{J}, \quad m \in \{0, \dots, L + 1\}.$$

Proof of Lemma 3.6. **Step 1.** As a consequence of Plancherel theorem, for any $f, g, h \in C([0, T_0] : H^4)$

$$\begin{split}\|\tilde{T}_m^{\sigma;\mu,\nu}[f, g; h]\|_{L^2} \lesssim 2^m &\sup_{s\in[2^m-1,2^{m+4}]} \min\big\{\|f(s)\|_{L^2}\|e^{-is\widetilde{\Lambda}_\nu}g(s)\|_{L^\infty}\|h(s)\|_{L^\infty},\\ &\|e^{-is\widetilde{\Lambda}_\mu}f(s)\|_{L^\infty}\|g(s)\|_{L^2}\|h(s)\|_{L^\infty}, \|e^{-is\widetilde{\Lambda}_\mu}f(s)\|_{L^\infty}\|e^{-is\widetilde{\Lambda}_\nu}g(s)\|_{L^\infty}\|h(s)\|_{L^2}\big\}.\end{split} \tag{3.43}$$

[6]Proposition 5.1 in [18], which is the main technical result in that paper, requires the hypothesis (2.45).

This bound will be used repeatedly in the proof of the lemma. Moreover, as a consequence of Definition 3.2 and Lemma 3.3, for any $(k',j')\in\mathcal{J}$ and $s\in[0,T_0]$,

$$\begin{aligned}
&\|f^{\mu}_{k',j'}(s)\|_{L^2}+\|f^{\nu}_{k',j'}(s)\|_{L^2}\lesssim(2^{\alpha k'}+2^{10k'})^{-1}2^{2\beta k'}2^{-(1-\beta)j'},\\
&\sum_{j'\geq\max(-k',0)}\big[\|f^{\mu}_{k',j'}(s)\|_{L^2}+\|f^{\nu}_{k',j'}(s)\|_{L^2}\big]\lesssim\min(2^{(1+\beta-\alpha)k'},2^{-(N_0-1)k'}),\\
&\sum_{j'\geq\max(-k',0)}\big[\|e^{-is\widetilde{\Lambda}_\mu}f^{\mu}_{k',j'}(s)\|_{L^\infty}+\|e^{-is\widetilde{\Lambda}_\nu}f^{\nu}_{k',j'}(s)\|_{L^\infty}\big]\lesssim\min(2^{(1/2-\beta-\alpha)k'},2^{-6k'})(1+|t|)^{-1-\beta}.
\end{aligned}\tag{3.44}$$

Notice also that, as a consequence of Definition 3.2,

$$\big\|\widetilde{\varphi}^{(k')}_{j'}\cdot P_{k'}H\big\|_{B^1_{k',j'}}\lesssim(2^{\alpha k'}+2^{10k'})\cdot 2^{3j'/2}2^{(1/2-\beta)\widetilde{k'}}\big\|\widetilde{\varphi}^{(k')}_{j'}\cdot P_{k'}H\big\|_{L^2},\tag{3.45}$$

for any $(k',j')\in\mathcal{J}$ and $H\in L^2$. Therefore, using (3.41) and the last three bounds, the left-hand side of (3.42) is dominated by

$$C2^m(1+2^k)(2^{\alpha k}+2^{10k})\cdot 2^{3j/2}2^{(1/2-\beta)\widetilde{k}}\cdot 2^{-2m-\beta m}2^{-(N_0-4)\max(k,0)}.$$

This suffices to prove (3.42) unless

$$3j/2\geq m+(N_0-20)k_++D,\tag{3.46}$$

where $D\geq 0$ is a large constant.

Assume now that (3.46) holds. We define three sets

$$\begin{aligned}
\mathcal{S}_1&=\{((k_1,j_1),(k_2,j_2),k_3)\in\mathcal{J}\times\mathcal{J}\times\mathbb{Z}:\ \max(k_1,k_2,k_3)\geq 2j/N_0\},\\
\mathcal{S}_2&=\{((k_1,j_1),(k_2,j_2),k_3)\in\mathcal{J}\times\mathcal{J}\times\mathbb{Z}:\ \min(k_1,k_2,k_3)\leq -10j\},\\
\mathcal{S}_3&=\{((k_1,j_1),(k_2,j_2),k_3)\in\mathcal{J}\times\mathcal{J}\times\mathbb{Z}:\ \max(j_1,j_2)\geq 10j\}.
\end{aligned}$$

It is easy to see that one can also use the bounds (3.43)–(3.45) to show that if (3.46) holds, then

$$\sum_{((k_1,j_1),(k_2,j_2),k_3)\in\mathcal{S}_p}2^{\max(k_1,k_2,k_3,0)}\|\widetilde{\varphi}^{(k)}_j\cdot P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2};P_{k_3}h]\|_{B^1_{k,j}}\lesssim 2^{-\beta^4 m},\tag{3.47}$$

for $p=1,2,3$.

Therefore, it suffices to prove that

$$2^{\max(k_1,k_2,k_3,0)}\big\|\widetilde{\varphi}^{(k)}_j\cdot P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2};P_{k_3}h]\big\|_{B^1_{k,j}}\lesssim 2^{-\beta^4(m+j)}\tag{3.48}$$

provided that

$$3j/2\geq m+(N_0-20)k_++D,\quad -10j\leq k_1,k_2,k_3\leq 2j/N_0,\quad \max(j_1,j_2)\leq 10j.\tag{3.49}$$

Step 2. The Cauchy-Schwarz inequality shows that, for any $\xi\in\mathbb{R}^3$,

$$\begin{aligned}
\Big|\mathcal{F}P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2};P_{k_3}h](\xi)\Big|&\lesssim\int_{\mathbb{R}}q_m(s)\|f^{\mu}_{k_1,j_1}(s)\|_{L^2}\|f^{\nu}_{k_2,j_2}(s)\|_{L^2}\|P_{k_3}h(s)\|_{L^\infty}ds\\
&\lesssim 2^{-\max(0,k_3)}2^{-\beta m}\sup_{s\in[2^m-1,2^{m+2}]}\|f^{\mu}_{k_1,j_1}(s)\|_{L^2}\|f^{\nu}_{k_2,j_2}(s)\|_{L^2}\\
&\lesssim 2^{-\max(0,k_1,k_2,k_3)}2^{-\beta m}2^{-(1-\beta)(j_1+j_2)}.
\end{aligned}\tag{3.50}$$

We first assume that

$$j+4k/3\leq D.\tag{3.51}$$

In this case, from (3.29), it suffices to prove that

$$2^{\max(k_1,k_2,k_3,0)}2^{\alpha k}2^{(1+\beta)j}2^{3k/2}\|\mathcal{F}P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2};P_{k_3}h]\|_{L^\infty}\lesssim 2^{-\beta^4(m+j)},$$

which follows from (3.50). Given (3.51), this suffices to prove the desired inequality (3.48).

Assume now that

$$j+4k/3\geq D \qquad \text{and} \qquad j\leq m+D. \tag{3.52}$$

Using again (3.43), (3.41), and (3.44), we have

$$\begin{aligned}2^{\max(k_1,k_2,k_3,0)}&(2^{\alpha k}+2^{10k})2^{(1+\beta)j}\big\|\widetilde{\varphi}^{(k)}_j\cdot P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^\mu_{k_1,j_1},f^\nu_{k_2,j_2};P_{k_3}h]\big\|_{L^2}\\&\lesssim 2^{\max(k_1,k_2,k_3,0)}(2^{\alpha k}+2^{10k})2^{(1+\beta)m}\cdot 2^m2^{-m(2+2\beta)}2^{-(N_0-4)\max(k,k_1,k_2,k_3,0)}\\&\lesssim 2^{-\beta m/2}.\end{aligned}$$

Moreover, using (3.50),

$$2^{\max(k_1,k_2,k_3,0)}(2^{\alpha k}+2^{10k})2^{(1/2-\beta)\widetilde{k}}\big\|\mathcal{F}P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^\mu_{k_1,j_1},f^\nu_{k_2,j_2};P_{k_3}h]\big\|_{L^\infty}\lesssim 2^{-\beta m}2^{11\max(k_1,k_2,k_3,0)}.$$

Recalling (3.29) and the restrictions (3.49) and (3.52), the desired bound (3.48) follows in this case.

Assume now that

$$j+4k/3\geq D \qquad \text{and} \qquad j\geq m+D \qquad \text{and} \qquad j\leq 3\min(j_1,j_2)/2+D. \tag{3.53}$$

Using again (3.43), (3.41), and (3.44), we have

$$\begin{aligned}2^{\max(k_1,k_2,k_3,0)}&(2^{\alpha k}+2^{10k})2^{(1+\beta)j}\big\|\widetilde{\varphi}^{(k)}_j\cdot P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^\mu_{k_1,j_1},f^\nu_{k_2,j_2};P_{k_3}h]\big\|_{L^2}\\&\lesssim 2^{11\max(k_1,k_2,k_3,0)}2^{3(1+\beta)\min(j_1,j_2)/2}\cdot 2^m\|f^\mu_{k_1,j_1}\|_{L^2}2^{3k_2/2}\|f^\nu_{k_2,j_2}\|_{L^2}\|P_{k_3}h\|_{L^\infty}\\&\lesssim 2^{11\max(k_1,k_2,k_3,0)}2^{-\beta m}2^{3(1+\beta)\min(j_1,j_2)/2}2^{-(1-\beta)(j_1+j_2)}\\&\lesssim 2^{-\beta j}.\end{aligned}$$

Moreover, using (3.49) and (3.50),

$$2^{\max(k_1,k_2,k_3,0)}(2^{\alpha k}+2^{10k})2^{(1/2-\beta)\widetilde{k}}\big\|\mathcal{F}P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^\mu_{k_1,j_1},f^\nu_{k_2,j_2};P_{k_3}h]\big\|_{L^\infty}\lesssim 2^{-\beta j}2^{11\max(k_1,k_2,k_3,0)}.$$

Recalling also the restrictions (3.49) and (3.52), the desired bound (3.48) follows in this case.

Finally, assume that

$$j+4k/3\geq D \qquad \text{and} \qquad j\geq m+D \qquad \text{and} \qquad j\geq 3\min(j_1,j_2)/2+D. \tag{3.54}$$

Without loss of generality we may assume that $j_1\leq j_2$, therefore $3/2j_1+D\leq j$. Notice that

$$\begin{aligned}P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^\mu_{k_1,j_1},f^\nu_{k_2,j_2};P_{k_3}h](x)=c\int_{\mathbb{R}}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}&q_m(s)e^{ix\xi}e^{is[\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta-\theta)]}\\&\times\varphi_k(\xi)\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta-\theta,s)\widehat{P_{k_3}h}(\theta,s)dsd\eta d\theta d\xi.\end{aligned}$$

We use integration by parts in ξ, see Lemma A.2 in [18], and the bounds

$$\sup_{\theta\in\mathbb{R}^3}|D^\rho_\theta\widehat{f^\mu_{k_1,j_1}}(\theta)|\lesssim_{|\rho|}(2^{\alpha k_1}+2^{10k_1})^{-1}2^{-(1/2-\beta)\widetilde{k_1}}2^{|\rho|j_1},$$

which follow directly from the definition of the Z norm (see Lemma 3.3). It follows that

$$\big|\widetilde{\varphi}^{(k)}_j(x)\cdot P_k\tilde{T}^{\sigma;\mu,\nu}_m[f^\mu_{k_1,j_1},f^\nu_{k_2,j_2};P_{k_3}h](x)\big|\lesssim 2^{-10j},$$

from which (3.47) follows easily. □

We can complete now the proof of Proposition 3.5.

Proof of Proposition 3.5. The functions $\Lambda_i, \Lambda_e, H_\varepsilon, H_1, |\nabla|, R$ satisfy standard symbol-type estimates

$$\begin{aligned}
&|D^\alpha_\xi \Lambda_e(\xi)| + |D^\alpha_\xi H_\varepsilon(\xi)| + |D^\alpha_\xi H_1(\xi)| \lesssim_\alpha (1+|\xi|)^{1-|\alpha|},\\
&|D^\alpha_\xi \Lambda_i(\xi)| + |D^\alpha_\xi |\nabla|(\xi)| \lesssim_\alpha |\xi|^{1-|\alpha|},\\
&|D^\alpha_\xi R(\xi)| \lesssim_\alpha (1+|\xi|)^{-2-|\alpha|}.
\end{aligned}$$

Moreover, the Z norm is stable under the action of Calderon–Zygmund operators, see Lemma A.1 in [18]. Therefore, using also the identities (3.25) and (3.17) and the bootstrap assumption (3.35), the functions $\rho, n, g, h, A_j, u_j, v_j$ can be written in the form

$$e^{-it\Lambda_i} f_i^+(t) + e^{it\Lambda_i} f_i^-(t) + e^{-it\Lambda_e} f_e^+(t) + e^{it\Lambda_e} f_e^-(t) + e^{-it\Lambda_b} f_b^+(t) + e^{it\Lambda_b} f_b^-(t), \tag{3.55}$$

for certain functions $f_i^+, f_i^-, f_e^+, f_e^-, f_b^+, f_b^-$ satisfying

$$\sup_{t\in[0,T_0]} \|(f_i^+(t), f_i^-(t), f_e^+(t), f_e^-(t), f_b^+(t), f_b^-(t))\|_{Z\cap H^{N_0}} \lesssim \delta_1. \tag{3.56}$$

In passing from the full nonlinearities $\mathcal{N}_\sigma$ to the quadratic nonlinearities $\mathcal{N}_\sigma^2$ we replaced

$$\begin{aligned}
&u_j q_i(\widetilde{\rho}) \quad \text{with} \quad u_j(1-c_i)\rho; && v_j q_e(\widetilde{n}) \quad \text{with} \quad v_j(1-c_e)n;\\
&\frac{\partial_j(|u|^2)}{2\widetilde{\gamma}_i \widetilde{h}_i(\widetilde{\rho})} \quad \text{with} \quad \frac{\partial_j(|u|^2)}{2}; && \frac{\partial_j(|v|^2)}{2\widetilde{\gamma}_e \widetilde{h}_e(\widetilde{n})} \quad \text{with} \quad \frac{\partial_j(|v|^2)}{2};\\
&\frac{\widetilde{h}_i'(\widetilde{\rho})\partial_j\widetilde{\rho}}{\widetilde{h}_i'(0)\widetilde{\gamma}_i} - \partial_j\rho \quad \text{with} \quad \partial_j\Big(\frac{d_i}{2c_i}\rho^2 - \frac{\varepsilon|u|^2}{2C_b}\Big); && \frac{\widetilde{h}_e'(\widetilde{n})\partial_j\widetilde{n}}{\widetilde{h}_e'(0)\widetilde{\gamma}_e} - \partial_j n \quad \text{with} \quad \partial_j\Big(\frac{d_e}{2c_e}n^2 - \frac{\varepsilon|v|^2}{2C_b}\Big).
\end{aligned} \tag{3.57}$$

These substitutions are justified informally, by the definitions at the beginning of the subsection and by the formulas (3.5).

To justify these substitutions rigorously we use first Lemma 3.3 and the representations (3.55)-(3.56) to conclude that, for any $t \in [0, T_0]$,

$$\|(\rho(t), n(t), u(t), v(t))\|_{H^{N_0}} + (1+|t|)^{1+\beta}\|(\rho(t), n(t), u(t), v(t))\|_{W^{\infty,5}} \lesssim \delta_1. \tag{3.58}$$

Using standard algebra properties of the spaces H^{N_0} and $W^{\infty,5}$, it follows that

$$\|F(\rho(t), n(t), u(t), v(t))\|_{H^{N_0}} + (1+|t|)^{1+\beta}\|F(\rho(t), n(t), u(t), v(t))\|_{W^{\infty,5}} \lesssim \delta_1, \tag{3.59}$$

for any smooth function F satisfying

$$F(0) = 0, \qquad \sum_{|\alpha|\leq N_0+1} |D^\alpha F(0)| \leq A, \tag{3.60}$$

provided that δ_1 is sufficiently small relative to A. In particular, using (3.5),

$$\|(\widetilde{\rho}(t), \widetilde{n}(t), \widetilde{\gamma}_i(t)-1, \widetilde{\gamma}_e(t)-1)\|_{H^{N_0}} + (1+|t|)^{1+\beta}\|(\widetilde{\rho}(t), \widetilde{n}(t), \widetilde{\gamma}_i(t)-1, \widetilde{\gamma}_e(t)-1)\|_{W^{\infty,5}} \lesssim \delta_1. \tag{3.61}$$

Let a denote generic functions satisfying the bounds

$$\|a\|_{H^{N_0}} + (1+|t|)^{1+\beta}\|a\|_{W^{\infty,5}} \lesssim \delta_1. \tag{3.62}$$

Using the formulas (3.5) and the bounds above, we write

$$\begin{aligned}
\widetilde{\gamma}_i(t) - 1 &= \frac{\varepsilon|u(t)|^2}{2C_b}(1+a), & \widetilde{\gamma}_e(t) - 1 &= \frac{\varepsilon|v(t)|^2}{2C_b}(1+a),\\
\rho(t) - \widetilde{\rho}(t) &= \frac{\varepsilon|u(t)|^2}{2C_b}(1+a), & n(t) - \widetilde{n}(t) &= \frac{\varepsilon|v(t)|^2}{2C_b}(1+a),
\end{aligned}$$

Therefore, using also the algebra properties of the spaces H^{N_0} and $W^{5,\infty}$,

$$\begin{aligned}u_j(t)q_i(\widetilde{\rho}(t)) &= u_j(t)q_i\Big(\rho(t)-\frac{\varepsilon|u(t)|^2}{2C_b}(1+a)\Big)\\&= u_j(t)(1-c_i)\Big(\rho(t)-\frac{\varepsilon|u(t)|^2}{2C_b}(1+a)\Big)+u_j(t)\Big(\rho(t)-\frac{\varepsilon|u(t)|^2}{2C_b}(1+a)\Big)^2\Big(\frac{q_i''(0)}{2}+a\Big),\\ \frac{\partial_j(|u(t)|^2)}{2\widetilde{\gamma}_i(t)\widetilde{h}_i(\widetilde{\rho}(t))} &= \frac{\partial_j(|u(t)|^2)}{2}(1+a),\end{aligned}$$

and

$$\begin{aligned}\frac{\widetilde{h}_i'(\widetilde{\rho}(t))\partial_j\widetilde{\rho}(t)}{\widetilde{h}_i'(0)\widetilde{\gamma}_i(t)} &= \partial_j\Big(\frac{c_i\widetilde{\rho}(t)+(d_i/2)\widetilde{\rho}(t)^2(1+a)}{c_i}\Big)\Big(1-\frac{\varepsilon|u(t)|^2}{2C_b}(1+a)\Big)\\&=\Big(1-\frac{\varepsilon|u(t)|^2}{2C_b}(1+a)\Big)\partial_j\Big[\rho(t)-\frac{\varepsilon|u(t)|^2}{2C_b}(1+a)+\frac{d_i}{2c_i}(1+a)\Big(\rho(t)-\frac{\varepsilon|u(t)|^2}{2C_b}(1+a)\Big)^2\Big].\end{aligned}$$

Similar formulas hold for the nonlinearities $v_j(t)q_e(\widetilde{n}(t))$, $\frac{\partial_j(|v(t)|^2)}{2\widetilde{\gamma}_e(t)\widetilde{h}_e(\widetilde{n}(t))}$, and $\frac{\widetilde{h}_e'(\widetilde{n}(t))\partial_j\widetilde{n}(t)}{\widetilde{h}_e'(0)\widetilde{\gamma}_e(t)}$. Therefore the substitutions in (3.57) are justified, in the sense that the nonlinearities $\mathcal{N}_\sigma(t)-\mathcal{N}_\sigma^2(t)$ can be written as finite sums of functions of the form $\Gamma_1 g_1(t)\cdot\Gamma_2 g_2(t)\cdot\Gamma_3 g_3(t)$, where $g_1,g_2\in\{\rho,n,u_1,u_2,u_3,v_1,v_2,v_3\}$, g_3 satisfies (3.62), and $\Gamma_1,\Gamma_2,\Gamma_3\in\{I,\partial_1,\partial_2,\partial_3\}$ are such that at most one is a derivative. For Proposition 3.5 it suffices to prove that

$$\Big\|\mathcal{F}^{-1}\Big[\int_0^t e^{is\Lambda_\sigma(\xi)}\mathcal{F}\big[\Gamma_1 g_1(s)\cdot\Gamma_2 g_2(s)\cdot\Gamma_3 g_3(s)\big](\xi)\,ds\Big]\Big\|_Z\lesssim\delta_1^2.$$

The Z norm is dominated by a suitable $B^1_{k,j}$ norm, see Definition 3.2, so the inequality above follows using the representations (3.55)-(3.56). This completes the proof of the proposition. □

References

[1] S. Alinhac, Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux. (French) [Life spans of the classical solutions of two-dimensional axisymmetric compressible Euler equations] Invent. Math. 111 (1993), no. 3, 627–670.

[2] J. A. Bittencourt Fundamentals of plasma physics, 3rd edition, 2004, Springer ISBN-13: 978-1441919304.

[3] S. Cordier and E. Grenier. Quasineutral limit of an Euler-Poisson system arising from plasma physics. Comm. Partial Differential Equations, 25 (5-6):1099–1113, 2000.

[4] F. Chen, introduction to plasma physics, 1995, Springer, ISBN-13: 978-0306307553.

[5] G.-Q. Chen, J. Jerome and D. Wang, Compressible Euler-Maxwell equations. Proceedings of the Fifth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics (Maui, HI, 1998). Transport Theory Statist. Phys. 29 (2000), no. 3-5, 311–331.

[6] P. Crispel, P. Degond and M. M.-H. Vignal, An asymptotic preserving scheme for the two-fluid Euler-Poisson model in the quasi neutral limit, J. Comp. Phys. 223 (2007) 208–234.

[7] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, Comm. Pure Appl. Math. 39 (1986), 267–282.

[8] D. Christodoulou, The Formation of Shocks in 3-Dimensional Fluids, EMS Monographs in Mathematics, EMS Publishing House, 2007.

[9] J.-L. Delcroix and A. Bers, Physique des plasmas, InterEditions/ CNRS Editions, Paris, 1994.

[10] P. Degond, F. Deluzet and D. Savelief, Numerical approximation of the Euler-Maxwell model in the quasineutral limit, J. Comput. Phys. 231 (2012), no. 4, 1917–1946.

[11] D. Gérard-Varet, D. Han-Kwan and F. Rousset, Quasineutral limit of the Euler-Poisson system for ions in a domain with boundaries, Indiana Univ. Math. J., to appear.

[12] P. Germain and N. Masmoudi, Global existence for the Euler-Maxwell system, preprint arXiv:1107.1595

[13] P. Germain, N. Masmoudi and B. Pausader, Non-neutral global solutions for the electron Euler-Poisson system in 3D. SIAM J. Math. Anal., 45-1 (2013), 267–278.

[14] P. Germain, N. Masmoudi, and J. Shatah, Global solutions for 3D quadratic Schrödinger equations, Int. Math. Res. Not. (2009), 414–432.

[15] P. Germain, N. Masmoudi, and J. Shatah, Global solutions for the gravity water waves equation in dimension 3, Ann. of Math., to appear.
[16] P. Germain, N. Masmoudi and J. Shatah, Global existence for capillary water-waves, Comm. Pure. Appl. Math., to appear.
[17] Y. Guo, Smooth irrotational flows in the large to the Euler-Poisson system in $\mathbb{R}^{3+1}$. Comm. Math. Phys. 195 (1998), no. 2, 249–265.
[18] Y. Guo, A. D. Ionescu, and B. Pausader, Global solutions of the Euler–Maxwell two-fluid system in 3D, preprint, arXiv:1303.1060.
[19] Y. Guo and B. Pausader, Global smooth ion dynamics in the Euler-Poisson system, Comm. Math. Phys. 303 (2011), 89–125.
[20] Y. Guo and X. Pu, KdV limit of the Euler-Poisson system. arXiv:1202.1830.
[21] Y. Guo and S. Tahvildar-Zadeh, Formation of Singularities in Relativistic Fluid Dynamics and in Spherically Symmetric Plasma Dynamics, Contemporary Mathematics, Vol. 238, 151–161, (1999).
[22] S. Gustafson, Stephen, K. Nakanishi, and T.-P. Tsai, Scattering theory for the Gross-Pitaevskii equation in three dimensions, Commun. Contemp. Math. 11 (2009), 657–707.
[23] Z. Hani, F. Pusateri, and J. Shatah, Scattering for the Zakharov system in 3 dimensions, Comm. Math. Phys. to appear.
[24] A. D. Ionescu and B. Pausader, The Euler-Poisson system in 2D: global stability of the constant equilibrium solution, Int. Math. Res. Not., 2013 (2013), 761–826.
[25] A. D. Ionescu and B. Pausader, Global solutions of quasilinear systems of Klein–Gordon equations in 3D, J. Eur. Math. Soc., to appear.
[26] A. D. Ionescu and F. Pusateri, Nonlinear fractional Schrödinger equations in one dimensions, J. Funct. Anal., to appear, arXiv:1209.4943.
[27] A. D. Ionescu and F. Pusateri, Global solutions for the gravity water waves system in 2d, preprint arXiv:1303.5357.
[28] J. Jang, The two-dimensional Euler-Poisson system with spherical symmetry. J. Math. Phys. 53 (2012), no. 2, 023701, 4 pp.
[29] J. Jang, D. Li and X. Zhang, Smooth global solutions for the two dimensional Euler-Poisson system, Forum Mathematicum, to appear.
[30] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. 28 (1979), 235–268.
[31] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Rational Mech. Anal. 58 (1975), 181–205.
[32] S. Klainerman, Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions, Comm. Pure Appl. Math. 38, 631–641 (1985).
[33] S. Klainerman, The null condition and global existence to nonlinear wave equations. Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 293—326, Lectures in Appl. Math. 23, Amer. Math. Soc., Providence, RI, 1986.
[34] D. Lannes, F. Linares, J.-C. Saut, The Cauchy problem for the Euler-Poisson system and derivation of the Zakharov-Kuznetsov equation, "Perspectives in Phase Space Analysis of PDE's" to appear in Birkhäuser series "Progress in Nonlinear Differential Equations and Their Applications".
[35] D. Li and Y. Wu, The Cauchy problem for the two dimensional Euler-Poisson system. arXiv:1109.5980
[36] Y. Nejoh and H. Sanuki, Large amplitude Langmuir and ion?acoustic waves in a relativistic two-fluid plasma, Phys. Plasmas 1, 2154 (1994).
[37] Y. J. Peng, Global existence and long-time behavior of smooth solutions of two-fluid Euler-Maxwell equations. Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), no. 5, 737–759.
[38] X. Pu, Dispersive Limit of the Euler-Poisson System in Higher Dimensions. arXiv:1204.5435.
[39] J. Shatah, Normal forms and quadratic nonlinear Klein-Gordon equations, Comm. Pure Appl. Math. 38 (1985), 685–696.
[40] T. Sideris, Formation of singularities in three-dimensional compressible fluids. Comm. Math. Phys. 101, (1985), 475–485.
[41] B. Texier, Derivation of the Zakharov equations. Arch. Ration. Mech. Anal. 184 (2007), no. 1, 121–183.
[42] N. L. Tsintsadze, Effects of electron mass variations in a strong electromagnetic wave, 1990 Phys. Scr. 1990 41

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